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CARTESIAN CLOSEDNESS IN CATEGORIES OF PARTIAL ALGEBRAS

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Abstract: We study categories of partial algebras of the same type. In these categories we define a binary operation of exponentiation for objects and investigate its behaviour. We discover two cartesian closed initially structured subcategories in every category of partial algebras of the same type.

It is well known that concrete categories having well-behaved function spaces, i.e. being initially structured and cartesian closed, play an important role in applications to many branches of mathematics. It is therefore worthwhile to look for such categories also among categories of general algebraic systems. In this note we focus our interest onto categories of partial algebras. As for generality, partial algebras lie between total (i.e. universal) algebras and relational systems. Therefore, when studying partial algebras, we can extend considerations known for total algebras or restrict those known for relational systems. However, such an extension or restriction is often not quite trivial and many new particular considerations have to be done for partial algebras.

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In our study of partial algebras we shall be sustained by results of [12] and [13] concerning categories of total algebras and by those of [9] concerning categories of relational systems. We shall define and investigate a binary operation of exponentiation for objects of categories of partial algebras of the same type. This operation can not be restricted to categories of total algebras in general. However, in some special cases such a restriction is possible and we then receive certain results for total algebras as consequences of results proved for partial algebras. Throughout the paper, all categories are considered to be concrete categories over the category of sets, i.e. to be categories of structured stes and structure-compatible maps. For any object A of a category we denote by $|A|$ the underlying set of A.

Definition 1. [11] Let $\mathbf{K}$ be a category with finite products and $\mathbf{S}, \mathbf{T}$ be full isomorphism closed subcategories of $\mathbf{K}$. Let $\mathbf{T}$ be finitely productive in $\mathbf{K}$. We say that $\mathbf{T}$ is *exponential* for $\mathbf{S}$ in $\mathbf{K}$ provided that for any two objects $A \in \mathbf{S}$ and $B \in \mathbf{T}$ there exists an object $A^B \in \mathbf{K}$ with $|A^B| = \mathrm{Mor}_{\mathbf{K}}(B, A)$ such that

(i) $A^B \in \mathbf{S} \cap \mathbf{T}$

(ii) the pair (A^B, e), where $e : B \times A^B \to A$ is the evaluation map (given by $e(y, f) = f(y)$), is a co-universal map for A with respect to the functor $B \times - : \mathbf{T} \to \mathbf{K}$.

If a category $\mathbf{T}$ is exponential for $\mathbf{K}$ in $\mathbf{K}$, then $\mathbf{T}$ will be called an *exponential subcategory* of $\mathbf{K}$ (cf. [7]). If $\mathbf{K}$ is an exponential subcategory of itself, then $\mathbf{K}$ is cartesian closed [2], i.e. the functor $B \times - : \mathbf{K} \to \mathbf{K}$ has a right adjoint for each object $B \in \mathbf{K}$ (and vice versa whenever in $\mathbf{K}$ all constant maps are morphisms). Especially, if $\mathbf{T}$ is exponential for $\mathbf{S}$ in $\mathbf{K}$ and if also $\mathbf{S}$ is finitely productive in $\mathbf{K}$, then $\mathbf{S} \cap \mathbf{T}$ is cartesian closed.

The objects A^B from the definition are unique up to the isomorphisms that are identity maps, and thus they are unique whenever $\mathbf{K}$ is transportable. These objects fulfil the first exponential law, i.e. the law $(A^B)^C \simeq A^{B \times C}$ (where $\simeq$ denotes the isomorphism in $\mathbf{K}$).

The concepts concerning partial algebras are taken from [1]. Throughout the paper, Ω will designate an arbitrary, but fixed set, and τ will designate an arbitrary, but fixed family of sets $\tau = (K_\lambda; \lambda \in \Omega)$. The family τ will be called a *type*. By a *partial algebra* of type τ we understand a pair $\langle X, (p_\lambda; \lambda \in \Omega) \rangle$ where X is a set and p_λ is a partial K_λ-ary operation on X (i.e. a partial map $p_\lambda : X^{K_\lambda} \to X$) for each $\lambda \in \Omega$. For any $\lambda \in \Omega$ we denote by D_{p_λ} the domain of the operation p_λ, i.e. the subset of X^{K_λ} having the property that $p_\lambda(x_k; k \in K_\lambda)$ is defined iff $(x_k; k \in K_\lambda) \in D_{p_\lambda}$. If $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ and $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ are partial algebras of type τ, then by a *homomorphism* of G into H we mean any map $f : X \to Y$ such that $p_\lambda(x_k; k \in K_\lambda) = x \Rightarrow q_\lambda(f(x_k); k \in K_\lambda) = f(x)$ for each $\lambda \in \Omega$. The set of all homomorphisms from G into H will be denoted by $\mathrm{Hom}(G, H)$. We denote by Pal_τ the category of all partial algebras of type τ with homomorphisms as morphisms. Obviously, Pal_τ is a transportable category with products (given by the direct products — see [1]).

