

APPROXIMATION PROPERTIES OF A GENERALIZATION OF BLEIMANN, BUTZER AND HAHN OPERATORS

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Abstract: In this paper we deal with a generalization of Bleimann, Butzer and Hahn operators which is obtained by replacing the binomial coefficients with some general ones satisfying a suitable recursive relation. We present their decomposition as sum of elementary operators and study the convergence of these new operators together with some quantitative estimates.

In 1996 M. Campiti and G. Metafune [5] presented and deeply studied a generalization of the classical Bernstein operators. Their work took its motivation from the development of the study of connections between approximation processes and evolution problems through semi-group theory. In some cases the introduction of new types of operators became necessary, for a unified treatment of this subject see [3].

Let $C[0, \infty)$ be the space of continuous functions on the unbounded interval $[0, \infty)$ and let $f \in C[0, \infty)$. Bleimann, Butzer and Hahn [4] introduced a linear and positive operator defined by

$$(1) \quad (L_n f)(x) = (1+x)^{-n} \sum_{k=0}^n f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \quad n \geq 1.$$

It is important to mention that this finite sum arises in a natural way and not as a truncation process of an operator defined by means of an infinite sum, as would be possible for the Favard, Meyer-König and Zeller, Szász-Mirakyan or Baskakov operators. The operator in question was further studied by many other authors both of its new properties were found and some generalizations were given. We recall here [1], [2] representing some of the most recent papers in this direction.

The purpose of this note is to apply the conception of Campiti and Metafuno for the operator defined by (1) and examine the main properties of this new approximation process. Actually, we replace the binomial coefficients by general ones satisfying similar recursive properties, more exactly the sequences of the constant value 1 at the sides of Pascal's triangle are replaced by arbitrary values. We obtain a decomposition of the Bleimann, Butzer and Hahn operator as a sum of elementary operators. Also, we prove that the sequences of the linear operators obtained here, converge towards an operator multiplied by an analytic function.

1. Construction of the operators

Let's take two sequences of real numbers $\alpha = (\alpha_n)_{n \geq 1}$, $\beta = (\beta_n)_{n \geq 1}$ and introduce the coefficients $a_{n,k}$ which satisfy the following recursive formulae:

$$(2) \quad \begin{aligned} a_{n,0} &= \alpha_n, & a_{n,n} &= \beta_n, & n &\geq 1 \\ a_{n+1,k} &= a_{n,k} + a_{n,k-1}, & 1 &\leq k \leq n. \end{aligned}$$

We observe that these numbers are determined uniquely by the sequences α and β . Also, if $\alpha_i = \beta_i = 1$ for $1 \leq i \leq n$ then $a_{n,k} = \binom{n}{k}$ for every $k \in \{0, 1, \dots, n\}$. We consider the operators $L_n^{(\alpha, \beta)}$ having the form:

$$(3) \quad (L_n^{(\alpha, \beta)} f)(x) = (1+x)^{-n} \sum_{k=0}^n a_{n,k} x^k f\left(\frac{k}{n-k+1}\right),$$

where f belongs to $C[0, \infty)$. We can make the following remarks:

- (i) $L_n^{(\alpha, \beta)}$ is a linear operator and depends linearly on the given sequences α, β ;

- (ii) if α, β are non-negative then $L_n^{(\alpha, \beta)}$ becomes a positive operator as well;
- (iii) $L_n^{(1, 1)}$ is identical to the Bleimann, Butzer and Hahn operator;
- (iv) if M is an upper bound for the sequences α and β then $L_n^{(\alpha, \beta)} f \leq \leq ML_n f$ for any positive $f \in C[0, \infty)$.

In order to investigate our operator we shall need the following relations which are verified by L_n , see [4]. For any $x \geq 0$:

$$\begin{aligned}
 (L_n e_0)(x) &= 1, & n &\geq 1 \\
 (4) \quad (L_n e_1)(x) &= x - x \left(\frac{x}{1+x} \right)^n, & n &\geq 1 \\
 |(L_n e_2)(x) - x^2| &\leq \frac{2x(1+x)^2}{n+2}, & n &\geq N(x)
 \end{aligned}$$

where $N(x) = 24(1+x)$ and e_k are the three test functions $e_k(t) = t^k, k = 0, 1, 2$.

