Mathematica Pannonica
9/2 (1998), 235–242

INTEGRAL OPERATORS WHICH PRESERVE THE SUBORDINATION

Margit **Pap**

Faculty of Mathematics and Computer Science, Babeş–Bolyai University, 3400 Cluj-Napoca, str. Kogălniceanu 1, Romania

Received: February 1998

MSC 1991: 30 C 45

Keywords: Subordination, integral operator, subordination chain.

Abstract: In this paper we find several conditions which imply that the integral operator given by formula (3) preserves the subordination. Our Theorem generalizes previous results of papers [4] and [6].

Let $H = H(U)$ denote the class of analytic functions in the unit disc U. For a positive integer n let $A_n = \{f \in H;\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots\}$. Let f and g be analytic in U. We say that f is *subordinate* to g, written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let $D = \{\varphi|\ \varphi$ analytic in $U, \varphi(z) \neq 0,\ z \in U,\ \varphi(0) = 1\}$, and let $B = \{h \in H(U):\ h(0) = 0,\ h'(0) = 1,\ h(z) \neq 0$ when $z \in U \setminus \{0\}$ and $h'(z) \neq 0,\ z \in U\}$.

In [4] the author determines conditions under which

$$f \prec g \quad \text{implies } A_h(f) \prec A_h(g), \tag{1}$$

where $h \in B$, $\tilde{K} \subset H(U)$, $A_h : \tilde{K} \to H(U)$, $A_h(f) = F$ and

$$F(z) = \left[\beta \int_0^z f^\beta(t) h^{-1}(t) h'(t) dt\right]^{1/\beta}. \tag{2}$$

In this paper we consider the integral operator $I : A_n \to A_n$, $I(f) = F$, where

$$F(z)=\left[\frac{\beta}{\phi(z)}\int_0^z f^\beta(t)\varphi(t)t^{-1}dt\right]^{1/\beta} \tag{3}$$

with Re $\beta>0$, $\varphi,\phi\in D$. We determine conditions such that an implication similar to (1) holds for this operator.

1. Preliminaries

Let α be a real number satisfying $|\alpha|<\pi/2$. A function $f\in A_n$ is α-spirallike if

$$\text{Re }\ [e^{i\alpha}zf'(z)/f(z)]>0.$$

It is well-known that if f is α-spirallike, then it is univalent. If $\varphi\in$ $\in D$ let K_φ denote the class of all functions from A_n which satisfy the following condition

$$\beta\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}\prec Q_{\beta,n}(z), \tag{4}$$

where $Q_{\beta,n}(z)$ are the well-known "open door" functions which map the unit disc on the complex plane slit along the half lines Re $w=0$, $|\text{Im }\ w|>C_n$, where

$$C_n(\beta)=\frac{n}{\text{Re }\ \beta}\left[|\beta|\sqrt{1+2\text{Re }\ \beta/n}+\text{Im }\ \beta\right].$$

Lemma 1. *Let $\phi,\varphi\in D$. Let β be a complex number with* Re $\beta>0$ *and $f\in A_n$. Suppose that*

$$\beta\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}\prec Q_{\beta,n}(z).$$

If $F=I(f)$ is defined by formula (3), then $F\in A_n$, $F(z)/z\neq 0$ for $z\in U$.

A more general form of this Lemma can be find in [5].

Lemma 2. *Let $\Psi_n\{\beta\}$ denote the set of functions $\psi:\mathbb{C}^2\to\mathbb{C}$ with the following property:* Re $\psi(\rho i,\sigma)\leq 0$ *when $\rho,\sigma\in I\!R$, $\sigma\leq-\frac{n}{2}\frac{|\beta-i\rho|^2}{\text{Re }\ \beta}$, $n\geq 1$. Let $p(z)=\beta+a_{n+1}z^{n+1}+\dots$ be an analytic function in U. If $\psi\in\Psi_n[\beta]$, then* Re $[\psi(p(z),zp'(z))]>0$ *implies* Re $p(z)>0$.

