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A SEPARATION THEOREM FOR M_ϕ-CONVEX FUNCTIONS

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Abstract: Some theorems about separation of two real functions by the function which is convex or affine with respect to the weighted quasi–arithmetic means are presented.

Introduction

It is shown in [2] that every real functions f and g, defined on an interval $I \subset \mathbb{R}$ and satisfying the inequality

$$f(tx + (1-t)y) \le tg(x) + (1-t)g(y)\,,$$

for all x, $y \in I$, and $t \in (0,1)$, can be separated by a convex function (cf. Th. A). Applying a Helly's theorem, the authors of [5] proved that if, besides the above inequality, the functions f, g satisfy the reverse inequality with f and g interchanged, then there exists an affine function which separates these functions (cf. Th. B).

In Section 1 we quote these results and we show that Th. B is a consequence of Th. A. Moreover, we discuss the inequality

$$f(tx + (1-t)y) \le tg_1(x) + (1-t)g_2(y)$$

with three functions defined again on a real interval I and we show that, in general, there is no a separating convex function between f and $\min(g_1, g_2)$.

The main results of this paper are given in Section 3 and 4 where we transfer the Ths. A and B to the class M_ϕ–convex and M_ϕ–affine functions (M_ϕ denotes the family of the weighted quasi–arithmetic means of the generator ϕ).

1. Remarks on separation theorems for convex and affine functions

We begin with recalling the following

Theorem A ([2]). *Real functions f and g, defined on a real interval I, satisfy the inequality*

$$f(tx + (1-t)y) \le tg(x) + (1-t)g(y),$$

for all x, $y \in I$ and $t \in (0,1)$ if, and only if, there exists a convex function $h : I \subset \mathbb{R}$ such that $f \le h \le g$.

As a simple consequence we obtain

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval. If f, g_1, $g_2 : I \subset \mathbb{R}$ satisfy the inequality*

(1) $\quad f(tx + (1-t)y) \le tg_1(x) + (1-t)g_2(y), \quad x, y \in I, \quad t \in (0,1),$

then there exists a convex function $h : I \longmapsto \mathbb{R}$ such that $f \le h \le$ $\le \max(g_1, g_2)$.

Remark 1. If f, g_1, $g_2 : I \subset \mathbb{R}$ satisfy the inequality (1), then obviously that $f \le \min(g_1, g_2)$. In this connection a question arises whether there exists a convex function $h : I \longmapsto \mathbb{R}$ such that $f \le h \le \min(g_1, g_2)$. Taking $I = \mathbb{R}$, g_1, $g_2 : \mathbb{R} \longmapsto \mathbb{R}$, $g_1(x) = x^2$, $g_2(x) = (x-1)^2$, $x > 0$, and $f = \min(g_1, g_2)$, it is easy to see that the answer is negative.

However, we can prove the following

Proposition. *Let $I \subset \mathbb{R}$ be an interval, and suppose that the functions f, $g_1, \dots, g_n : I \mapsto \mathbb{R}$ satisfy the inequality*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \le \sum_{i=1}^{n} t_i g_i(x_i), \qquad \sum_{i=1}^{n} t_i = 1, \quad t_i \ge 0, \quad x_i \in I.$$

If $g_n \le \min(g_1, \dots, g_{n-1})$, then there exists a convex function $h :$ $: I \longmapsto \mathbb{R}$ such that $f \le h \le \min(g_1, \dots, g_{n-1})$. If moreover, $g_n =$ $= \min(g_1, \dots, g_{n-1})$ then the converse implication also holds true.

Proof. Take arbitrary $x, y \in I$, $t \in [0,1]$, and $i = 1, \dots, n-1$. Setting $t_i = t$, $t_n = 1-t$, and $t_j = 0$, $j = 1, \dots, n-1$, $j \neq i$; $x_i = x$, $x_n = y$, we get

$$f(tx + (1-t)y) \leq tg_i(x) + (1-t)g_n(y), \quad i = 1, \dots, n-1.$$

It follows that

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g_n(y) \leq tg(x) + (1-t)g(y),$$

where $g = \min(g_1, \dots g_{n-1})$. Now Th. A completes the proof. $\Diamond$

Applying Helly's theorem on the existence a straight line intersecting a family of parallel compact segment in $\mathbb{R}^2$, the authors of [5] proved the following

Theorem B. *Let $I \subset \mathbb{R}$ be an interval. The functions $f, g : I \longmapsto \mathbb{R}$ satisfy the system of inequalities*

$$\begin{cases} f(tx + (1-t)y) & \leq tg(x) + (1-t)g(y) \\ g(tx + (1-t)y) & \geq tf(x) + (1-t)f(y) \end{cases} \quad x, y \in I, \quad t \in (0,1),$$

if, and only if, there exists an affine function $h : I \longmapsto \mathbb{R}$ such that $f \leq h \leq g$.

