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HYPERBOLAS AND ORTHOLOGIC TRIANGLES

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Abstract: We obtain several characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas of a triangle ABC using the family of triangles with vertices on Euler lines of triangles BCP, CAP, and ABP for a variable point P in the plane and the notion of orthologic triangles.

1. Introduction

Among conics which pass through the vertices A, B, C of the triangle ABC and its orthocentre H the most interesting are Feuerbach, Kiepert, and Jarabek hyperbolas. These are equilateral hyperbolas that go through the incentre I, the centroid G, and the circumcentre O, respectively. They have been extensively studied in the past. The following are some more recent papers that consider them: [1], [2], [6], [7], [5], [11], [19], [18], and [22].

In this paper we shall present new characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas associated to a triangle ABC. We shall use the same method for all three hyperbolas. Our idea is to associate to every point P and every real number $\lambda \neq -1$ a triangle $P_a^\lambda P_b^\lambda P_c^\lambda$ and to look for triangles XYZ having the property that P

lies on a hyperbola if and only if the triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and XYZ are orthologic.

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars at vertices of ABC onto sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto sides BC, CA, and AB of ABC are concurrent at the point $[XYZ, ABC]$.

In this definition and throughout this paper all triangles are non-degenerate, that is, their vertices are not collinear. The last assumption implies that in our approach we must exclude some points P so that ours are characterisations of three named hyperbolas without a small number of their points.

In order to describe triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ more precisely, we need the following definitions. Let $\lambda \neq -1$ be a real number. For points A and B, let $[A, B; \lambda]$ be a point A when $A = B$ and a unique point P on the line AB such that $|AP|/|PB| = \lambda$ when $A \neq B$. For a triangle ABC, let $W(ABC)$ denote the complement in the plane of the union of the side lines BC, CA, AB. For a point P in $W(ABC)$, let G_a^P, G_b^P, G_c^P and O_a^P, O_b^P, O_c^P denote centroids and circumcentres of triangles BCP, CAP, ABP, respectively. Let

$$P_a^\lambda = [O_a^P, G_a^P; \lambda], \quad P_b^\lambda = [O_b^P, G_b^P; \lambda], \quad \text{and} \quad P_c^\lambda = [O_c^P, G_c^P; \lambda],$$

and let $\mathcal{F}_\lambda$ denote the function which associates to a point P a triangle $P_a^\lambda P_b^\lambda P_c^\lambda$.

In the Section 3 we shall prove that the points P_a^λ, P_b^λ, and P_c^λ are collinear if and only if P lies on a plane quartic denoted here by Q_λ. Hence, the domain of the function $\mathcal{F}_\lambda$ is the complement $W_\lambda(ABC)$ of Q_λ in $W(ABC)$. Let $V_\lambda(ABC)$ denote the complement of the circumcircle γ_0 of ABC in $W_\lambda(ABC)$.

Let γ be a curve in the plane. Let { be a function from a subset S of the plane that associates to each point P of S a triangle $\mathcal{F}(P)$. A triangle XYZ is $(\mathcal{F}, \gamma)$-*simple in* S provided XYZ is orthologic to $\mathcal{F}(P)$ if and only if a point P is in the set $\gamma \cap S$.

Let γ_F, γ_J, and γ_K denote the Feuerbach, Jarabek, and Kiepert hyperbola of the triangle ABC, respectively. With the above definitions and notation we can formulate the results of this paper as contributions to the following problem.

Problem. For $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, find $(\mathcal{F}_\lambda, \gamma)$-simple in $V_\lambda(ABC)$ triangles.

Observe that when we know that a triangle XYZ is $(\mathcal{F}_\lambda, \gamma)$-simple in $V_\lambda(ABC)$ for $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, then we have the following **characterization** of γ:

The hyperbola γ is the closure of all points P in $V_\lambda(ABC)$ such that the triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and XYZ are orthologic.

The triangles XYZ which we prove in this paper to provide solutions to the above problem are all endpoints of segments of controlled length perpendicular to sides of ABC. A more formal description uses the following notation.

For a triple $h = (s_1, s_2, s_3)$ of real numbers and for triangles ABC and XYZ, let $[ABC, XYZ, h]$ denote a triangle UVW such that UX, VY, WZ are perpendicular to BC, CA, AB and the directed distances $|UX|$, $|VY|$, $|WZ|$ are equal to s_1, s_2, s_3, respectively. When $s_1 = 0$, we put $U = X$, and we do similar assignments when s_2 and s_3 are zero.