Definition 2. For any pair of objects $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle, H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle \in \mathrm{Pal}_\tau$ we put $G^H = \langle \mathrm{Hom}(H, G), (r_\lambda; \lambda \in \Omega)\rangle$ where, for each $\lambda \in \Omega$, r_λ is the K_λ-ary partial operation on $\mathrm{Hom}(H, G)$ given by $r_\lambda(f_k; k \in K_\lambda) = f$ iff $f \in \mathrm{Hom}(H, G)$ is a unique homomorphism with the property that $q_\lambda(y_k; k \in K_\lambda) = y \Rightarrow p_\lambda(f_k(y_k); k \in K_\lambda) = f(y)$. The objects G^H will be called the *powers* of G and H.

The above defined powers of partial algebras of the same type do not fulfil the first exponential law in general. Nevertheless, they fulfil some other usual exponential laws. For example, the following statement is valid ($\prod$ and $\simeq$ denote the direct product and isomorphism in Pal_τ):

Proposition 1. *Let $G_i \in \mathrm{Pal}_\tau$ for each $i \in I$ and $H \in \mathrm{Pal}_\tau$ be objects. Then $(\prod_{i\in I} G_i)^H \simeq \prod_{i\in I} G_i^H$.*

Proof. Let $G_i = \langle X_i, (p^i_\lambda; \lambda \in \Omega)\rangle$ for each $i \in I$ and let $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$. For each $\lambda \in \Omega$ let $p_\lambda, s_\lambda, r^i_\lambda$ $(i \in I), r_\lambda$ be the K_λ-ary partial operations of the partial algebras $\prod_{i\in I} G_i, (\prod_{i\in I} G_i)^H, G_i^H, \prod_{i\in I} G_i^H$ respectively. We denote by pr_i $(i \in I)$ the i-th projection $\mathrm{pr}_i : \prod_{i\in I} G_i \to G_i$. For any $f \in \mathrm{Hom}(H, \prod_{i\in I} G_i)$ put $\varphi(f) = g$ where $g : I \to \bigcup_{i\in I} \mathrm{Hom}(H, G_i)$ is the map given by $g(i) = pr_i \circ f$ whenever $i \in I$. It can easily be seen that φ is a bijection of $\mathrm{Hom}(H, \prod_{i\in I} G_i)$ onto $\prod_{i\in I} \mathrm{Hom}(H, G_i)$. Let $\lambda \in \Omega, (f_k; k \in K_\lambda) \in D_{s_\lambda}$ and $s_\lambda(f_k; k \in K_\lambda) = f$. Next, let $(y_k; k \in K_\lambda) \in D_{q_\lambda}$ and $q_\lambda(y_k; k \in K_\lambda) = y$. Then for each $i \in I$ we have $p^i_\lambda(\varphi(f_k)(i)(y_k); k \in K_\lambda) = p^i_\lambda(\mathrm{pr}_i(f_k(y_k)); k \in K_\lambda) = \mathrm{pr}_i(p_\lambda(f_k(y_k); k \in K_\lambda)) = \mathrm{pr}_i(f(y)) = \varphi(f)(i)(y)$. As $\varphi(f)(i)$ is clearly unique (because f is unique), for each $i \in I$ we have $r^i_\lambda(\varphi(f_k)(i); k \in K_\lambda) = \varphi(f)(i)$. Consequently, $r_\lambda(\varphi(f_k); k \in$