It turns out that Th. B is a consequence of Th. A. In fact, applying Th. A to the first of the inequalities we get a convex function $h_1 : I \longmapsto \mathbb{R}$ such that $f \leq h_1 \leq g$. Writing the second inequality in the equivalent form

$$(-g)(tx + (1-t)y) \leq t(-f)(x) + (1-t)(-f)(y), \quad x, y \in I, \quad t \in (0,1),$$

and applying again Th. A we obtain a concave function $h_2 : I \longmapsto \mathbb{R}$ such that $f \leq h_2 \leq g$.

Now there are three possible cases: either the graphs of h_1 and h_2 have two different common points or they have only one common point or there is no points of intersection of the graphs of h_1 and h_2.

Taking in the first case the straight line through the both common points; in the second case a straight line through the common point which lies between the graphs of h_1 and h_2, and, in the third case, any straight line between the graphs of h_1 and h_2, we get the desired affine function h.

2. Definitions and some properties of $\mathbf{M}_\phi$-convex functions

Let $I \subset \mathbb{R}$ be an interval. For a fixed continuous and strictly monotonic function $\phi : I \longmapsto \mathbb{R}$ and for any fixed $t \in (0,1)$, we define $M_{\phi,t} : I^2 \longmapsto I$ by the formula

(2) $$M_{\phi,t}(x,y) = \phi^{-1}(t\phi(x) + (1-t)\phi(y)), \quad x,\ y \in I.$$

The function $M_{\phi,t}$ is a mean in I i.e., for all $x,\ y \in I$,

$$\min(x,y) \le M_{\phi,t}(x,y) \le \max(x,y),$$

and it is called a *weighted quasi-arithmetic mean* (cf. [1], p. 287 and [3], p. 189). Note that, for any interval $J \subset I$,

$$M_{\phi,t}(J \times J) \subset J, \qquad t \in (0,1).$$

This property allows us to introduce the following

Definition 1. Let a subinterval J of I and $t \in (0,1)$ be fixed. A function $w : J \longmapsto I$ is said to be

(i) $M_{\phi,t}$*–convex* if $w(M_{\phi,t}(x,y)) \le M_{\phi,t}(w(x), w(y))$, $x,\ y \in J$;

(ii) $M_{\phi,t}$*–concave* if $w(M_{\phi,t}(x,y)) \ge M_{\phi,t}(w(x), w(y))$, $x,\ y \in J$;

(iii) $M_{\phi,t}$*–affine* if $w(M_{\phi,t}(x,y)) = M_{\phi,t}(w(x), w(y))$, $x,\ y \in J$.

Definition 2. A function $w : J \longmapsto I$ is called M_ϕ*–convex* if for every $t \in (0,1)$ it is $M_{\phi,t}$-convex. Analogously we define M_ϕ*-concave* and M_ϕ*-affine* functions.

Remark 2. Let $I = \mathbb{R}$ and let $\phi : I \longmapsto \mathbb{R}$ be given by

$$\phi(u) = au + b, \qquad u \in I,$$

where $a,\ b \in \mathbb{R}$, $a \ne 0$, are fixed. It is easy to see that $M_{\phi,\frac{1}{2}}$-convexity of a function w is equivalent to the Jensen convexity of w, and, for every fixed $t \in (0,1)$, the $M_{\phi,t}$-convexity of w reduces to its t-convexity (cf. [4]). Moreover, M_ϕ-convexity of a function coincides with its classical convexity. Thus the notion of the M_ϕ-convexity generalizes the classical convexity.

In the sequel the following criterion of the M_ϕ-convexity will be useful.

Lemma 1. *Let $\phi : J \longmapsto \mathbb{R}$ be continuous and strictly decreasing. Then $u : \phi(J) \longmapsto J$ is concave if, and only if, the function $\phi^{-1} \circ u \circ \phi$ is M_ϕ-convex on J.*

Proof. By the concavity of u we have

$$u(tr + (1-t)s) \ge tu(r) + (1-t)u(s), \quad r,\ s \in \phi(I), \quad t \in (0,1).$$

Setting here $r = \phi(x)$, $s = \phi(y)$, for $x,\ y \in J$, and applying the decreasing monotonicity of ϕ, we get

$$w\left(\phi^{-1}\left(t\phi(x) + (1-t)\phi(y)\right)\right) \le \phi^{-1}\left(t\phi\left(w(x)\right) + (1-t)\phi\left(w(y)\right)\right),$$

for all $x,\ y \in J$, and $t \in (0,1)$, where $w := \phi^{-1} \circ u \circ \phi$. This shows that w is M_ϕ-convex on J. The converse implication is obvious. ◇