For an expression ε in terms of side lengths a, b, and c of the triangle ABC and a real number h, let $\varepsilon[h]$ denote the triple $(h\,\varepsilon, h\,\varphi(\varepsilon), h\,\psi(\varepsilon))$. In other words, the coordinates $\varepsilon[h]_1$, $\varepsilon[h]_2$, $\varepsilon[h]_3$ of $\varepsilon[h]$ are products with h of ε, the first cyclic permutation of ε, and the second cyclic permutation of ε, respectively. For example, $a[h] = (h\,a, h\,b, h\,c)$ and if $w_a = \frac{b+c-a}{2}$, $w_b = \frac{c+a-b}{2}$, and $w_c = \frac{a+b-c}{2}$, then $w_a[h] = (h\,w_a, h\,w_b, h\,w_c)$.

With this notation at hand, we can describe our task in this paper as a search for expressions ε and points X, Y, and Z in the plane of the triangle ABC such that the triangles $[ABC, XYZ, \varepsilon[h]]$ are $(\mathcal{F}_\lambda, \gamma)$-simple in $V_\lambda(ABC)$ for γ either γ_F, γ_J, or γ_K.

Recall that the triangles $[ABC, XYZ, \varepsilon[h]]$ have already been used for characterizations of Kiepert and Feuerbach hyperbolas. Indeed, the original description of the Kiepert hyperbola is that it is a locus of centres of perspective of triangles ABC and $X_hY_hZ_h$, where $X_hY_hZ_h = [ABC, A_mB_mC_m, a[h]]$ and A_m, B_m, C_m are midpoints of sides (see [6]).

Another application of triangles $X_hY_hZ_h$ on vertices of similar isosceles triangles build on sides of ABC is a result in [12] which shows that triangles ABC and $X_hY_hZ_h$ are orthologic and the point $[ABC, X_hY_hZ_h]$ traces the Kiepert hyperbola as h goes through reals.

The description of the Feuerbach hyperbola due to Kariya [10] is that it is a locus of centres of perspective of triangles ABC and $P_hQ_hR_h$, where $P_hQ_hR_h = [ABC,\, A_pB_pC_p,\, 1[h]]$ and A_p, B_p, C_p are projections of the incentre onto sides.

Another application of triangles $P_hQ_hR_h$ whose vertices are intersections of circles concentric to the incircle with perpendiculars through incentre to sides is a result which shows that triangles ABC and $P_hQ_hR_h$ are orthologic and the point $[ABC,\, P_hQ_hR_h]$ traces the Feuerbach hyperbola as h goes through reals.

2. Preliminaries on complex numbers

We shall use complex numbers because they lead to the simplest expressions. Hence, our proofs are entirely algebraic. Every book on the use of complex numbers in geometry from the references below give excellent and adequate introductions to this technique of proof. In this section we give only the most basic notions and conventions.

A point P in the Gauss plane is represented by a complex number p. This number is called the *affix* of P and we write $\tilde{P} = p$ or $P(p)$ to indicate this. The complex conjugate of p is denoted $\bar{p}$. However, we shall be avoiding this notation by using next letter (now letter q) for the complex conjugate and sometimes write $P(p,\, q)$ or $\tilde{P} = (p,\, q)$ in order to describe affix of a point and to describe its complex conjugate. In order to avoid quotients, we shall use z^* for $1/z$.

In the sections on the Kiepert and Jarabek hyperbolas, we follow the standard assumption that the vertexes A, B, and C of the reference triangle are represented by numbers u, v, and w on the unit circle so that the circumcentre O of ABC is the origin. Hence, the affix of O is number 0 (zero) and complex conjugates of u, v, and w are $1/u$, $1/v$, and $1/w$ (or, in our notation, u^*, v^*, and w^*).

Most interesting points, lines, circles, curves, ... associated with the triangle ABC are expressions that involve symmetric functions of u, v, and w that we denote as follows.

$$\sigma = u + v + w, \quad \tau = v\,w + u\,w + u\,v, \quad \mu = u\,v\,w,$$

$$\sigma_a = -u + v + w, \quad \sigma_b = u - v + w, \quad \sigma_c = u + v - w,$$

$$\tau_a = -v\,w + w\,u + u\,v, \quad \tau_b = v\,w - w\,u + u\,v, \quad \tau_c = v\,w + w\,u - u\,v,$$

$$\mu_a = v\,w, \quad \mu_b = w\,u, \quad \mu_c = u\,v, \quad \delta_a = v - w,$$

$$\delta_b = w - u, \quad \delta_c = u - v, \quad \zeta_a = v + w, \quad \zeta_b = w + u, \quad \zeta_c = u + v.$$

For each $k \geq 2$, σ_k, σ_{ka}, σ_{kb}, and σ_{kc} are derived from σ, σ_a, σ_b, and σ_c with the substitution $u = u^k$, $v = v^k$, $w = w^k$. In a similar fashion we can define analogous expressions using letters τ, μ, δ, and ζ. We shall use corresponding small Latin letters to denote analogous symmetric functions in a, b, and c (lengths of sides of ABC). For example, $m = a\,b\,c$, $s = a + b + c$, $t = b\,c + c\,a + a\,b$, $z_a = b + c$, and $s_{2a} = b^2 + c^2 - a^2$.