$\in K_\lambda) = \varphi(f)$ which means that $\varphi : (\prod\limits_{i\in I} G_i)^H \to \prod\limits_{i\in I} G_i^H$ is a homomorphism. Clearly, the inverse map φ^{-1} to φ is given by $\varphi^{-1}(g) = f$ where $f(y)(i) = g(i)(y)$ for all $y \in Y$ and $i \in I$. Let $(g_k; k \in K_\lambda) \in D_{r_\lambda}$ and $r_\lambda(g_k; k \in K_\lambda) = g$. Then for each $i \in I$ we have $p^i_\lambda(\varphi^{-1}(g_k)(y_k)(i); k \in K_\lambda) = p^i_\lambda(g_k(i)(y_k); k \in K_\lambda) = g(i)(y)$ because $r^i_\lambda(g_k(i); k \in K_\lambda) = g(i)$. But $g(i)(y) = \varphi^{-1}(g)(y)(i)$ and consequently $p_\lambda(\varphi^{-1}(g_k)(y_k); k \in K_\lambda) = \varphi^{-1}(g)(y)$. As $\varphi^{-1}(g)$ is clearly unique (because g is unique), $s_\lambda(\varphi^{-1}(g_k); k \in K_\lambda) = \varphi^{-1}(g)$ which means that $\varphi^{-1} : \prod\limits_{i\in I} G_i^H \to (\prod\limits_{i\in I} G_i)^H$ is a homomorphism. ◊

A partial algebra $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called *idempotent* if all its operations are idempotent, i.e. if for any $x \in X$ and any $\lambda \in \Omega$ the family $(x_k; k \in K_\lambda)$ given by $x_k = x$ for each $k \in K_\lambda$ fulfils $(x_k; k \in K_\lambda) \in D_{p_\lambda}$ and $p_\lambda(x_k; k \in K_\lambda) = x$. We denote by IPal_τ the full subcategory of Pal_τ whose objects are precisely the idempotent partial algebras of type τ. Clearly, IPal_τ is an initially structured category (in the sense of [6]).

Theorem 1. IPal_τ *is an exponential subcategory of* Pal_τ.

Proof. Clearly, IPal_τ is productive in Pal_τ. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle \in \mathrm{Pal}_\tau$ and $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle \in \mathrm{IPal}_\tau$ be objects. Then, obviously, $G^H \in \mathrm{IPal}_\tau$. Let $e : H \times G^H \to G$ be the evaluation map and, for each $\lambda \in \Omega$, let r_λ and s_λ be the K_λ-ary partial operations of G^H and $H \times G^H$, respectively. Let $\lambda \in \Omega, ((y_k, f_k); k \in K_\lambda) \in D_{s_\lambda}$ and $s_\lambda((y_k, f_k); k \in K_\lambda) = (y, f)$. Then $p_\lambda(e(y_k, f_k); k \in K_\lambda) = p_\lambda(f_k(y_k); k \in K_\lambda) = f(y) = e(y, f)$ because $q_\lambda(y_k; k \in K_\lambda) = y$ and $r_\lambda(f_k; k \in K_\lambda) = f$. Therefore $e : H \times G^H \to G$ is a homomorphism. Let $K = \langle Z, (t_\lambda; \lambda \in \Omega)\rangle \in \mathrm{IPal}_\tau$ be an object and $\varphi : H \times K \to G$ a homomorphism. For any $z \in Z$ and $y \in Y$ put $\widehat{\varphi}(z)(y) = \varphi(y, z)$. Now, if $\lambda \in \Omega$ and $q_\lambda(y_k; k \in K_\lambda) = y$, then $p_\lambda(\widehat{\varphi}(z)(y_k); k \in K_\lambda) = p_\lambda(\varphi(y_k, z); k \in K_\lambda) = \varphi(y, z) = \widehat{\varphi}(z)(y)$ for each $z \in Z$. Consequently, $\widehat{\varphi}(z) : H \to G$ is a homomorphism for each $z \in Z$. Let $\lambda \in \Omega, (z_k, k \in K_\lambda) \in D_{t_\lambda}$ and $t_\lambda(z_k; k \in K_\lambda) = z$. Then from $q_\lambda(y_k; k \in K_\lambda) = y$ it follows that $p_\lambda(\widehat{\varphi}(z_k)(y_k); k \in K_\lambda) = p_\lambda(\varphi(y_k, z_k); k \in K_\lambda) = \varphi(y, z) = \widehat{\varphi}(z)(y)$. Because of the idempotency of $H, \widehat{\varphi}(z)$ is a unique homomorphism of H into G having this property. Therefore $r_\lambda(\widehat{\varphi}(z_k); k \in K_\lambda) = \widehat{\varphi}(z)$ and we have shown that $\widehat{\varphi} \in \mathrm{Hom}(K, G^H)$. Clearly, $e \circ (\mathrm{id}_y \times \widehat{\varphi}) = \varphi$ and $\widehat{\varphi}$ is a unique homomorphism of K into G^H fulfilling this equality. Consequently, (G^H, e) is a co-universal map for G with respect to the functor $H \times - : \mathrm{IPal}_\tau \to \mathrm{Pal}_\tau$. $\Diamond$