A more general form of this Lemma can be found in [1] and [2].

A function $L(z,t)$, $z\in U$, $t\geq 0$ is a *subordination chain* if $L(\cdot,t)$ is analytic and univalent in U for all $t\geq 0$ and $L(z,t_1)\prec L(z,t_2)$, when $0\leq t_1<t_2$.

Lemma 3 [3, p.159]. *The function* $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, *with* $a_1(t) \neq 0$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\mathrm{Re}\ \left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \quad z \in U,\ t \geq 0.$$

2. Main results

Theorem. *Let* $\varphi, \phi \in D$, *let* β *be a complex number so that* Re $\beta > 0$ *and let* $f \in K_\varphi$, *where*

$$K_\varphi = \left\{ f \in A_n |\ \beta\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} \prec Q_\beta(z) \right\}.$$

If $g \in A_n$ *satisfies*

$$\mathrm{Re}\ \left(\beta\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)}\right) > 0, \tag{5}$$

then

$$(\varphi(z))^{1/\beta} \prec (\varphi(z))^{1/\beta} g(z) \tag{6}$$

implies that

$$(\phi(z))^{1/\beta} I(f)(z) \prec (\phi(z))^{1/\beta} I(g)(z),$$

where $I(f) = F$ *and* F *is given by formula* (3).

Proof. We denote $g_1(z) = (\varphi(z))^{1/\beta} g(z)$ and observe that $g_1(0) = 0$, $g_1'(0) = (\varphi(0))^{1/\beta} g'(0) = (\varphi(0))^{1/\beta} \neq 0$ and

$$\mathrm{Re}\ \left(\beta\frac{zg_1'(z)}{g_1(z)}\right) = \mathrm{Re}\ \left(\beta\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)}\right) > 0$$

from which follows that g_1 is a spirallike function, consequently it is univalent.

Let denote $G(z) = I(g)(z)$ and $F(z) = I(f)(z)$. If we take the logarithmical derivative of (2) we obtain that

$$\frac{\beta zG'(z)}{G(z)} + \frac{z\phi'(z)}{\phi(z)} + \frac{z\left[\beta\frac{zG'(z)}{G(z)} + \frac{z\phi'(z)}{\phi(z)}\right]'}{\beta\frac{zG'(z)}{G(z)} + \frac{z\phi'(z)}{\phi(z)}} = \beta\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)}. \tag{7}$$

Let denote

$$P(z) = \beta\frac{zG'(z)}{G(z)} + \frac{z\phi'(z)}{\phi(z)}, \quad P(0) = \beta.$$

From (5) and the conditions (3) it follows that

$$\text{Re}\ \left(P(z)+\frac{zP'(z)}{P(z)}\right)>0,$$

from which, according to Lemma 1 we obtain that Re $P(z)>0$. If we set $G_1(z)=(\phi(z))^{1/\beta}G(z)$, then $G_1(0)=0$, $G_1'(0)\neq 0$ and

$$\text{Re}\ \beta\frac{zG_1'(z)}{G_1(z)}=\text{Re}\ \left(\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}\right)>0,$$

from which we conclude that G_1 is a spirallike function, consequently it is univalent. From (3) we obtain that

$$g_1(z)=g(z)\varphi^{1/\beta}(z)=\frac{G(z)\phi^{1/\beta}(z)}{\beta^{1/\beta}}\left[\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}\right]^{1/\beta}.$$

Let

$$L(z,t)=(1+t)^{1/\beta}\frac{G(z)\phi^{1/\beta}(z)}{(\beta+\gamma)^{1/\beta}}\left[\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}\right]^{1/\beta}.$$