The expressions which appear in triangle geometry usually depend on sets that are of the form $\{a, b, c, \dots, x, y, z\}$ (that is, union of triples of letters). Let φ and ψ stand for permutations $|\, b, c, a, \dots \dots, y, z, x \,|$ and $|\, c, a, b, \dots, z, x, y \,|$.

Let $f = f(x, y, \dots)$ be an expression that depends on a set $S = \{x, y, \dots\}$ of variables and let $\varrho : S \to S$ be a permutation of S. Then f^ϱ is a short notation for $f(\varrho(x), \varrho(y), \dots)$. For permutations $\varrho, \dots, \xi$ of S we shall use $\mathbb{S}_{\varrho,\dots,\xi}\, f$ and $\mathbb{P}_{\varrho,\dots,\xi}\, f$ to shorten $f + f^\varrho + \dots + f^\xi$ and $f\,f^\varrho \dots f^\xi$. Finally, $\mathbb{S} f$ and $\mathbb{P} f$ replace $\mathbb{S}_{\varphi,\psi}\, f$ and $\mathbb{P}_{\varphi,\psi}\, f$.

Let $\langle k, m, n\rangle$ be a notation for $-k\,\mu + \mathbb{S}\, u^2(m\,v + n\,w)$. Let S be area of ABC.

Since points, lines, conics, ... associated to a triangle often appear in triples in which two members are build from a third by appropriate permutation, we shall often give only one of them while the other two (*relatives*) are obtained from it by cyclic permutations.

Let us close these preliminaries with few words on analytic geometry that we shall use.

In triangle geometry lines play an important role so that we have special notation $[f, g, h]$ for the set of all points $P(p, q)$ that satisfy the equation $f\,p + g\,q + h = 0$. When g is a complex conjugate of f and h is a real number, this set is a line.

Let $X(x, a)$, $Y(y, b)$, and $Z(z, c)$ be three points and let ℓ be a line $[f, g, h]$ in the plane. Then the line XY is $[a - b,\, y - x,\, b\,x - a\,y]$, the parallel to ℓ through X is $[f, g, -g\,a - f\,x]$ and the perpendicular to ℓ through X is $[f, -g, g\,a - f\,x]$, where g is a complex conjugate of f. The conditions for points X, Y, and Z to be collinear and for lines $[f, g, h]$, $[k, m, n]$, and $[r, s, t]$ to be concurrent are

$$\begin{vmatrix} 1 & a & x \\ 1 & b & y \\ 1 & c & z \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} f & g & h \\ k & m & n \\ r & s & t \end{vmatrix} = 0.$$

3. Statements of results

Let $n_F = 6$ and $n_J = n_K = 5$. For $X = K, F, J$, let X_i, for $i = 1, \ldots, n_X$, denote the following expressions.

$$K_1 = a, \; K_2 = a\, z_a^*, \; K_3 = a^2\, z_a^*, \; K_4 = a\, s_a\, z_a^*, \; K_5 = a^3.$$

$$F_1 = 1, \; F_2 = s_a^*, \; F_3 = a\, s_a, \; F_4 = a\, s_a^*, \; F_5 = s_{2a}\, z_a\, s_a^*, \; F_6 = z_a\, s_a^*.$$

$$J_1 = a^*, \; J_2 = s_a\, z_a^*, \; J_3 = a\, s_{2a}^*, \; J_4 = z_a^*\, s_{2a}^*, \; J_5 = a^*\, s_{2a}^*.$$

For $i = J_3, J_4, J_5$ we must assume in addition that ABC has no right angle.

Theorem 1. *Let $\lambda \neq -1$ and $h \neq 0$ be real numbers. For any triangle PQR homothetic to the triangle ABC, for $X = K, J, F$, and for $i = 1, \ldots, n_X$, the triangle*

$$[ABC, \, PQR, \, X_i[h]]$$

is $(\mathcal{F}_\lambda, \gamma_X)$-simple in $V_\lambda(ABC)$.

