# A DISTRIBUTIONALLY CHAOTIC TRIANGULAR MAP WITH ZERO SEQUENCE TOPOLOGICAL ENTROPY

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**Abstract**: Recently, the notion of distributional chaos for a continuous map $\Phi$ from a metric space $(M, d)$ into itself has been introduced. In the case of a compact interval it has been proved that $\Phi$ is distributionally chaotic if and only if it has positive topological entropy. So, the following natural question arises: Is this property true for a general compact metric space? In this paper we show that the answer is negative, by presenting an example of a (triangular) map which is distributionally chaotic but has, not only zero topological entropy, but also zero sequence topological entropy for any increasing sequence of positive integers.

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# 1. Introduction

In the last decades several different definitions of chaos have been proposed. Among them the definition of Li and Yorke [5] seems to capture our intuitive notion of the meaning of chaotic behaviour. In an inessentially modified version of the original one, it is the following: a map $f$ of a metric space $(M, d)$ into itself is *chaotic* if it admits a scrambled set, i.e. a subset $S$ of $M$ containing at least two points, such that, for any $x, y \in S$ with $x \neq y$,

$$\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0.$$

Nevertheless chaos in this sense is not stable since arbitrary small perturbations of chaotic functions can destroy Li-Yorke chaos.

Among other definitions of chaos we mention that of Devaney [1], and that which assumes positive topological entropy, as the characterization of a chaotic map. In [7] these different definitions of chaos are critically examined in order to determine whether or not they satisfy some *natural* requirements: the conclusion suggested by the authors is that only positive topological entropy appears to satisfy all the criteria.

However, the numerical value of the topological entropy can be a quite misleading indicator of the actual extent of chaotic behaviour. To avoid this disadvantage, in [6] the following new notion of *distributional chaos* was introduced and, for spaces with finite diameter, a related *measure of chaos*.

Let $\Phi$ be a map from a metric space $(M, d)$ into itself. For any pair $(x, y)$ of points in $M$, we define a sequence $\delta_{xy}$ by:

$$\delta_{xy}(m) = d(\Phi^m(x), \Phi^m(y)), \quad m = 0, 1, \cdots.$$

Next, for any pair $(x, y)$ and any positive integer $n$, define the functions $F_{xy}^{(n)}$ on the real line by

$$F_{xy}^{(n)}(t) = \frac{1}{n} \#\{m : 0 \leq m \leq n - 1 \quad \text{and} \quad \delta_{xy}(m) < t\},$$

where $\#S$ is the number of elements in the set $S$. Each $F_{xy}^{(n)}$ is a left-continuous *distribution function*.

Now consider the functions $F_{xy}$ and $F_{xy}^*$ defined by

$$F_{xy}(t) = \liminf_{n \to \infty} F_{xy}^{(n)}(t), \quad F_{xy}^*(t) = \limsup_{n \to \infty} F_{xy}^{(n)}(t).$$

For any pair $(x, y)$, $F_{xy}$ and $F_{xy}^*$ are distribution functions with $F_{xy}(t) \leq$ $\leq F_{xy}^*(t)$ for all real $t$. We normalize these functions to be left-continuous and then adopt the convention that $F_{xy} < F_{xy}^*$ means $F_{xy}(t) <$ $< F_{xy}^*(t)$ for $t$ in some non degenerate interval.

We shall refer to $F_{xy}$ as the *lower* and $F_{xy}^*$ as the *upper*, *distribution functions* of $x$ and $y$ relative to the map $\Phi$.

**Definition** [6]. A function $\Phi$ mapping a metric space $(M, d)$ into itself is *distributionally chaotic* (briefly *d-chaotic*) if there is a pair of points $(x, y)$ in $M$ such that $F_{xy} < F_{xy}^*$. If $M$ has finite diameter $d_M$, the *measure of chaos* of $\Phi$ is the number $\mu(\Phi)$ given by

$$\mu(\Phi) = \sup_{x,y \in M} \frac{1}{d_M} \int_0^\infty \left( F_{xy}^*(t) - F_{xy}(t) \right) dt.$$

The main relation between positive topological entropy and distributional chaos is given by the following

**Theorem 1** [7]. *A continuous function $f$ mapping a compact interval into itself is d-chaotic if and only if it has positive topological entropy.*

So, the following natural question arises: Is Th. 1 still true for a general compact metric space?

In the present paper we exhibit an example showing that for triangular maps of the square, *d*-chaoticity is not equivalent to positive topological entropy.

## 2. Construction of the example

For any increasing sequence $A$ of positive integers and any continuous function $\Phi$, $h_A(\Phi)$ will denote the corresponding sequence topological entropy of $\Phi$ (see [4]). Note that if $A$ is the sequence of all positive integers, then $h_A$ is just the usual topological entropy $h$.