2. Results

Firstly, we point out a decomposition of $L_n^{(\alpha, \beta)}$ in elementary operators.

Theorem 1. For $f \in C[0, \infty)$ and $n \geq 1$ the following identity

$$(L_n^{(\alpha, \beta)} f)(x) = \sum_{m=1}^n \alpha_m (U_{m, n} f)(x) + \sum_{m=1}^n \beta_m (V_{m, n} f)(x)$$

holds, where

$$(5) \quad (U_{m, n} f)(x) = \begin{cases} \sum_{k=1}^{n-m} \binom{n-m-1}{k-1} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right), & m < n \\ (1+x)^{-n} f(0) & m = n \end{cases}$$

and

$$(6) \quad (V_{m, n} f)(x) = \begin{cases} \sum_{k=m}^{n-1} \binom{n-m-1}{k-m} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right), & m < n \\ x^n (1+x)^{-n} f(n), & m = n. \end{cases}$$

Proof. Our purpose is to determine the elementary operators $U_{m, n}, V_{m, n}$ which are associated to the sequences α and β .

At the first step, we choose $\beta = 0$ and $\alpha = \delta_m, m \geq 1$, where $\delta_m = (\delta_{m, n})_{n \geq 1}, \delta_{m, n}$ being Kronecker's symbol. We obtain $U_{m, n} f = L_n^{(\delta_m, 0)} f$ and the coefficients $a_{n, k} = a_{n, k}^{(m, 0)}$ have the following form:

$$a_{n,k}^{(m,0)} = \begin{cases} 1, & n = m, k = 0 \\ \binom{n-m-1}{k-1}, & n > m, 1 \leq k \leq n - m \end{cases}$$

and in all other cases: $n < m$; $n = m, k \geq 1$; $n > m, k = 0$ and $k \geq n - m = 1$; the coefficients vanish. The above identities and (3) lead us to (5).

At the second step, we choose $\alpha = 0$ and $\beta = \delta_m, m \geq 1$, where $\delta_m = (\delta_{m,n})_{n \geq 1}$. It results $V_{m,n}f = L_n^{(0,\delta_m)}f$. In this case, the numbers $a_{n,k} = a_{n,k}^{(0,m)}$ are indicated below:

$$a_{n,k}^{(0,m)} = \begin{cases} 1, & n = m, k = n \\ \binom{n-m-1}{k-m}, & n > m, m \leq k \leq n - 1 \end{cases}$$

and in all other cases: $n < m$; $n = m, k \leq n - 1$; $n > m, k \leq m - 1$ and $k = n$; the coefficients vanish. By (3) we get (6). \diamond

Taking $f = e_0$ in (5) and (6) we easily obtain the following useful identities:

$$(7) \quad (U_{m,n}e_0)(x) = \begin{cases} x(1+x)^{-m-1}, & m < n \\ (1+x)^{-n}, & m = n \end{cases}$$

$$(8) \quad (V_{m,n}e_0)(x) = \begin{cases} x^m(1+x)^{-m-1}, & m < n \\ x^n(1+x)^{-n}, & m = n. \end{cases}$$

Substituting (7) and (8) in Th. 1, we can state:

Lemma 1. *The following identity*

$$(L_n^{(\alpha,\beta)}e_0)(x) = \sum_{m=1}^{n-1} (\alpha_m x + \beta_m x^m)(1+x)^{-m-1} + (\alpha_n + \beta_n x^n)(1+x)^{-n},$$

holds.

In what follows, we consider that the sequences α and β are bounded and under this assumption we can introduce the functions u, v, w , where

$$(9) \quad u(x) = \sum_{m \geq 1} \frac{\alpha_m x}{(1+x)^{m+1}}, \quad v(x) = \sum_{m \geq 1} \frac{\beta_m x^m}{(1+x)^{m+1}}$$

$$\text{and } w(x) = u(x) + v(x), \quad x \geq 0.$$

Also, we shall use the modulus of continuity of a function f defined as $\omega(f, \delta) := \sup\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x - y| \leq \delta\}, \delta \geq 0$.

In order to estimate the convergence of our operator we need the following result.

