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MALDISTRIBUTION IN HIGHER DIMENSIONS

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Abstract: The concept of maldistribution was recently introduced by Myerson [3]. We give some new criteria for maldistribution in higher dimensions.

1. Introduction

We first introduce the following notation. For $\mathbf{x} = (x_1, \dots, x_s) \in \in \mathbb{R}^s$ let

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$$[\mathbf{0}, \mathbf{x}) = \{(t_1, \dots, t_s) \in \mathbb{R}^s : 0 \le t_i < x_i \text{ for } 1 \le i \le s\},$$

$$[\mathbf{0}, \mathbf{x}) = \{(t_1, \dots, t_s) \in \mathbb{R}^s : 0 \le t_i \le x_i \text{ for } 1 \le i \le s\},$$

The function $g : [0,1]^s \to [0,1]$ is said to be a *distribution function* if $g(\mathbf{x}) = P([\mathbf{0}, \mathbf{x}))$ for every $\mathbf{x} \in [0,1]^s$, where P is a probability measure on $[0,1)^s$.

Two distribution functions g_1 and g_2 are identified if they have the same values of integrals

$$\int_{[0,1]^s} \int_{[0,1]^s} F(\mathbf{x}, \mathbf{y}) \mathrm{d}g_1(\mathbf{x}) \mathrm{d}g_1(\mathbf{y}) = \int_{[0,1]^s} \int_{[0,1]^s} F(\mathbf{x}, \mathbf{y}) \mathrm{d}g_2(\mathbf{x}) \mathrm{d}g_2(\mathbf{y})$$

for any continuous function F on $[0,1]^s \times [0,1]^s$. An alternative definition can be found in [4, p. xii].

Let $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ in $[0,1)^s$, $\mathbf{x}_n = (x_{n1} \dots, x_{ns})$ be a given sequence. We set

$$A([\mathbf{0}, \mathbf{x}), \omega_N) = \#\{n \le N : \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}.$$

Now, let $g : [0,1]^s \to [0,1]$ be a given distribution function. If there exists an increasing sequence of natural numbers $(N_k)_{k=1}^{\infty}$, such that

$$\lim_{k \to \infty} A([\mathbf{0}, \mathbf{x}), \omega_{N_k})/N_k = g(\mathbf{x}) \text{ for } \mathbf{x} \in [0,1]^s,$$

then $g(\mathbf{x})$ is called a *distribution function of* ω. As a standard monograph on distribution functions and uniform distribution we refer to [2].

Let $G(\omega)$ be the set of all distribution functions of ω.

For a given continuous function $F(\mathbf{x}, \mathbf{y})$ on $[0,1]^s \times [0,1]^s$ $G(F)$ denotes the set of all distribution functions $g : [0,1]^s \to [0,1]$ satisfying

$$\int_{[0,1]^s} \int_{[0,1]^s} F(\mathbf{x}, \mathbf{y}) \mathrm{d}g(\mathbf{x}) \mathrm{d}g(\mathbf{y}) = 0.$$

We will specify the following distribution function $c_{\boldsymbol{\alpha}} : [0,1]^s \to$ $\to [0,1]$:

$$c_{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{cases} 1, & \text{for } \mathbf{x} \in [\boldsymbol{\alpha}, \mathbf{1}] \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore we consider two types of sequences ω, namely sequences of the first class defined by

$$G(\omega) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in [0,1]^s\}; \tag{1}$$

and sequences of the second class defined by

$$\begin{aligned} G(\omega) =& \{t c_{\boldsymbol{\alpha}}(\mathbf{x}) + (1-t) c_{\boldsymbol{\beta}}(\mathbf{x}) : t \in [0,1], \\ & \boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1]^s, \alpha_i \ne \beta_i \Rightarrow \alpha_i = 1, \beta_i = 0 \text{ for } i = 1, \dots, s\}. \end{aligned} \tag{2}$$

The study of sequences ω satisfying

$$G(\omega) \supset \{c_{\mathbf{ff}}(\mathbf{x}) : \boldsymbol{\alpha} \in [0,1]^s\} \tag{3}$$

was initiated by Myerson. In [3] he gives an exposition on uniform distribution and introduces "maldistribution" a notion describing a very different situation. . A sequence $\omega = (x_n)$ is called *maldistributed* if for any interval $J \subseteq [0,1)$

$$\limsup_{N\to\infty} \frac{1}{N} \# \{n \leq N \mid x_n \in J\} = 1,$$

which is equivalent to (3). Furthermore different criteria for uniform maldistribution are given there and some very general examples show the existence of such sequences in the one- and multidimensional case. Strauch [5] has obtained a characterization of sequences satisfying (1) in the one-dimensional case.