Remark. Since there can be at most two values of the parameter h for which the vertices of the "triangle" $[ABC, PQR, X_i[h]]$ are collinear, we must exclude these values in addition to the value $h = 0$. In the above statement this is implicit in the assumption that we consider only nondegenerate triangles.

In the above theorem the triangle PQR can be, for example, the triangle ABC, the complementary triangle $A_m B_m C_m$, the anticomplementary triangle $A_a B_a C_a$, the Euler triangle $A_f B_f C_f$, and the opposite triangle $A_s B_s C_s$, where A_m, B_m, C_m denote midpoints of sides of the triangle ABC, A_a, B_a, C_a intersections of parallels through vertices to sides, A_f, B_f, C_f midpoints of segments joining vertices with the orthocentre H, and A_s, B_s, C_s reflections of vertices at the circumcentre O.

Theorem 2. *For any triangle PQR homothetic to the triangle ABC, for $X = K, F, J$, and for $i, j = 1, \ldots, n_X$, the triangle*

$$[ABC, \, [ABC, \, PQR, \, X_i[h]], \, X_j[k]]$$

is $(\mathcal{F}_\lambda, \gamma_X)$-simple in $V_\lambda(ABC)$ for all real numbers h and all real num-

bers k except the value $-h\,a_{ij}$, *where* a_{ij} *is the* $(i,\,j)$ *entry of the matrix* M_X *with*

$$M_K = \begin{bmatrix} 1 & \frac{2z}{y} & \frac{2z}{v} & \frac{2z}{u} & 2x \\ \frac{y}{2z} & 1 & \frac{y}{v} & \frac{y}{u} & \frac{xy}{z} \\ \frac{v}{2z} & \frac{v}{y} & 1 & \frac{v}{u} & \frac{vx}{z} \\ \frac{u}{2z} & \frac{u}{y} & \frac{u}{v} & 1 & \frac{ux}{z} \\ \frac{1}{2x} & \frac{z}{xy} & \frac{z}{vx} & \frac{z}{ux} & 1 \end{bmatrix},$$

$$x = s_2^*, \quad y = s^2, \quad z = z_a\, z_b\, z_c,$$

$$u = 4\,s\,t - 4\,m - s^3, \quad v = s^3 - 2\,s\,t + 2\,m,$$

$$M_F = \begin{bmatrix} 1 & \frac{w}{v} & \frac{s}{x} & \frac{y}{x} & \frac{w}{z} & \frac{y}{u} \\ \frac{v}{w} & 1 & \frac{v}{xy} & \frac{v}{sx} & \frac{v}{z} & \frac{v}{su} \\ \frac{x}{s} & \frac{xy}{v} & 1 & \frac{y}{s} & \frac{xy}{z} & \frac{xy}{su} \\ \frac{x}{y} & \frac{sx}{v} & \frac{s}{y} & 1 & \frac{sx}{z} & \frac{x}{u} \\ \frac{z}{w} & \frac{z}{v} & \frac{z}{xy} & \frac{z}{sx} & 1 & \frac{z}{su} \\ \frac{u}{y} & \frac{su}{v} & \frac{su}{xy} & \frac{u}{x} & \frac{su}{z} & 1 \end{bmatrix},$$

$$x = 2\,m, \quad y = s_a\, s_b\, s_c, \quad z = s_{2a}\, s_{2b}\, s_{2c},$$

$$u = 4\,s\,t - 6\,m - s^3, \quad v = u + 2\,m, \quad w = 16\,S^2,$$

$$M_J = \begin{bmatrix} 1 & \frac{z}{ms} & \frac{y}{4m^2} & \frac{yz}{msu} & \frac{y}{sx} \\ \frac{ms}{z} & 1 & \frac{sy}{4mz} & \frac{y}{u} & \frac{my}{xz} \\ \frac{4m^2}{y} & \frac{4mz}{sy} & 1 & \frac{4mz}{su} & \frac{4m^2}{sx} \\ \frac{msu}{yz} & \frac{u}{y} & \frac{su}{4mz} & 1 & \frac{mu}{xz} \\ \frac{sx}{y} & \frac{xz}{my} & \frac{sx}{4m^2} & \frac{xz}{mu} & 1 \end{bmatrix},$$

$$x = s_a\, s_b\, s_c, \quad y = s_{2a}\, s_{2b}\, s_{2c}, \quad z = z_a\, z_b\, z_c,$$

$$u = 4\,s\,t - 4\,m - s^3.$$

Remark. Observe that some important triangles related to the triangle ABC are of the form $[ABC,\, A_m B_m C_m,\, K_1[h]]$ for a suitable constant

h. For example, the first Brocard triangle $A_b B_b C_b$ (for $h = 2\,S/s_2$), the Toricelli triangles $A_v B_v C_v$ and $A_u B_u C_u$ on vertices of equilateral triangles build on sides either towards outside or towards inside (for $h = \pm\sqrt{3}/2$), and Napoleon triangles $A_{vn} B_{vn} C_{vn}$ and $A_{un} B_{un} C_{un}$ on centres of these equilateral triangles (for $h = \pm\sqrt{3}/6$).