Lemma 2. *If $f \in C[0, \infty)$ and $M = \max\{\sup_{n \geq 1} |\alpha_n|, \sup_{n \geq 1} |\beta_n|\}$ then:*

$$(10) \quad \begin{aligned} & |(L_n^{(\alpha, \beta)} f)(x) - f(x)(L_n^{(\alpha, \beta)} e_0)(x)| \leq \\ & \leq M \left(1 + 2x(1+x)^2 + 2nx^2 \left(\frac{x}{x+1} \right)^n \right) \omega \left(f, \frac{1}{\sqrt{n}} \right). \end{aligned}$$

Proof. By using the property of M and the relation (2) it is clear that

$$(11) \quad |a_{n,k}| \leq M \binom{n}{k}.$$

On the other hand, we recall a known property of ω : for every $\delta > 0$,

$$(12) \quad |f(y) - f(x)| \leq \left(1 + \frac{1}{\delta^2} (x-y)^2 \right) \omega(f, \delta).$$

Finally, starting from (4) we can deduce:

$$(13) \quad \begin{aligned} (L_n \psi_x)(x) &= (L_n e_2)(x) - 2x(L_n e_1)(x) + x^2(L_n e_0)(x) \leq \\ &\leq \frac{2x(1+x)^2}{n+2} + 2x^2 \left(\frac{x}{x+1} \right)^n, \end{aligned}$$

where $\psi_x(t) = (t-x)^2, t \geq 0$.

By using (3), (11), (12), (13) we can write successively:

$$\begin{aligned} & |(L_n^{(\alpha, \beta)} f)(x) - f(x)(L_n^{(\alpha, \beta)} e_0)(x)| \leq \\ & \leq \sum_{k=0}^{\infty} |a_{n,k}| \frac{x^k}{(1+x)^n} \left| f \left(\frac{k}{n-k+1} \right) - f(x) \right| \leq \\ & \leq M \sum_{k=0}^n \binom{n}{k} \frac{x^k}{(1+x)^n} \left(1 + \frac{1}{\delta^2} \left(\frac{k}{n-k+1} - x \right)^2 \right) \omega(f, \delta) = \\ & = M \left(1 + \frac{1}{\delta^2} (L_n \psi_x)(x) \right) \omega(f, \delta) \leq \\ & \leq M \left\{ 1 + \frac{1}{\delta^2} \left(\frac{2x(1+x)^2}{n+2} + 2x^2 \left(\frac{x}{x+1} \right)^n \right) \right\} \omega(f, \delta). \end{aligned}$$

Taking $\delta = 1/\sqrt{n}$, the relation (10) follows. \diamond

Lemma 3. For any $x > 0$ the following inequality

$$|(L_n^{(\alpha, \beta)} e_0)(x) - w(x)| \leq \frac{1+x^{n+1}}{(1+x)^{n+1}} \Delta(n)$$

holds, where w is defined at (9) and

$$\Delta(n) = \max \left\{ \sup_{m>n} |\alpha_m - \alpha_n|, \sup_{m>n} |\beta_m - \beta_n| \right\}.$$

Proof. We remind that our general hypothesis is the sequences α, β are bounded.

For $x > 0$, by using Lemma 1 and the expression of w , we can write successively:

$$\begin{aligned}
 & |(L_n^{(\alpha, \beta)} e_0)(x) - w(x)| = \\
 & = \left| \frac{\alpha_n}{(1+x)^{n+1}} - \sum_{m=n+1}^{\infty} \frac{\alpha_m x}{(1+x)^{m+1}} + \frac{\beta_n x^{n+1}}{(1+x)^{n+1}} - \sum_{m=n+1}^{\infty} \frac{\beta_m x^m}{(1+x)^{m+1}} \right| = \\
 & = \left| \alpha_n - \sum_{k=1}^{\infty} \frac{\alpha_{n+k} x}{(1+x)^k} + x^n \left(\beta_n x - \sum_{k=1}^{\infty} \beta_{n+k} \left(\frac{x}{1+x} \right)^k \right) \right| (1+x)^{-n-1} = \\
 & = \left| x \sum_{k=1}^{\infty} \frac{\alpha_n - \alpha_{n+k}}{(1+x)^k} + x^n \sum_{k=1}^{\infty} (\beta_n - \beta_{n+k}) \left(\frac{x}{1+x} \right)^k \right| (1+x)^{-n-1} \leq \\
 & \leq \Delta(n) \sum_{k=1}^{\infty} \left(\frac{x}{(1+x)^k} + x^n \left(\frac{x}{1+x} \right)^k \right) = \frac{1+x^{n+1}}{(1+x)^{n+1}} \Delta(n). \quad \diamond
 \end{aligned}$$

Theorem 2. Let $f \in C[0, \infty)$. If the sequences α and β converge then

$$(14) \quad \lim_{n \rightarrow \infty} L_n^{(\alpha, \beta)} f = wf,$$

uniformly on any interval $[a, b] \subset (0, \infty)$, where w is defined at (9).