By a simple computation we obtain that

$$z\frac{\partial L/\partial z}{\partial L/\partial t}=(1+t)\left[\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}+\frac{z\left[\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}\right]'}{\beta\frac{zG'(z)}{G(z)}+\frac{z\phi'(z)}{\phi(z)}}\right]=$$

$$=(1+t)\left(\beta\frac{zg'(z)}{g(z)}+\frac{z\varphi'(z)}{\varphi(z)}\right)$$

and according to condition (3), it follows that Re $z\dfrac{\partial L/\partial z}{\partial L/\partial t}>0$. Taking into account that

$$L(z,t)=a_1(t)z+\dots,\quad G'(0)\neq 0$$

it follows that

$$\lim_{t\to\infty}|a_1(t)|=\lim_{t\to\infty}\left|\frac{\partial L(0,t)}{\partial t}\right|=\lim_{t\to\infty}(1+t)^{1/\beta}|G'(0)|=\infty.$$

According to Lemma 3 we obtain that $L(z,t)$ is a subordination chain i.e. $L(z,s)\prec L(z,t)$ when $0\le s<t$, $z\in U$. Let denote $F_1(z)=$ $=\phi^{1/\beta}(z)F(z)$. We can assume that G_1 is regular and univalent on the closed disc $\overline{U}$. If not, then we can replace $F(z)$ by $F_{1r}(z)=F_1(rz)$ and $G_1(z)$ by $G_{1r}(z)=G_1(rz)$ where $0<r<1$, and $G_{1r}(z)$ is regular and univalent on $\overline{U}$. We would then prove $F_{1r}(z)\prec G_{1r}(z)$ for all $0<r<1$ and by letting $r\to 1^-$ we have $F(z)\prec G(z)$. Suppose that F_1 is not subordinate to G_1; then there exist $z_0\in U$ and $\zeta_0\in\partial U$ such that

$$F_1(z_0) = G_1(\zeta_0), \quad F_1(|z| < |z_0|) \subset G_1(U), \tag{8}$$
$$z_0 F_1'(z_0) = (1+t)\zeta_0 G_1'(\zeta_0),$$

for more details see paper [2]. Direct computations show that

$$\beta \frac{zF_1'(z)}{F_1(z)} = \frac{\beta z F'(z)\phi^{1/\beta}(z) + z\phi'(z)\phi^{1/\beta - 1}(z)F(z)}{\phi^{1/\beta}(z)F(z)} = \tag{9}$$
$$= \beta \frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)}.$$

and

$$\beta \frac{zG_1'(z)}{G_1(z)} = \beta \frac{zG'(z)}{G(z)} + \frac{z\phi'(z)}{\phi(z)}. \tag{10}$$

If we set $z = z_0$ in (9) and $z = \zeta_0$ in (10), then we deduce from (8) that

$$\beta \frac{z_0 F'(z_0)}{F(z_0)} + \frac{z_0\phi'(z_0)}{\phi(z_0)} = (1+t)\left[\beta \frac{\zeta_0 G'(\zeta_0)}{G(\zeta_0)} + \frac{\zeta_0 \phi'(\zeta_0)}{\phi(\zeta_0)}\right], \quad t \geq 0.$$

Because

$$g_1(z) = L(z,0) \prec L(z,t) \quad \text{for} \quad t \geq 0$$

we obtain that

$$f_1(z_0) = (\varphi(z_0))^{1/\beta} f(z_0) = \frac{F(z_0)(\phi(z_0))^{1/\beta}}{\beta^{1/\beta}} \left[\beta \frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)}\right]^{1/\beta} =$$

$$= \frac{G(\zeta_0)(\phi(\zeta_0))^{1/\beta}}{\beta^{1/\beta}} \left[(1+t)\left(\beta \frac{\zeta_0 G'(\zeta_0)}{G(\zeta_0)} + \frac{\zeta_0\phi'(\zeta_0)}{\phi(\zeta_0)}\right)\right]^{1/\beta} = L(\zeta_0, t) \notin g_1(U)$$

which contradicts the assumption $f_1(z) \prec g_1(z)$, hence $F_1(z) \prec G_1(z)$. ◊

For $\varphi = \phi = 1$ we obtain the following:

Corollary 1. *If β is a complex number with* Re $\beta > 0$,

$$f \in K_1 = \left\{ f |\ f \in A_n,\ \beta \frac{zf'(z)}{f(z)} \prec Q_\beta(z) \right\}$$

and $g \in A_n$ with Re $\left(\frac{\beta z g'(z)}{g(z)}\right) > 0$ *(i.e. g is spirallike), then $f(z) \prec$ $\prec g(z)$ implies that*

$$\left(\beta \int_0^z f^\beta(t) t^{-1} dt\right)^{1/\beta} \prec \left(\beta \int_0^z g^\beta(t) t^{-1} dt\right)^{1/\beta}.$$

The implication of Cor. 1 was proved in [6], Th. 1 under more restrictive conditions for β and f, namely: $\beta > 0$ and Re $\left(\frac{zf'(z)}{f(z)}\right) >$ > 0.

Example 1. Let $g(z) = z/(1+\lambda z)$, where $|\lambda| = 1$. In this case $\frac{zg'(z)}{g(z)} = \frac{1}{1+\lambda z}$, consequently Re $\beta\frac{zg'(z)}{g(z)} > 0$, for all $\beta > 0$. That in this case in account to Cor. 1 if $f \in K_1$ and $f(z) \prec \frac{z}{1+\lambda z}$, $\beta > 0$ then

$$\left(\beta\int_0^1 f^\beta(t)t^{-1}dt\right)^{1/\beta} \prec \left(\beta\int_0^1 \frac{t^{\beta-1}}{(1+\lambda t)^\beta}dz\right)^{1/\beta}.$$

If we take $\beta = 1$, then this subordination reduces to the following result

$$\int_0^z f(t)t^{-1}dt \prec \frac{1}{\lambda}\ln(1+\lambda z).$$

For $\lambda = 1$ we have $f(z) \prec \frac{z}{1+z}$ (i.e. Re $f(z) < \frac{1}{2}$) and $f \in K_1$, so that

$$\int_0^z f(t)t^{-1}dt \prec \ln(1+z) = w(z).$$

Because $w(U)$ is the strip in the complex plane given by $|\text{Im } w| < \frac{\pi}{2}$, this result is equivalent with the following: if $f \in K_1$ and Re $f(z) < \frac{1}{2}$ then

$$\left|\text{Im } \int_0^z f(t)t^{-1}dt\right| < \frac{\pi}{2}.$$

Example 2. Let $g(z) = Mz$, where $M \in \mathbb{C}^*$, then for $\beta > 0$

$$\text{Re } \beta\frac{zg'(z)}{g(z)} = \beta > 0.$$

In this case according to Cor. 1 if $f \in K_1$ and $f(z) \prec Mz$ (i.e. $|f(z)| < |M|$) then

$$\left(\beta\int_0^z f^\beta(t)t^{-1}dt\right)^{1/\beta} \prec \left(\beta\int_0^z (Mt)^\beta t^{-1}dt\right)^{1/\beta} = Mz$$

i.e.

$$\left|\left(\int_0^z f^\beta(t)t^{-1}dt\right)^{1/\beta}\right| < \frac{|M|}{\beta^{1/\beta}}.$$

Let h be a function from B, then

$$f(z) = \frac{zh'(z)}{h(z)} \in D.$$

If in formula (3) we let $\phi = 1$ and $\varphi(z) = \frac{zh'(z)}{h(z)}$, then

$$A_h(f)(z) = F(z) = \left(\beta \int_0^z f^\beta(t) \frac{h'(t)}{h(t)} dt\right)^{1/\beta},$$

and on account of Th.1 we obtain the following corollary.

Corollary 2. *Let β be a complex number with* Re $\beta > 0$. *Let h be a function from B and let f be a function from A_n so that*

$$\beta \frac{zf'(z)}{f(z)} + \frac{zh''(z)}{h'(z)} + 1 - \frac{zh'(z)}{h(z)} \prec Q_\beta(z), \quad z \in U$$

and let g be a function from A_n so that

$$\text{Re} \left(\beta \frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + 1 - \frac{zh'(z)}{h(z)}\right) > 0, \quad z \in U.$$

Then

$$(11) \qquad \left(\frac{zh'(z)}{h(z)}\right)^{1/\beta} f(z) \prec \left(\frac{zh'(z)}{h(z)}\right)^{1/\beta} g(z), \quad z \in U$$

implies that $A_h(f)(z) \prec A_h(g)(z)$.