The orthic triangle $A_o B_o C_o$ and the three images triangle $A_r B_r C_r$ whose vertices are reflections of A, B, and C at opposite sides of ABC are of the form $[ABC,\, ABC,\, J_1[h]]$. Also, the tangential triangle $A_t B_t C_t$ (formed by tangents to the circumcircle at vertices of ABC) has the form $[ABC,\, A_m B_m C_m,\, J_3[h]]$.

Let ν denote the expression $a\,(b^2 + c^2 - a^2)$.

Theorem 3. *Let $k \neq 0$, h, and $\lambda \neq -1$ be real numbers. For any triangle PQR homothetic to the triangle ABC, for $X = K$, F, J, and for $j = 1, \ldots, n_X$, the triangle*

$$\big[ABC,\, \big[ABC,\, PQR,\, \nu[h]\big],\, X_j[k]\big]$$

is $(\mathcal{F}_\lambda,\, \gamma_X)$-simple in $V_\lambda(ABC)$.

Theorem 4. *Let k, $h \neq 0$, and $\lambda \neq -1$ be real numbers. For any triangle PQR homothetic to the triangle ABC, for $X = K$, F, J, and for $j = 1, \ldots, n_X$, the triangle*

$$\big[ABC,\, \big[ABC,\, PQR,\, X_j[h]\big],\, \nu[k]\big]$$

is $(\mathcal{F}_\lambda,\, \gamma_X)$-simple in $V_\lambda(ABC)$.

Let

$$I_K = \big\{b,\, K,\, Kb,\, Km,\, u,\, ub,\, v,\, vb,\, un,\, vn\big\},$$

$$I_J = \big\{h,\, o,\, r,\, t,\, tr,\, w,\, Hh,\, Ho,\, O,\, Ot\big\},$$

and

$$I_F = \big\{e,\, ep,\, er,\, k,\, kr,\, p,\, pp,\, Ie,\, Ik,\, Ip,\, Oi\big\}.$$

For each element i of these three sets we define a triangle $A_i B_i C_i$ by describing the vertex A_i. The vertices B_i and C_i have analogous descriptions. Let A_e be the centre of the A-excircle, A_{ep} the projection of A_e onto BC, the point A_{er} is the reflection of A_e at BC, the vertex A_k is the second intersection of the bisector of the angle A with the circumcircle, A_{kr} is the reflection of A_k at BC, the point A_p is the projection of the incentre I onto BC, the vertex A_{pp} is the projection of A_p onto AI, A_{Oi} is a projection onto BC of any point different from O on line IO joining the incentre with the circumcentre, A_{Ip} is a projection onto $B_p C_p$ of any point different from central point X_{65} [9] on line IO, A_{Ie} and A_{Ik} are projections onto B_eC_e and B_kC_k of any point on IO different from the incentre I, the point A_b is the projection of the Grebe-Lemoine point K onto the perpendicular bisector of BC, the vertices A_K, A_{Kb}, and A_{Km} are the projection of any point different from the circumcentre O on the line KO onto BC, B_bC_b, and OA_m, vertices A_u and A_{un} are the vertex and the centre of the equilateral triangle build on BC towards inside, A_v and A_{vn} are the vertex and the centre of the equilateral triangle build on BC towards outside, A_{ub} and A_{vb} are projections of the Grebe-Lemoine points of $A_uB_uC_u$ and $A_vB_vC_v$ onto perpendicular bisectors of B_uC_u and B_vC_v, A_h is the second intersection of altitude line AH with the circumcircle, A_o is the projection of A onto BC, the point A_r is the reflection of A at BC, the intersection of tangents to the circumcircle at B and C is A_t, the reflection of A_t at BC is A_{tr}, A_w is the intersection of common tangents of the A-excircle with B-excircle and C-excircle, A_{Hh} and A_{Ho} are the projections onto B_hC_h and B_oC_o of any point X on the Euler line of ABC different from the orthocentre H, and A_O and A_{Ot} are the projections onto BC and B_tC_t of any point X on the Euler line of ABC different from the circumcentre O.