The purpose of this paper is to give criterions for the classes (1) and (2). Explicit examples of sequences satisfying (1) and (2), respectively, are given.

2. Results

The starting point of this paper is the following proposition.

Proposition 1. *Let $(M_i)_{i=1}^{\infty}$ be a sequence of positive numbers satisfying $\lim\limits_{k\to\infty} \sum_{i=1}^{k-1} M_i/M_k = 0$. For a given sequence $\sigma = (\mathbf{y}_k)_{k=1}^{\infty}$ in $[0,1]^s$, let the sequence $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ in $[0,1]^s$ be constructed by $\mathbf{x}_n = \mathbf{y}_k$ for $\sum_{i=1}^{k-1} M_i \leq n < \sum_{i=1}^{k} M_i$. Finally, let $\mathbf{H}_\sigma \subset [0,1]^s \times [0,1]^s$ denote the set of all limit points of the sequence $((\mathbf{y}_{k-1}, \mathbf{y}_k))_{k=2}^{\infty}$. Then*

$$G(\omega) = \{t c_{\mathbf{ff}}(\mathbf{x}) + (1-t) c_{\mathbf{fi}}(\mathbf{x}) : t \in [0,1], (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbf{H}_\sigma\}.$$

Proof. For $N = \sum_{i=1}^{k-1} M_i + \theta_k M_k$, $0 \leq \theta_k < 1$, we have

$$A([\mathbf{0},\mathbf{x}), \omega_N) = A([\mathbf{0},\mathbf{x}), (\mathbf{y}_{k-1}, \mathbf{y}_k)) + o(N).$$

Moreover, we write

$$A([\mathbf{0},\mathbf{x}), (\mathbf{y}_{k-1}, \mathbf{y}_k)) = \begin{cases} M_{k-1}, & \text{if } \mathbf{y}_{k-1} \in [\mathbf{0},\mathbf{x}) \text{ and } \mathbf{y}_k \notin [\mathbf{0},\mathbf{x}), \\ \theta_k M_k, & \text{if } \mathbf{y}_{k-1} \notin [\mathbf{0},\mathbf{x}) \text{ and } \mathbf{y}_k \in [\mathbf{0},\mathbf{x}), \\ M_{k-1} + \theta_k M_k, & \text{if } \mathbf{y}_{k-1}, \mathbf{y}_k \in [\mathbf{0},\mathbf{x}). \end{cases}$$

Assume that $A([\mathbf{0},\mathbf{x}), \omega_N)/N \to g(\mathbf{x})$, $\mathbf{x} \in [0,1]^s$ for selected sequences of indices $N = \sum_{i=1}^{k-1} M_i + \theta_k M_k$, $k = k(N)$. Then we can further chose N such that $(\mathbf{y}_{k-1}, \mathbf{y}_k) \to (\boldsymbol{\alpha}, \boldsymbol{\beta})$, $M_{k-1}/(M_{k-1} + \theta_k M_k) \to t$, and $\theta_k M_k/(M_{k-1}+\theta_k M_k) \to (t-1)$, for some $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and t. Thus $g(\mathbf{x}) = t c_{\boldsymbol{\beta}}(\mathbf{x}) + (1-t) c_{\boldsymbol{\alpha}}(\mathbf{x})$.

On the other hand we can construct a sequence $N_k = \sum_{i=1}^{k-1} M_i + \theta_k M_k$ satisfying $M_{k-1}/(M_{k-1}+\theta_k M_k) \to t$ for any $t \in [0,1]$ provided that $(\mathbf{y}_{k-1}, \mathbf{y}_k) \to (\boldsymbol{\alpha}, \boldsymbol{\beta})$. Then $A([\mathbf{0}, \mathbf{x}), \omega_{N_k})/N_k \to t c_{\boldsymbol{\beta}}(\mathbf{x}) + (1-t) c_{\boldsymbol{\alpha}}(\mathbf{x})$. ◊

Corollary 1. *For a given* $\mathbf{H} \subset [0,1]^s$ *suppose that there exist a sequence* $\sigma = (\mathbf{y}_k)_{k=1}^{\infty}$ *in* $[0,1]^s$ *such that*

(i) $\mathbf{H}$ *coincides with the set of limit points of* σ,
(ii) $\lim_{k\to\infty} \mathbf{y}_k - \mathbf{y}_{k-1} = \mathbf{0}$.