Similarly we prove the following

Lemma 2. *Let $\phi : J \longmapsto \mathbb{R}$ be continuous and strictly increasing. Then $u : \phi(J) \longmapsto J$ is convex if, and only if, the function $\phi^{-1} \circ u \circ \phi$ is M_ϕ-convex on J.*

3. Separation theorem for $\mathbf{M}_\phi$-convex functions

The main result of this section reads as follows:

Theorem 1. *Let I and J be intervals such that $J \subset I$ and suppose that $\phi : J \longmapsto \mathbb{R}$ is continuous and strictly monotonic. Then $f,\ g : J \longmapsto I$ satisfy the inequality*

(3) $$f\left(M_{\phi,t}(x,y)\right) \le M_{\phi,t}\left(g(x), g(y)\right), \qquad x,\ y \in J, \quad t \in (0,1),$$

if, and only if, there exists an M_ϕ-convex function $h : J \longmapsto I$ such that

(4) $$f(x) \le h(x) \le g(x)\,, \quad x \in J\,.$$

Proof. Assume that (3) holds true. First consider the case when ϕ is strictly decreasing. From (2) and (3), for all $x,\ y \in J$, and $t \in (0,1)$, we obtain

$$f\left(\phi^{-1}(t\phi(x) + (1-t)\phi(y))\right) \le \phi^{-1}\left(t\phi(g(x)) + (1-t)\phi(g(y))\right).$$

Choose arbitrary $r,\ s \in \phi(J)$. Substituting here $x = \phi^{-1}(r)$ and $y = \phi^{-1}(s)$ and making use of the decreasing monotonicity of ϕ we get

(5) $$(\phi\circ f\circ\phi^{-1})(tr+(1-t)s) \ge t(\phi\circ g\circ\phi^{-1})(r)+(1-t)(\phi\circ g\circ\phi^{-1})(s)$$

for all $r,\ s \in \phi(J)$ and $t \in (0,1)$. Define $\bar{f},\ \bar{g} : \phi(J) \longmapsto J$ by

(6) $$\bar{f} = \phi\circ f\circ\phi^{-1}\,, \quad \bar{g} = \phi\circ g\circ\phi^{-1}\,.$$

In view of (5) we have

$$\bar{f}(tr + (1-t)s) \ge t\bar{g}(r) + (1-t)\bar{g}(s), \qquad r,\ s \in \phi(J), \quad t \in (0,1).$$

Now, applying Th. A, we infer that there exists a concave function $\bar{h} : \phi(J) \longmapsto J$ such that

$$\bar{f}(r) \ge \bar{h}(r) \ge \bar{g}(r)\,, \quad r \in \phi(J)\,.$$

Putting here $r = \phi(x)$, $x \in J$, and making use of the decreasing monotonicity of ϕ, we get

$$f(x) \le \left(\phi^{-1}\circ\bar{h}\circ\phi\right)(x) \le g(x)\,, \qquad x \in J\,.$$

In view of Lemma 1, the function $h : J \longmapsto I$ defined by

$$h = \phi^{-1}\circ\bar{h}\circ\phi$$

is the desired M_ϕ–convex function.

Now consider the remaining case when ϕ is strictly increasing. A similar reasoning as in the previous part of the proof shows that

$$(\phi\circ f\circ\phi^{-1})(tr + (1-t)s) \le t(\phi\circ g\circ\phi^{-1})(r) + (1-t)(\phi\circ g\circ\phi^{-1})(s)$$

for all $r,\ s \in \phi(J)$, and $t \in (0,1)$, which means that

$$\bar{f}(tr + (1-t)s) \le t\bar{g}(r) + (1-t)\bar{g}(s)\,, \qquad r,\ s \in \phi(J), \quad t \in (0,1),$$

where $\bar{f},\ \bar{g} : \phi(J) \longmapsto J$ are defined by (6). Applying again the Th. A gives the existence of convex function $h : \phi(J) \longmapsto J$ such that

$$\bar{f}(r) \le \bar{h}(r) \le \bar{g}(r), \quad r \in \phi(J).$$

Putting here $r = \phi(x)$, $x \in J$, and making use of the increasing monotonicity of ϕ we obtain (4) with $h : J \longmapsto I$ defined by formula $h = \phi^{-1} \circ \bar{h} \circ \phi$. By Lemma 2, h is the desired M_ϕ-convex function.

The converse implication is an easy consequence of the fact that the weighted quasi-arithmetic mean is strictly monotonic with respect to each variable. ◊

Remark 3. Applying Th. 1 with $\phi : J \longmapsto \mathbb{R}$ defined by $\phi(u) = au + b$, $u \in J$, where $a, b \in \mathbb{R}$, $a \neq 0$, are fixed, we get the result obtained in [2].