Some of the cases in the following theorem are clearly consequences of the previous theorem (for example, the first Brocard triangle $A_bB_bC_b$ has the form $[ABC,\, A_mB_mC_m,\, K_1[k]]$, for a suitable $k \neq 0$). Moreover, in some cases we must make additional assumptions about the triangle ABC. For example, for $i = b$, the triangle ABC can not be equilateral and for $i = t$ and $i = w$ it can not have right angle.

Theorem 5. *For $X = K$, J, F, for $i \in I_X$, and for all real numbers h, the triangle $[ABC,\, A_iB_iC_i,\, \nu[h]]$ is $(\mathcal{F}_\lambda,\, \gamma_X)$-simple in $V_\lambda(ABC)$.*

For $X = K$, J, F, for any $i \in I_X$, and all $j, j' = 1, \ldots, n_X$ one can show that triangles

$$[ABC,\, [ABC,\, A_iB_iC_i,\, \nu[h]],\, X_j[k]],$$
$$[ABC,\, [ABC,\, A_iB_iC_i,\, X_j[h]],\, \nu[k]],$$
$$[ABC,\, [ABC,\, A_iB_iC_i,\, X_{j'}[h]],\, X_j[k]],$$

are $(\mathcal{F}_\lambda,\, \gamma_X)$-simple in $V_\lambda(ABC)$ for all real values of constants h and k except exactly one value of either h or k. The matrices of exceptions are similar to the matrices M_K, M_J, and M_F.

An important source of $(\mathcal{F}_\lambda,\, \gamma_X)$-simple in $V_\lambda(ABC)$ triangles is the following general result.

Theorem 6. *Let $Q \in V_\lambda(ABC)$ be a point different from the orthocentre H. The antipedal triangle $Q^a Q^b Q^c$ of Q with respect to ABC is orthologic with the triangle $P_a^\lambda P_b^\lambda P_c^\lambda$ if and only if P lies on a conic through the points A, B, C, H, and Q.*

Corollary. *For $X = K, J, F$, the antipedal triangle $Q^a Q^b Q^c$ with respect to ABC of any point Q on the hyperbola γ_X outside the circumcircle γ_0 and different from the orthocentre H is $(\mathcal{F}_\lambda, \gamma_X)$-simple in $V_\lambda(ABC)$.*

4. Proofs

Preliminaries. Let us first determine the affixes of points P_a^λ, P_b^λ, and P_c^λ. Since the affix of G_a^P is $3^*(p+\zeta_a)$ and the affix of O_a^P is $n_a^*\,\mu_a\,(p\,q-1)$, where $n_a = p + \mu_a\,q - \zeta_a$, it follows that P_a^λ has the affix

$$3^*(\lambda+1)^*\,n_a^*\,(3\,\mu_a\,(p\,q-1) + \lambda\,n_a\,(\zeta_a + p)).$$

The affixes of P_b^λ and P_c^λ are relatives of the affix of P_a^λ.

Observe that points P_a^λ, P_b^λ, P_c^λ are collinear if and only if the point P lies on a quartic Q_λ with equation $3\,\mu\,(2\,\lambda+3)\,(p\,q-1)^2 + \lambda^2\,n_a\,n_b\,n_c = 0$.

Triangles XYZ and PQR with affixes of vertices x, y, z, p, q, and r are orthologic if and only if $(XYZ,\,PQR) = 0$, where

$$(XYZ,\,PQR) = \mathbb{S}\left[x\,(\bar{q}-\bar{r}) + \bar{x}\,(q-r)\right].$$

Proof of Theorem 1 for $X = K$ and $i = 1$. Since triangles ABC and PQR are homothetic, there is a point $T(x,\,y)$ and a real number $\xi \neq -1$ such that $\tilde{P} = (\xi+1)^*\,(u+\xi\,x)$, $\tilde{Q} = \varphi(\tilde{P})$, and $\tilde{R} = \psi(\tilde{P})$.

Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\,PQR,\,K_1[h]]$. Then $\tilde{U} = \tilde{P} + I\,h\,(v-w)$, where $I = \sqrt{-1}$. Also, $\tilde{V} = \varphi(\tilde{U})$ and $\tilde{W} = \psi(\tilde{U})$.