Corollary 1. IPal_τ *is a cartesian closed category.*

Definition 3. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ be a partial algebra of type τ.

a) G is said to fulfil the *interchange law* if for any pair of elements $\lambda, \mu \in \Omega$ from $(x_{kl}; l \in K_\mu) \in D_{p_\mu}$ for each $k \in K_\lambda$, $(p_\mu(x_{kl}; l \in K_\mu); k \in K_\lambda) \in D_{p_\lambda}$ and $(x_{kl}; k \in K_\lambda) \in D_{p_\lambda}$ for each $l \in K_\mu$ it follows that $(p_\lambda(x_{kl}; k \in K_\lambda); l \in K_\mu) \in D_{p_\mu}$ and $p_\lambda(p\mu(x_{kl}; l \in K_\mu); k \in K_\lambda) = p_\mu(p_\lambda(x_{kl}; k \in K_\lambda); l \in K_\mu)$.

b) G is called *diagonal* if for any element $\lambda \in \Omega$ from $(x_{kl}; l \in K_\lambda) \in D_{p_\lambda}$ for each $k \in K_\lambda$ and $(p_\lambda(x_{kl}; l \in K_\lambda); k \in K_\lambda) \in D_{p_\lambda}$ it follows that $(x_{kk};\ k \in K_\lambda) \in D_{p\lambda}$ and $p_\lambda(p_\lambda(x_{kl}; l \in K_\lambda); k \in K_\lambda) = p_\lambda(x_{kk}; k \in K_\lambda)$.

We denote by CPal_τ or DPal_τ the full subcategory of Pal_τ whose objects are precisely the partial algebras of type τ fulfilling the interchange law or the diagonal partial algebras of type τ, respectively. Next, we put $\mathrm{CDPal}_\tau = \mathrm{CPal}_\tau \cap \mathrm{DPal}$ and $\mathrm{CDIPal}_\tau = \mathrm{CDPal}_\tau \cap \mathrm{IPal}_\tau$. The category CDIPal_τ is obviously initial structured.

Remark 1. For total algebras the notion of interchange law coincides with the notion of commutativity studied in [5]. On the other hand, the notion of diagonality in the case of total algebras is more general than the notion of diagonality defined and studied in [8] (the diagonality in [8] means both diagonality and idempotency).

The following assertion is a generalization of a result which is well known for total algebras (see e.g. [5]):

Lemma 1. *Let* $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle \in \mathrm{CPal}_\tau$ *and* $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle \in \mathrm{Pal}_\tau$ *be objects. Then there exists a subalgebra of the direct product* $G^{|H|}$ *whose underlying set is* $\mathrm{Hom}(H, G)$.

Proof. Let $G^{|H|} = \langle X^Y, (s_\lambda; \lambda \in \Omega)\rangle$. Let $\lambda \in \Omega$, $(f_k; k \in K_\lambda) \in (\mathrm{Hom}(H,G))^{K_\lambda}$ and $s_\lambda(f_k; k \in K_\lambda) = f$. Let $\mu \in \Omega$ and $(y_l; l \in K_\mu) \in D_{q_\mu}$. Then we have $f(q_\mu(y_l; l \in K_\mu)) = p_\lambda(f_k(q_\mu(y_l; l \in K_\mu)); k \in K_\lambda) = p_\lambda(p_\mu(f_k(y_l); l \in K_\mu); k \in K_\lambda) = p_\mu(p_\lambda(f_k(y_l); k \in K_\lambda); l \in K_\mu) = p_\mu(f(y_l); l \in K_\mu)$. Hence $f \in \mathrm{Hom}(H, G)$. $\Diamond$

We denote by $[H, G]$ the subalgebra of $G^{|H|}$ from Lemma 1.

Proposition 2. *Let* $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle \in \mathrm{CDPal}_\tau$ *and* $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle \in \mathrm{IPal}_\tau$ *be objects. Then* $G^H = [H, G]$.