We recall that a *triangular map of the square* is a continuous map $F : I^2 \to I^2$ of the form $F(x, y) = \big(f(x), g(x, y)\big) = \big(f(x), g_x(y)\big)$, where $I := [0, 1]$ and $g_x : I \to I$ is a family of maps depending continuously on $x$. The map $f$ of the corresponding dynamical system is called the *base* for $F$.

We will consider triangular maps $G : I^2 \to I^2$ with a linear base and such that $G(0, y) = (0, y)$ for all $y \in I$, i.e. maps such that $(x, y) \mapsto \mapsto (\lambda x, g_x(y))$ and $g_0 = \mathrm{Id}$, where $\lambda \in (0, 1)$ is fixed. Denote the class of such maps by $\mathcal{T}_\lambda$. As proved in [2], any $G$ in $\mathcal{T}_\lambda$ has zero topological entropy.

In the proof of the next theorem the following notation will be useful: Given a fixed $\lambda$ in $(0, 1)$ and an increasing sequence $\{n_i\}_{i \geq 1}$ of positive integers, let $G_i$ for each $i \geq 1$, be a restriction to the rectangle $[\lambda^{n_i}, 1] \times I$ of a function in $\mathcal{T}_\lambda$ such that, for all $y \in I$,

$$G_i(1, y) = (\lambda, y) \quad \text{and} \quad G_i(\lambda^{n_i}, y) = (\lambda^{n_i+1}, y), \quad i = 1, 2 \cdots .$$

Next, let $s_k = \sum_{j=1}^{k} n_j$ and let $\{[G_1, G_2, \cdots, G_{k+1}]\}_{k \geq 1}$ be the sequence of functions recursively defined by

$$[G_1] = G_1,$$

and for any $k \geq 1$, $[G_1, G_2, \cdots, G_{k+1}]$ is the function defined on the rectangle $[\lambda^{s_{k+1}}, 1] \times I$ by

$$[G_1, G_2, \cdots, G_{k+1}](x, y) = \begin{cases} [G_1, G_2, \cdots, G_k](x, y), & x \in [\lambda^{s_k}, 1], \\ \varphi_k^{-1}\big(G_{k+1}(\varphi_k(x, y))\big), & x \in [\lambda^{s_{k+1}}, \lambda^{s_k}], \end{cases}$$

where $\varphi_k$ is the function defined on $[\lambda^{s_{k+1}}, \lambda^{s_k}] \times I$ by $\varphi_k(x, y) = = (\lambda^{-s_k} x, y)$.

Finally, if $G^\infty$ is the identity map on $\{0\} \times I$, we denote by $[G_1, G_2, \cdots]$ the map $G$ given by $\big(\lim_{k \to \infty}[G_1, G_2, \cdots, G_k]\big) \cup G^\infty$. Moreover, if the portions $G_k$ are chosen so that $G = [G_1, G_2, \cdots]$ is continuous on $\{0\} \times I$, then $G \in \mathcal{T}_\lambda$.

Our result is the following

**Theorem 2.** *There exists $G \in \mathcal{T}_\lambda$ such that:*

(i) *$G$ has a two-point scrambled set,*

(ii) *$G$ has no scrambled set with more than two points,*

(iii) *$h_A(G) = 0$ for any sequence $A$,*

(iv) *for $u = (0,0)$ and $v = (0,1)$,*

$$F_{uv}(t) = 0, \quad F_{uv}^*(t) \geq t, \quad t \in (0,1),$$

*where $F_{uv}$ and $F_{uv}^*$ are the lower and the upper distribution functions of $u$ and $v$ relative to $G$.*

**Proof.** Instead of a map from $\mathcal{T}_\lambda$ we will construct a topologically conjugate map $G$ defined on $[0, \infty] \times I$ (where $\infty$ stands for $+\infty$) by $G(x, y) = (x + 1, g_x(y))$ if $x < \infty$, and $G(\infty, y) = (\infty, y)$. Moreover, we may assume that $[0, \infty] \times I$ is equipped with the metric $|(x_1, y_1) - (x_2, y_2)| = \max\big\{\big|\lambda^{x_1} - \lambda^{x_2}\big|, |y_1 - y_2|\big\}$, which makes $[0, \infty] \times I$ homeomorphic to $I^2$ with the usual metric. This will simplify our notation and, on the other hand, will not change the considered properties (including the sequence topological entropy), since they are invariant with respect to homeomorphic transformations, cf. [4].