Then there exists a sequence $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ *in* $[0,1]^s$ *such that*

$$G(\omega) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in \mathbf{H}\}.$$

Proof. According to (i) and (ii), we have $\mathbf{H}_\sigma = \{(\boldsymbol{\alpha}, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbf{H}\}$. Hence, Cor. 1 is a consequence of Prop. 1. ◊

Corollary 2. *Let* $\mathbf{H} \subset [0,1]^s$ *be such that there exists a continuous function* $\phi; [0,1] \to [0,1]^s$, *for which* $\mathbf{H} = \{\phi(t) : t \in [0,1]\}$. *Then there exists a sequence* $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ *in* $[0,1]^s$ *such that* $G(\omega) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in \mathbf{H}\}$.

Proof. For the proof, we note that (i) and (ii) from Cor. 1 hold for $\mathbf{y}_k = \phi(y_k)$, $k = 1, 2, \dots$, where

$$y_k = \left\{(-1)^{[\sqrt{k}]}\sqrt{k}\right\}.$$

Here $[x]$ denotes the integral part and $\{x\}$ the fractional part of x. The density of $(y_k)_{k=1}^{\infty}$ and $\lim_{k\to\infty} y_k - y_{k-1} = 0$ are proved (in a more general form) in Ex. 1, below. ◊

In the multidimensional case the fundamental First Theorem of Helly and Second Theorem of Helly (see [4, p. xiii]) are also valid. Following the same reasoning as in [5] the following analogues of one-dimensional results can be proved:

Proposition 2. *Let* $F(\mathbf{x}, \mathbf{y})$ *be continuous function defined on* $[0,1]^s \times [0,1]^s$. *For any* $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ *in* $[0,1]^s$ *we have*

$$G(\omega) \subset G(F) \iff \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(\mathbf{x}_m, \mathbf{x}_n) = 0.$$

The **proof** of this proposition can be given in a way analogous to the proof of [5, Prop. 11].

Criterion. *Let* $\mathbf{H} \subset [0,1]^s$ *be a closed set, and let* $F(\mathbf{x},\mathbf{y})$ *be a continuous function defined on* $[0,1]^s \times [0,1]^s$ *satisfying, for every* $\mathbf{x},\mathbf{y} \in [0,1]^s$,

(i) $F(\mathbf{x},\mathbf{y}) \geq 0$, *and*
(ii) $F(\mathbf{x},\mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$.

Then, for any sequence $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ *in* $[0,1]^s$, $G(\omega) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in \mathbf{H}\}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(\mathbf{x}_m, \mathbf{x}_n) = 0,$$

where $\mathbf{H}$ *coincides with the set of all limit points of the sequence*

$$\left(\frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n\right)_{N=1}^{\infty}.$$

Proof. For continuous F satisfying (i) and (ii), we have $G(F) = \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in [0,1]^s\}$. Moreover, if $G(\omega) \subset \{c_{\boldsymbol{\alpha}}(\mathbf{x}) : \boldsymbol{\alpha} \in [0,1]^s\}$, then, for selected N,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n = \int_{[0,1]^s} \mathbf{x} \, \mathrm{d}c_{\boldsymbol{\alpha}}(\mathbf{x}) = \boldsymbol{\alpha}.$$

The proof can now be easily completed . ◊

Remark 1. In applications, we can work with $F(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||_1$, where $||\mathbf{x}||_1 = \sum_{i=1}^{s} |x_i|$.

In the following we present two general results which will be used later.

Definition. Let $m_1, \ldots, m_s$ be integers ≥ 2. A sequence $(\mathbf{x}_n) = (x_{n1}, \ldots, x_{ns})$ in $\mathbb{R}^s$ is called $(m_1, \ldots, m_s)$-*uniformly distributed*, if the sequence $(x_{n1}, \ldots, x_{ns}, [x_{n1}] \bmod m_1, \ldots, [x_{ns}] \bmod m_s)$ is uniformly distributed in $[0,1]^s \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}$.