Let us observe that points U, V, and W will be collinear if and only if h is different from $12^*\,S^*\,(\xi+1)^*\,(s_2 \pm 2\,\sqrt{s_4 - t_2})$, where S denotes the area of ABC.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and UVW is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\,UVW) = 2\,I\,h\,eq_K\,eq_0\,(\lambda+1)^*\,\mathbb{P}\,u^*\,n_a^*,$$

where $eq_0 = 1 - p\,q$ and

$$eq_K =(\tau^2 - 3\mu\sigma)p^2 - \mu^2(\sigma^2 - 3\tau)q^2 + \\ +(4\mu\sigma^2 - \sigma\,\tau^2 - 3\mu\,\tau)p - \mu(4\tau^2 - \sigma^2\tau - 3\mu\,\sigma)q + \tau^3 - \mu\,\sigma^3.$$

Notice that $eq_0 = 0$ is the equation of the circumcircle of ABC while $eq_K = 0$ is the equation of the Kiepert hyperbola of ABC since the vertices $A(u,\, u^*)$, $B(v,\, v^*)$, and $C(w,\, w^*)$, the orthocentre $H(\sigma,\, \tau\,\mu^*)$, and the centroid $G(3^*\,\sigma,\, 3^*\,\tau\,\mu^*)$ satisfy it. This shows that UVW is $(\mathcal{F}_\lambda,\, \gamma_K)$-simple in $V_\lambda(ABC)$ for all $h \neq 0$ except for at most two additional values of h found above when points U, V, and W are collinear.

Proof of Theorem 1 for $X = J$ and $i = 1$. We first determine $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$ as above. Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\, PQR,\, J_1[h]]$. Then $\tilde{U} = \tilde{P} + I\,h\,v\,w\,(v - w)^*$, $\tilde{V} = \varphi(\tilde{U})$, and $\tilde{W} = \psi(\tilde{U})$.

Let us observe that points U, V, and W will be collinear if and only if h is different from $4^*\,S^*\,(\xi + 1)^*\,s_2^*\,m\,(3\,m \pm \sqrt{3\,m_2 - \mathbb{S}\,a^4\,s_{2a}})$.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and UVW is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, UVW) = I\,h\,eq_J\,eq_0\,(\lambda + 1)^*\,\mathbb{P}\,n_a^*,$$

where $eq_J = \sigma\,p^2 - \mu\,\tau\,q^2 + (\tau - \sigma^2)\,p + (\tau^2 - \mu\,\sigma)q$.

The equation of the Jarabek hyperbola of ABC is $eq_J = 0$ since the vertices $A(u,\, u^*)$, $B(v,\, v^*)$, and $C(w,\, w^*)$, the orthocentre $H(\sigma,\, \tau\,\mu^*)$, and the circumcentre $O(0,\, 0)$ satisfy it.

Proof of Theorem 1 for $X = F$ and $i = 1$. In contrast with the previous two sections, in order to avoid square roots, here we shall assume that the vertices A, B, and C of the base triangle have affixes u^2, v^2, and w^2, with the same assumption about u, v, and w. Let ϱ denote a transformation which replaces variables u, v, and w with u^2, v^2, and w^2.

This time $\tilde{P} = (\xi + 1)^*\,(u^2 + \xi\,x)$, $\tilde{Q} = \varphi(\tilde{P})$, and $\tilde{R} = \psi(\tilde{P})$.

Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\, PQR,\, F_1[h]]$. Then $\tilde{U} = \tilde{P} + h\,v\,w$, $\tilde{V} = \varphi(\tilde{U})$, and $\tilde{W} = \psi(\tilde{U})$.

Let us observe that points U, V, and W will be collinear if and only if h is different from $4^*\,S^*\,(\xi + 1)^*\,(m \pm \sqrt{m\,(s^3 - 4\,m\,t + 9\,m)})$.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and UVW is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, UVW) = h\,eq_F\,eq_0\,(\lambda + 1)^*\,\mathbb{P}\,\delta_a\,u^*\,\varrho(n_a)^*,$$

where

$$eq_F = \tau\, p^2 - \mu_3\, \sigma\, q^2 + (\mu\, \sigma + 2\, \mu\, \sigma^2 - \sigma^2\, \tau)\, p + \\ + \mu\, (\sigma\, \tau^2 - 2\, \sigma^2\, \mu - \mu\, \tau)\, q + \tau^3 - \sigma^3\, \mu.$$

Observe that $eq_F = 0$ is the equation of the Feuerbach hyperbola of ABC since the vertices $A(u^2,\, 1/u^2)$, $B(v^2,\, 1/v^2)$, and $C(w^2,\, 1/w^2)$, the orthocentre $H(\varrho(\sigma,\, \tau\, \mu^*))$, and the incentre $I(-\tau,\, -\sigma\, \mu^*)$ satisfy it.

Proof of Theorem 2 for $X = K$, $i = 1$, and $j = 5$. Let $UVW = [ABC,\, PQR,\, K_1[h]]$ and $LMN = [ABC,\, UVW,\, K_5[k]]$. We know $\tilde{U}$, $\tilde{V}$, and $\tilde{W}$ from the proof of Thm. 1, so that it is not difficult to see that $\tilde{L} = \tilde{U} + I\, k\, \delta_a^3\, \mu_a^*$, $\tilde{M} = \varphi(\tilde{L})$, and $\tilde{N} = \psi(\tilde{L})$.