For any integer $k > 1$ define continuous maps $\mu_k, \nu_k : I \to I$ by

$$\mu_k(t) = \max\{0, t - 1/k\}, \quad \nu_k(t) = \min\{2t, t + 1/k, 1\}, \quad t \in I.$$

Then

$\mu_k^i(I) = [0, 1 - i/k]$ for $0 \le i \le k$, $(\nu_k^i \circ \mu_k^i)(I) = I$ for $0 \le i \le k - 1$.
For any positive integer $k$ define a map $G_k$ on $[0, 2k] \times I$ by $G_k(x, y) = (x + 1, g_{k,x}(y))$ where

$$g_{k,x}(y) = \begin{cases} \max\{0, y + h_k(x)\} & \text{if } x \in [0, k] \\ \min\{y + h_k(x)\min\{ky, 1\}, 1\} & \text{if } x \in [k, 2k] \end{cases}$$

and the function $h_k(x)$ is piecewise linear and connects the points
$(0, 0), (1, -1/k), (k - 1, -1/k), (k + 1, 1/k), (2k - 1, 1/k)$ and $(2k, 0)$
(note that $g_{k,x} = \text{Id}$ for $x = 0, k, 2k$, $g_{k,x}(y) = \mu_k(y)$ for $x \in [1, k - 1]$,
and $g_{k,x}(y) = \nu_k(y)$ for $x \in [k + 1, 2k - 1]$).

The map $G_k$ has the following phase diagram (with $k = 7$) in
which we depicted the trajectories going through the points of the segments $([0, 1] \cup [k, k + 1]) \times \{i/k\}$, $0 \le i \le k$ and $[k, k + 1] \times \{1/(k2^i)\}$,
$1 \le i \le k - 1$. All trajectories are piecewise linear, except for the trajectory of the segment $[k, k+1] \times \{1/(k2^{k-1})\}$ in the interval $[2k, 2k+1]$
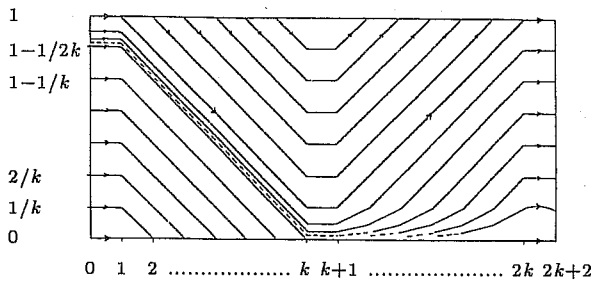(see Fig. 1).



Fig.1

Finally, let

$$1 < k(1) < k(2) < \cdots$$

be a sequence of integers (characterized below) and let $G = [G_{k(1)}, G_{k(2)}, \cdots]$. Clearly $G$ is continuous and topologically conjugate to a map of $\mathcal{T}_\lambda$.

Put $n(0) = 0$, $n(i) = 2(k(1) + k(2) + \cdots + k(i))$ for $i > 0$, and choose the sequence $\{k(i)\}_{i>0}$ so that

$$k(i) \quad \text{divides} \quad k(i + 1) \quad \text{and} \quad \frac{n(i)}{k(i + 1)} \longrightarrow 0 \quad \text{monotonically.}$$

In [2] it has been proved that the function $G$ satisfies (i)–(iii).
We now prove (iv). Let $\bar{t} \in (0, 1)$; we have to prove that $F_{uv}(\bar{t}) = 0$. To do this take $j$ such that $p := k(j) > \frac{1}{1-\bar{t}}$ and let $s_i = n(i) + $

$+ \frac{k(i+1)}{p}$, $i \geq j$. For all $n = n(i) + t$, $1 \leq t \leq \frac{k(i+1)}{p}$, $i \geq j$, we have $|G^n(u) - G^n(v)| \geq 1 - \frac{1}{p} > \bar{t}$. Thus we get

$$0 \leq F_{uv}^{(s_i)}(\bar{t}) \leq \frac{n(i)}{s_i} < \frac{pn(i)}{k(i+1)} \to 0 \quad \text{as} \quad i \to \infty.$$

To prove that $F_{uv}^*(t) \geq t$ for $t \in (0,1)$, it is enough to show that this inequality holds on a dense subset of $(0,1)$. Fix an integer $j > 1$ and consider the points of the form $m/k(j)$, $1 \leq m < k(j)$, and let $r_i = n(i) + k(i+1)$, $i \geq j$. The number of integers $n = n(i) + t, 1 \leq t \leq k(i+1)$, for which $|G^n(u) - G^n(v)| < \frac{m}{k(j)}$ is $\frac{mk(i+1)}{k(j)} - 1$, thus

$$F_{uv}^{(r_i)}\left(\frac{m}{k(j)}\right) \geq \frac{\frac{mk(i+1)}{k(j)} - 1}{r_i} \to \frac{m}{k(j)} \quad \text{as} \quad i \to \infty.$$

Since the points $m/k(j)$, $j > 1$, $1 \leq m < k(j)$ are dense in $(0,1)$, the proof of the theorem is complete. $\Diamond$

**Remark.** In the paper [3] some other examples of distributionally chaotic triangular maps with zero topological entropy are constructed; all these examples have base maps of $2^\infty$ type. The example presented here is of special interest since the function $G$ not only has zero topological entropy but also has zero sequence topological entropy and, moreover, its base map is linear.

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