Proposition 3. *A sequence* $(\mathbf{x}_n)$ *in* $\mathbb{R}^s$ *is* $(m_1, \ldots, m_s)$*-uniformly distributed, if and only if* $(x_{n1}/m_1, \ldots, x_{ns}/m_s)$ *is uniformly distributed modulo* 1.

Proof. The necessary part of the theorem is clear, because $m\{x/m\} = [x] \bmod m + \{x\}$. We insert this equality into the components of $(x_{n1}/m_1, \ldots, x_{ns}/m_s)$ and use the uniform distribution of $(x_{n1}, \ldots, x_{ns}, [x_{n1}] \bmod m_1, \ldots, [x_{ns}] \bmod m_s)$.

For the sufficient part we let $\chi_{k,m}$ be the characteristic function of the residue class $k \bmod m$. Then we have

$$\frac{1}{N}\sum_{n<N}\chi_{k_1,m_1}([x_{n1}])\cdots\chi_{k_s,m_s}([x_{ns}])\exp\left(2\pi i(h_1x_{n1}+\cdots+h_sx_{ns})\right)=$$

$$=\frac{1}{N}\sum_{n<N}\chi_{\left[\frac{k_1}{m_1},\frac{k_1+1}{m_1}\right)}\left(\frac{x_{n1}}{m_1}\right)\cdots\chi_{\left[\frac{k_s}{m_s},\frac{k_s+1}{m_s}\right)}\left(\frac{x_{ns}}{m_s}\right)\times$$

$$\times\exp\left(2\pi i\left(h_1m_1\left\{\frac{x_{n1}}{m_1}\right\}+\cdots+h_sm_s\left\{\frac{x_{ns}}{m_s}\right\}\right)\right)\to 0,$$

which is a consequence of $\chi_{k,m}([x]) = \chi_{\left[\frac{k}{m},\frac{k+1}{m}\right)}\left(\frac{x}{m}\right)$ and the Weyl-criterion for uniform distribution. ◊

Corollary 3. *Let $1, \alpha_1, \dots, \alpha_s$ be linearly independent over the rationals and set $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$. Then the sequence $(n\boldsymbol{\alpha})$ is $(m_1, \dots, m_s)$-uniformly distributed for any choice of $(m_1, \dots, m_s)$.*

Remark 2. If a sequence $(x_{n1}, \dots, x_{ns})$ is $(2, \dots, 2)$-uniformly distributed, then $((-1)^{[x_{n1}]}x_{n1}, \dots, (-1)^{[x_{ns}]}x_{ns})$ is uniformly distributed modulo 1.

Proposition 4. *Let a_i, b_i, c_i be real numbers with $a_i, b_i \neq 0$ $(i = 1, \dots, s)$ and $0 < u_i < 1$ and let v_i be given such that $0 < u_1v_1 < u_2v_2 < \cdots < u_sv_s$ and $u_iv_i \notin \mathbb{Z}$ for all $i = 1, \dots, s$. Set*

$$x_{ni} = (a_i[b_in^{v_i}] + c_i)^{u_i},$$

and $\mathbf{x}_n = (x_{n1}, \dots, x_{ns})$. Then the s-dimensional sequence $\omega = (\mathbf{x}_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1.

Proof. Let $\mathbf{h} = (h_1, \dots, h_s) \in \mathbb{Z}^s \setminus \{0\}$. We have to show that the one-dimensional sequence

$$\tilde{x}_n = \sum_{i=1}^{s} h_ix_{ni}$$

is u.d. mod 1. Let r be the maximal index $(1 \leq r \leq s)$ such that $h_r \neq 0$ and $h_i = 0$ for all $i > r$. We have for $1 \leq i \leq r$:

$$(a_i[b_in^{v_i}] + c_i)^{u_i} = (a_ib_i)^{u_i}n^{v_iu_i} + O\big(n^{v_i(u_i-1)}\big),$$

by the Binomial Theorem. Thus we get

$$\tilde{x}_n = \sum_{i=1}^{r} h_ix_{ni} =$$

$$= h_r(a_rb_r)^{u_r}n^{u_rv_r} + \cdots + h_1(a_1b_1)^{u_1}n^{u_1v_1} + \sum_{i=1}^{r} O\big(n^{v_i(u_i-1)}\big).$$