Proof. In concordance with Lemmas 2 and 3 we can write:

$$\begin{aligned}
 & |(L_n^{(\alpha, \beta)} f)(x) - w(x)f(x)| \leq \\
 & \leq |(L_n^{(\alpha, \beta)} f)(x) - f(x)(L_n^{(\alpha, \beta)} e_0)(x)| + |f(x)| |(L_n^{(\alpha, \beta)} e_0)(x) - w(x)| \leq \\
 & \leq M \left(1 + 2x(1+x)^2 + 2nx^2 \left(\frac{x}{x+1} \right)^n \right) \omega \left(f, \frac{1}{\sqrt{n}} \right) + \\
 & + \frac{1+x^{n+1}}{(1+x)^{n+1}} \Delta(n) |f(x)| \leq K_n(b) \omega \left(f, \frac{1}{\sqrt{n}} \right) + \Delta(n) \|f\|_{C[a, b]},
 \end{aligned}$$

where $K_n(b) = M \left(1 + 2b(1+b)^2 + 2nb^2 \left(\frac{b}{b+1} \right)^n \right)$. Respect with n , $K_n(b)$ is bounded and the convergence of α and β implies that $\Delta(n)$ tends to zero, consequently (14) follows. \diamond

We notice that choosing $\alpha = \beta = 1$, the relation (9) implies $w(x) = 1$ and thus we arrive at the well-known result $\lim_{n \rightarrow \infty} L_n f = f$, the convergence being uniform on each compact subinterval of $(0, \infty)$.

Returning at Th. 1, we are able to present a property of elementary operators $U_{m,n}$ and $V_{m,n}$; it will be shown that $(U_{m,n})_{n \geq 1}, (V_{m,n})_{n \geq 1}$ define a pointwise approximation process on $C[0, \infty)$.

Theorem 3. If $f \in C[0, \infty)$ and $U_{m,n}, V_{m,n}$ are defined by (5) respectively (6), then we have:

- (i) $\lim_{n \rightarrow \infty} (U_{m,n}f)(x) = \frac{x}{(1+x)^{m+1}} f(x),$
- (ii) $\lim_{n \rightarrow \infty} (V_{m,n}f)(x) = \frac{x^m}{(1+x)^{m+1}} f(x),$
for $x \geq 0$ and any natural number $m \geq 1$.

Proof. By using (5) and making the changes $n - m - 1 := p$ and $k - 1 := i$ we can write:

$$\begin{aligned} &\lim_{n \rightarrow \infty} (U_{m,n}f)(x) = \\ &= \lim_{p \rightarrow \infty} \frac{x}{(1+x)^{m+1}} \sum_{i=0}^p \binom{p}{i} \frac{x^i}{(1+x)^p} f\left(\frac{i+1}{m+p-i+1}\right) = \\ &= \frac{x}{(1+x)^{m+1}} \lim_{p \rightarrow \infty} (L_p^*f)(x), \end{aligned}$$

where L_p^* is obtained from Bleimann, Butzer, Hahn operator substituting the knots $i/(p - i + 1)$ with $(i + 1)/(m + p - i + 1)$. Following [2] we can prove that $\lim_{p \rightarrow \infty} (L_p^*f)(x) = f(x)$ at each point $x \in [0, \infty]$. This implies the aimed conclusion. We shall omit the proof of the second identity because it follows a similar technique.

Putting $\lim_{n \rightarrow \infty} (U_{m,n}f)(x) = (U_m f)(x)$ and $\lim_{n \rightarrow \infty} (V_{m,n}f)(x) = (V_m f)(x)$, as a consequence of Ths. 1 and 3, results the natural relation

$$\sum_{m=1}^{\infty} ((U_m f)(x) + (V_m f)(x)) = f(x). \quad \diamond$$

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