For β real, $\beta > 0$ the author of paper [4] established other conditions under which conclusion (11) is also true.

3. Particular cases

1) If we take $h(z) = ze^{\lambda z}$, $|\lambda| \le 1$, then

$$\frac{zh''(z)}{h'(z)} + 1 - \frac{zh'(z)}{h(z)} = \frac{\lambda z}{1 + \lambda z}.$$

Example 3. Let β complex number with Re $\beta > 0$. If f and g are functions from A_n so that

$$\beta \frac{zf'(z)}{f(z)} + \frac{\lambda z}{1 + \lambda z} \prec Q_\beta(z), \quad z \in U$$

and

$$\text{Re} \left(\beta \frac{zg'(z)}{g(z)} + \frac{\lambda z}{1 + \lambda z}\right) > 0, \quad z \in U.$$

Then

$$(1 + \lambda z)^{1/\beta} f(z) \prec (1 + \lambda z)^{1/\beta} g(t)$$

implies that

$$\left[\beta \int_0^z f^\beta(t) \frac{1 + \lambda t}{t} dt\right]^{1/\beta} \prec \left[\beta \int_0^z g^\beta(t) \frac{1 + \lambda t}{t} dt\right]^{1/\beta}.$$

2) If we consider

$$h(z) = z \exp \int_0^z \frac{e^{\lambda z} - 1}{t} dt, \quad \lambda \in \mathbb{C},$$

then $h(0) = 0$ and $(h(z) \neq 0$ when $0 < |z| < 1$ and $h'(z) = e^{\lambda z} \frac{h(z)}{z}$ which implies that $h'(z) \neq 0$ in U, $h'(0) = 1$, consequently $h \in B$. A simple computation yields that $\frac{zh'(z)}{h(z)} = e^{\lambda z}$ and

$$1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)} = \lambda z.$$

Example 4. Let $\beta \in \mathbb{C}$, Re $\beta > 0$, $\lambda \in \mathbb{C}$, and let f and g two functions from A_n so that

$$\beta \frac{zf'(z)}{f(z)} + \lambda z \prec Q_\beta(z), \quad z \in U$$

and

$$\text{Re} \left(\beta \frac{zg'(z)}{g(z)} + \lambda z \right) > 0, \quad z \in U.$$

Then

$$(e^{\lambda z})^{1/\beta} f(z) \prec (e^{\lambda z})^{1/\beta} g(z)$$

implies

$$\left(\beta \int_0^z f^\beta(t) \frac{e^{\lambda t}}{t} dt \right)^{1/\beta} \prec \left(\beta \int_0^z g^\beta(t) \frac{e^{\lambda t}}{t} dt \right)^{1/\beta}.$$

References

[1] MILLER, S. S. and MOCANU, P. T.: Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65** (1978), 285–305.

[2] MILLER, S. S. and MOCANU, P. T.: Differential subordinations and univalent functions, *Michigan Math. J.* **28** (1981), 167–171.

[3] POMMERENKE, Ch.: Univalent Functions, Vanderhoeck and Ruprecht, Gottingen, 1975.

[4] BULBOACĂ, T.: Integral operators which preserve the subordination, *Mathematica (Cluj)* **36(61)** (1996).

[5] MILLER, S. S. and MOCANU, P. T.: Integral operators on certain classes of analytic functions, Univalent Functions, Fractional Calculus and their Applications, Halstead Press, John Willey (1989), 153–166.

[6] MILLER, S. S., MOCANU, P. T. and READE, M. O.: Subordination preserving integral operators, *Transactions of the American Mathematical Society* **283**/2 (1984), 605–615.