Set $\varepsilon_n = \sum_{i=1}^{r} O\big(n^{v_i(u_i-1)}\big)$. Then, because of the assumption, ε_n tends to zero for $n \to \infty$. Set furthermore $u_iv_i = z_i$ and $h_i(a_ib_i)^{u_i} = A_i$. Since $\varepsilon_n \to 0$, we have to show that

$$f(n) := A_r n^{z_r} + A_{r-1} n^{z_{r-1}} + \cdots + A_1 n^{z_1}$$

is u.d. mod 1. Because of the assumption we have $A_r \neq 0$ and $0 < < z_1 < \cdots < z_r$. Thus we can apply [2, Th. 3.5] (this is a theorem which results from a combination of Fejér's theorem and the difference theorem) to the sequence $f(n)$ and the proof is complete. ◊

3. Examples

The existence of examples for (1) can be shown by using Cor. 2 and the well-known existence of Peano curve in $[0,1]^s$. We note that in [1] Peano curves are used to construct uniformly distributed sequences in the unit cube. An alternative example for (1) can be shown by using the construction as in Prop. 1.

Example 1. Let ω be defined as

$$\omega = \Big((-1)^{[[\log^{(j)} n]^{1/p_1}]}[\log^{(j)} n]^{1/p_1}, \ldots$$

$$\ldots, (-1)^{[[\log^{(j)} n]^{1/p_s}]}[\log^{(j)} n]^{1/p_s}\Big)_{n=1}^{\infty} \bmod 1$$

where $\log^{(j)} n$ denotes the jth iterated logarithm $\log \ldots \log n$, and $p_1, \ldots, p_s$ are coprime positive integers. Then, for $j > 1$, the sequence ω satisfies (1).

Proof. The line of construction of ω is the same as in Prop. 1. Precisely, for $\exp^{(j)} k < n < \exp^{(j)}(k+1)$, $(\exp^{(j)} k = \exp \ldots \exp k)$ we have $\mathbf{x}_n = \mathbf{y}_k$, where

$$\mathbf{y}_k = \Big((-1)^{[k^{1/p_1}]}k^{1/p_1}, \ldots, (-1)^{[k^{1/p_s}]}k^{1/p_s}\Big) \bmod 1.$$

Thus, in order to show (1) it suffices to prove that

(i) $(\mathbf{y}_k)_{k=1}^{\infty}$ is dense in $[0,1]^s$, and

(ii) $\lim_{k\to\infty} \mathbf{y}_k - \mathbf{y}_{k-1} = \mathbf{0}$.

Condition (i) follows from Props. 3, 4 and Remark 2. Condition (ii) follows from the expression

$$y_{ki} = \begin{cases} \{k^{1/p_i}\} & \text{for } k \in \cup_{n=0}^{\infty}\big[(2n)^{p_i}, (2n+1)^{p_i}\big) \\ 1 - \{k^{1/p_i}\} & \text{for } k \in \cup_{n=0}^{\infty}\big[(2n+1)^{p_i}, (2n+2)^{p_i}\big). \end{cases}$$

y_{ki}, for $k \in \big[(2n)^{p_i}, (2n+1)^{p_i}\big)$, increases from 0 to $\big((2n+1)^{p_i} - 1\big)^{1/p_i} - - 2n$ ($\to 1$ as $n \to \infty$) with differences $y_{ki} - y_{(k-1)i} = k^{1/p_i} - (k-1)^{1/p_i}$ ($\to 0$ as $k \to \infty$), and, for $k \in \big[(2n+1)^{p_i}, (2n+2)^{p_i}\big)$, y_{ki} decreases from 1 to $1 - \Big(\big((2n+2)^{p_i} - 1\big)^{1/p_i} - (2n+1)\Big)$ ($\to 0$ as $n \to \infty$) with differences $y_{ki} - y_{(k-1)i} = (k-1)^{1/p_i} - k^{1/p_i}$ ($\to 0$ as $k \to \infty$). Thus $\lim_{k\to\infty} y_{ki} - y_{(k-1)i} = 0$. ◊

Example 2. Let ω be defined as in previous example without the factors $(-1)^{[[\log^{(j)} n]^{1/p_i}]}$, $i = 1, \dots, s$, i.e.

$$\omega = \left([\log^{(j)} n]^{1/p_1}, \dots, [\log^{(j)} n]^{1/p_s}\right)_{n=1}^{\infty} \bmod 1.$$

Then ω satisfies (2).