Recall that a function $h : J \mapsto (0, \infty)$ is *geometrically convex* if

$$h\left(x^t y^{1-t}\right) \le (h(x))^t \, (h(y))^{1-t}, \quad x, y \in J, \ t \in (0,1).$$

Taking $I = (0, \infty)$, and $\phi(t) = \log t$ $(t > 0)$ in Th. 1 we obtain the following

Corollary 2. *Let $J \subset (0, \infty)$ be an interval and suppose that $f, g : J \longmapsto (0, \infty)$. Then*

$$f\left(x^t y^{1-t}\right) \le (g(x))^t \, (g(y))^{1-t}, \qquad x, y \in J, \quad t \in (0,1),$$

if, and only if, there exists a geometrically convex function $h : J \longmapsto (0, \infty)$ such that

$$f(x) \le h(x) \le g(x), \quad x \in J.$$

4. Separation theorem for M_ϕ-affine functions

In this section we prove the following

Theorem 2. *Let I, J be intervals such that $J \subset I$. Suppose that $\phi : J \longmapsto \mathbb{R}$ is a continuous and strictly monotonic, and $f, g : J \longmapsto I$. Then the following conditions are equivalent:*

(i) *there exists an M_ϕ–affine function $h : J \longmapsto I$ such that*

$$f(x) \le h(x) \le g(x), \quad x \in J;$$

(ii) *there exist an M_ϕ-convex function $h_1 : J \longmapsto I$ and an M_ϕ-concave function $h_2 : J \longmapsto I$ such that*

$$f(x) \le h_1(x) \le g(x), \quad x \in J, \qquad f(x) \le h_2(x) \le g(x), \quad x \in J;$$

(iii) *the functions f and g satisfy the system of inequalities:*

$$\begin{cases} f(M_{\phi,t}(x,y)) & \le M_{\phi,t}(g(x), g(y)) \\ g(M_{\phi,t}(x,y)) & \ge M_{\phi,t}(f(x), f(y)) \end{cases} \quad x, y \in I, \quad t \in (0,1).$$

Proof. Implication (i) $\Longrightarrow$ (ii) is a consequence of the fact that every affine function is both convex and concave.

The increasing monotonicity of the weighted quasi-arithmetic mean $M_{\phi,t}$ with respect to each variable yields the implication (ii) $\Longrightarrow$ $\Longrightarrow$ (iii).

To show the implication (iii) $\Longrightarrow$ (i) first assume that ϕ is strictly decreasing. Taking $\bar{f}$, $\bar{g} : \phi(J) \longmapsto J$ defined by (6) we can write the system (iii) in the form

$$\begin{cases} \bar{f}(tr + (1-t)s) & \geq t\bar{g}(r) + (1-t)\bar{g}(s) \\ \bar{g}(tr + (1-t)s) & \leq t\bar{f}(r) + (1-t)\bar{f}(s) \end{cases} \quad r,\, s \in \phi(J), \quad t \in (0,1).$$

Applying Th. B we infer that there exists an affine function $\bar{h} : \phi(J) \longmapsto \longmapsto J$ such that

$$\bar{g}(r) \leq \bar{h}(r) \leq \bar{f}(r)\,, \quad r \in \phi(J)\,.$$

Putting here $r = \phi(x)$, $x \in J$, and making use of the decreasing monotonicity of ϕ we get

$$f(x) \leq h(x) \leq g(x)\,, \quad x \in J\,,$$

where $h : J \longmapsto I$ is given by the formula $h = \phi^{-1} \circ \bar{h} \circ \phi$. Clearly, h is the desired M_ϕ–affine function.

Assume now that ϕ is strictly increasing. Similarly as in the previous case, the function $\bar{f}$, $\bar{g} : \phi(J) \longmapsto J$ defined by (6) satisfy the system of inequalities

$$\begin{cases} \bar{f}(tr + (1-t)s) & \leq t\bar{g}(r) + (1-t)\bar{g}(s) \\ \bar{g}(tr + (1-t)s) & \geq t\bar{f}(r) + (1-t)\bar{f}(s) \end{cases} \quad r,\, s \in \phi(J), \quad t \in (0,1).$$

The existence of the affine function $\bar{h} : \phi(J) \longmapsto J$ such that

$$\bar{f}(r) \leq \bar{h}(r) \leq \bar{g}(r)\,, \quad r \in \phi(J)\,.$$

is again a consequence of theorem B. Now it is easy to check that $h :$ $: J \longmapsto I$ given by $h = \phi^{-1} \circ \bar{h} \circ \phi$ satisfies the condition (i). $\diamondsuit$

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