Let us note that there exist at most two values of k when points L, M, and N are collinear. These values have rather complicated form.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and LMN is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, LMN) = I\, (\langle 6,\, 1,\, 1\rangle k + 2\, \mu\, h)\, eq_K\, eq_0\, (\lambda + 1)^*\, \mu^*{}_2\, \mathbb{P}\, n_a^*,$$

This shows that LMN is $(\mathcal{F}_\lambda,\, \gamma_K)$-simple in $V_\lambda(ABC)$ for all k except the value $2\, h\, s_2^*$ and at most two more values for which points L, M, and N are collinear.

Proof of Theorem 3 for $X = K$ and $i = 1$. Let $UVW = [ABC,\, PQR,\, \nu[h]]$ and $LMN = [ABC,\, UVW,\, K_1[k]]$. It is easy to check that $\tilde{U} = \tilde{P} + I\, h\, \zeta_a\, \mu^*\, \mathbb{P}\, \delta_a$, $\tilde{V} = \varphi(\tilde{U})$, and $\tilde{W} = \psi(\tilde{U})$. It follows that $\tilde{L} = \tilde{U} + I\, k\, \delta_a$, $\tilde{M} = \varphi(\tilde{L})$, and $\tilde{N} = \psi(\tilde{L})$.

Once again there exist at most two values of k when points L, M, and N are collinear. These values have complicated expressions in terms of side lengths.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and LMN is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, LMN) = 2\, I\, k\, eq_K\, eq_0\, (\lambda + 1)^*\, \mu^*\, \mathbb{P}\, n_a^*,$$

Proof of Theorem 5 for $X = F$ and $i = e$. Assume $\tilde{A} = u^2$, $\tilde{B} = v^2$, and $\tilde{C} = w^2$. Recall [13] that $\tilde{A}_e = \tau_a$, $\tilde{B}_e = \tau_b$, and $\tilde{C}_e = \tau_c$. Let U, V, W denote vertices of the triangle $[ABC,\, A_eB_eC_e,\, \nu[h]]$. It is easy to check that $\tilde{U} = \tau_a + I\, h\, \zeta_{2a}\, \mu^*{}_2\, \mathbb{P}\, \delta_{2a}$, $\tilde{V} = \varphi(\tilde{U})$, and $\tilde{W} = \psi(\tilde{U})$.

Once again there exist at most two values of h when points U, V, and W are collinear. These are $2^*\, S^*\, (-m \pm \sqrt{m\, (s^3 - 4\, s\, t + 9\, m)})\, \mathbb{P}\, s_a^*$. Finally,

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, UVW) = 2\, eq_F\, eq_0\, (\lambda + 1)^*\, \mu^*\, \mathbb{P}\, \delta_a\, \varrho(n_a)^*.$$

Proof of Theorem 6. Let $\tilde{Q} = (x, y)$. Then $\tilde{Q^a} = (\mu_a\, x\, y - x^2 + + \zeta_a\, x - 2\, \mu_a)(x + \mu_a\, y - \zeta_a)^*$, $\tilde{Q^b} = \varphi(\tilde{Q^a})$, and $\tilde{Q^c} = \psi(\tilde{Q^a})$. It follows that

$(P_a^\lambda P_b^\lambda P_c^\lambda,\, Q^a Q^b Q^c) = 2\,(x\,y-1)\, eq_Q\, eq_0\,(\lambda+1)^*\, \mathbb{P}\,\delta_a\, n_a^*\,(x+\mu_a\, y-\zeta_a)^*$,

where $eq_Q = a\,p^2 + b\,q^2 + c\,p + d\,q + e$,

$$a = \mu\, y^2 - x - \tau\, y + \sigma,$$
$$b = \mu\,(\sigma\, x - x^2 + \mu\, y - \tau),$$
$$c = x^2 - \mu\,\sigma\, y^2 + \zeta_a\,\zeta_b\,\zeta_c\, y - \sigma_2 - \tau,$$
$$d = \tau\, x^2 - \zeta_a\,\zeta_b\,\zeta_c\, x - \mu_2\, y^2 + \tau_2 + \mu\,\sigma,$$
$$e = (\sigma_2 + \tau)\, x - \sigma\, x^2 + \mu\,\tau\, y^2 - (\tau_2 + \mu\,\sigma)\, y.$$

It is obvious that $eq_Q = 0$ is an equation of a conic. One can easily check that it goes through A, B, C, H, and Q.

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