Proof. As in the previous case, for $\exp^{(j)} k < n < \exp^{(j)}(k+1)$, we have $\mathbf{x}_n = \mathbf{y}_k$, where

$$\mathbf{y}_k = \left(k^{1/p_1}, \dots, k^{1/p_s}\right) \bmod 1.$$

First we show that the sequence $((\mathbf{y}_{k-1}, \mathbf{y}_k))_{k=2}^{\infty}$ has two types of limit points:

(i) $(\boldsymbol{\alpha}, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \in [0,1]^s$ is arbitrary, and
(ii) $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} \neq \boldsymbol{\beta} \in [0,1]^s$ and $\alpha_i \neq \beta_i \Rightarrow \alpha_i = 1, \beta_i = 0$ for $i = 1, \dots, s$.

1^o. Let us assume $k \neq n^{p_i}$, for $n = 1, 2, \dots$ and $i = 1, 2, \dots, s$. Then there exist positive integers $n_1, \dots, n_s$ such that $k \in \left(n_1^{p_1}, (n_1+1)^{p_1}\right) \cap \dots \cap \left(n_s^{p_s}, (n_s+1)^{p_s}\right)$, and so,

$$y_{ki} - y_{(k-1)i} = \left(n_i^{p_i} + j\right)^{1/p_i} - \left(n_i^{p_i} + j - 1\right)^{1/p_i} \to 0,$$

as $k \to \infty$. By applying Prop. 4 to $\mathbf{y}_k$ one shows that $(\mathbf{y}_k)_{k=1}^{\infty}$ is u.d. in $[0,1]^s$. But the k with $k = n^{p_i}$ have zero density, and so we derive that the sequence $((\mathbf{y}_{k-1}, \mathbf{y}_k))_{k=2}^{\infty}$, $k \neq n^{p_i}$, for $n = 1, 2, \dots$ and $i = 1, 2, \dots, s$ has limit points of type (i).

2^o. Now we take $k = n_i^{p_i}$, for $i \in I$, and assume that $k \neq n^{p_j}$, for $n = 1, 2, \dots$ and $j \in \{1, 2, \dots, s\} \setminus I$. Then $y_{ki} = 0$ and

$$y_{(k-1)i} = \left(n_i^{p_i} - 1\right)^{1/p_i} - \left(n_i - 1\right) \to 1.$$

Put $\prod_{i\in I} p_i = A$ and $\{1, 2, \dots, s\} \setminus I = \{j_1 \dots, j_l\}$. Because of the assumption $A/p_{j_1}, \dots, A/p_{j_l}$ are pairwise different and nonintegers. Thus, using Prop. 4, the sequence $\left((n^A)^{1/p_{j_1}}, \dots, (n^A)^{1/p_{j_l}}\right)_{n=1}^{\infty}$ is u.d. mod 1. This property remains, if we restrict n to be $\neq m^{p_{j_i}}$, for $m = 1, 2, \dots$ and $i = 1, \dots, l$. This shows that the sequence $((\mathbf{y}_{k-1}, \mathbf{y}_k))_{k=2}^{\infty}$, where $k = n^A$, and $n \neq m^{p_j}$ for $m = 1, 2, \dots$ and $j \in \{1, 2, \dots, s\} \setminus I$, has a limit point of type (ii) with coordinates $\alpha_i = 1$ and $\beta_i = 0$ for $i \in I$, and $\alpha_j = \beta_j$ for $j \in \{1, 2, \dots, s\} \setminus I$ for arbitrary $\alpha_j \in [0,1]$.

Finally, we apply Prop. 1, and the proof is complete. ◊

References

[1] HLAWKA, E.: Über eine Klasse von Näherungspolygonen zur Peanokurve, *J. Number Th.* **43** (1993), 93–108.

[2] KUIPERS, L. and NIEDERREITER, H.: Uniform Distribution of Sequences, John Wiley & Sons, New York, 1974.

[3] MYERSON, G.: A sampler of recent developments in the distribution of sequences, Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991), Marcel Dekker, New York, *Lecture Notes in Pure and App. Math.* **147** (1993) 163–190.

[4] SHOHAT, J.A. and TAMARKIN, J.D.: The Problem of Moments. Mathematical Surveys, Amer. Math. Soc., Providence, Rhode Island, 1943.

[5] STRAUCH, O.: Uniformly maldistributed sequences in a strict sense, *Mh. Math.* (to appear).