Proof. Let $[H, G] = \langle \mathrm{Hom}(H, G), (s_\lambda; \lambda \in \Omega)\rangle$, $G^H = \langle \mathrm{Hom}(H, G), (\tau_\lambda; \lambda \in \Omega)\rangle$ and let $\lambda \in \Omega$ be an element. Let $s_\lambda(f_k; k \in K_\lambda) = f$ and let $(y_l; l \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_l; l \in K_\lambda) = y$. Then $f \in \mathrm{Hom}(H, G)$

and $f(y) = f(q_\lambda(y_l; l \in K_\lambda)) = p_\lambda(f_k(q_\lambda(y_l; l \in K_\lambda)); k \in K_\lambda) = = p_\lambda(p_\lambda(f_k(y_l); l \in K_\lambda); k \in K_\lambda) = p_\lambda(f_k(y_k); k \in K_\lambda)$. The idempotency of H now implies that f is a unique homomorphism of H into G with this property. We have shown that $s_\lambda(f_k; k \in K_\lambda) = f\rangle\tau_\lambda(f_k; k \in \in K_\lambda) = f$ for each $\lambda \in \Omega$. As the converse implication results from the idempotency of H, the proof is complete. ◊

Theorem 2. *The category* IPal_τ *is exponential for* CDPal_τ *in* Pal_τ*.*

Proof. From Prop. 2 it immediately follows that $G^H \in \mathrm{CDIPal}_\tau$ whenever $G \in \mathrm{CDPal}_\tau$ and $H \in \mathrm{IPal}_\tau$. Now the statement follows from Th. 1. ◊

Corollary 2. CDIPal_τ *is a cartesian closed category.*

Example. Some examples of total algebras fulfilling the interchange law and being diagonal and idempotent can be found in [12] and [13]. In order to give a non-trivial but simple example of a non-total partial algebra having the same properties, let $X = (X, \rho)$ be a set with a binary relation (i.e. a directed graph). Let ∇ be the binary partial operation on X for which $x\nabla y$ is defined $(x, y \in X)$ iff $x\rho y$ and then $x\nabla y = x$, and let Δ be the binary partial operation on X that is dual to ∇, i.e. $x\Delta y$ is defined iff $x\rho y$ and then $x\Delta y = y$. It can easily bee see that the algebra $(X, \nabla, \Delta) \in \mathrm{Pal}_{(2,2)}$ (where $2 = \{0, 1\}$) fulfills:

(1) $(X, \nabla, \Delta) \in \mathrm{CPal}_{(2,2)}$,
(2) ρ is reflexive iff $(X, \nabla, \Delta) \in \mathrm{IPal}_{(2,2)}$,
(3) if ρ is transitive, then $(X, \nabla, \Delta) \in \mathrm{CDPal}_{(2,2)}$,
(4) X is a preordered set iff $(X, \nabla, \Delta) \in \mathrm{CDPal}_{(2,2)}$.

Of course, the same is valid also for each of the algebras $(X, \nabla) \in \mathrm{Pal}_{(2)}$ and $(X, \Delta) \in \mathrm{Pal}_{(2)}$.

Remark 2. a) The Th. 2, in contrast to Th. 1, remains valid also when restricting our considerations to total algebras. The Cor. 2 then coincides with a statement from [13].

b) Let us replace the definition of diagonality of partial algebras with the following, stronger definition: a partial algebra $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called diagonal if for any $\lambda \in \Omega$ from $(x_{kl}; k \in K_\lambda) \in D_{p_\lambda}$ for each $k \in K_\lambda$ it follows that the condition $p_\lambda(p_\lambda(x_{kl}; l \in K_\lambda); k \in \in K_\lambda) = x$ is equivalent with $p_\lambda(x_{kk}; k \in K_\lambda) = x$. Then Prop. 2, Th. 2 and Cor. 2 remain valid and, moreover, for mono-K-ary partial algebras (K a set), i.e. partial algebras of type $\tau = (K)$, the diagonality implies the validity of the interchange law. Thus, this case, IPal_τ is exponential for DPal_τ in Pal_τ, and DIPal_τ is a cartesian closed category. Of course, for mono-K-ary total algebras both definitions of diagonality coincide. If K is a set with card $K = 2$, then the mono-K-ary total algebras fulfilling the interchange law are nothing else than the known medial groupoids (see e.g. [4]).

c) In [12] it is shown that for $\tau = (K)$, where K is a finite set, the full subcategory of the category DIPal_τ given by its total objects is cartesian closed. For arbitrary type τ the cartesian closedness of the full subcategory of the category CDIPal_τ given by its total objects is proved in [13]. A necessary and sufficient condition for a total algebra to be diagonal is given in [10].

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