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SELF-PSEUDOPROJECTIVE COMPLETELY DECOMPOSABLE ABELIAN GROUPS

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Abstract: The groups in the title are characterized as the completely decomposable groups C having a rank one summand X of idempotent type such that C and X are divisible by the same primes.

For every abelian group C, the class Gen (C) of homomorphic images of direct sums of copies of C is closed under direct sums and homomorphic images and thus (in general) lacks only closure under extensions in order to be a radical class. The function which assigns to each group the sum of all its subgroups from Gen (C) (which is also in Gen (C)) is an idempotent subfunctor of the identity (*socle)* which is a radical precisely when Gen (C) is closed under extensions. When X is a torsion-free group of rank one, Gen (X) is closed under extensions if and only if X has idempotent type. This follows from [4] and is explicitly stated in [2] where further information is obtained concerning the general question: When is a socle a radical?

It turns out that in general, Gen (C) is closed under extensions if and only if C is *self-pseudoprojective* in the sense that for every short exact sequence

$$0 \to K \to N \xrightarrow{g} L \to 0$$

with $K \in \mathrm{Gen}(C)$ and evqery $f : C \to L$ there is an endomorphism h of C and a homomorphism $k : C \to N$ such that $gk = fh \neq 0$. For this concept and other related generalizations of projectivity (in modules and abelian categories), see the paper of Wakamatsu [11], the thesis of Berning [1] or the survey of Wisbauer [12].

In this note we show that for a completely decomposable torsion-free group C, $\mathrm{Gen}(C)$ is closed under extensions if and only if C has a rank-one summand X of idempotent type such that C and X are divisible by the same primes (so that $\mathrm{Gen}(C) = \mathrm{Gen}(X)$ and the rank one groups of idempotent type are essentially the only completely decomposable self-pseudoprojectives).

Throughout, group always means abelian group; we mostly use the notation and conventions of [3], but for an enumeration $p_1, p_2, \dots$ of the primes, a sequence $(h_1, h_2, \dots)$ of non-negative integers or ∞ symbols will be called a *height-sequence* rather than a *characteristic.* If S is a set of primes, a group D will be called S-divisible if $pD = D$ for all $p \in S$. The group of rationals whose denominators have their prime factors in S will be called $\mathbb{Q}(S)$. We recall a few more items of notation. The type of a group element x or a rational group X is denoted by $t(x)$ or $t(X)$; for a torsion-free group G and type τ, $G(\tau)$ is the subgroup $\{x \in G : t(x) \geq \tau\}$; $\langle x \rangle$ is the cyclic subgroup generated by x, $\langle x \rangle_*$ the smallest pure subgroup containing x.

Theorem 1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a set of torsion-free groups of rank one. Then $\bigoplus_{\lambda \in \Lambda} X_\lambda$ is self-pseudoprojective if and only if there exists a set S of primes such that each X_λ is S-divisible and some $X_\lambda \cong \mathbb{Q}(S)$ (or, equivalently, if the set of types of the X_λ has a smallest element and this is idempotent).*

We shall prove this result in several stages.

Lemma 1. *Let X, Y be torsion-free groups of rank one with $t(X) \leq$ $\leq t(Y)$. Then $Y \in \mathrm{Gen}(X)$.*

Proof. If $y \in Y \setminus \{0\}$ has height sequence $(h_1, h_2, \dots)$ then there is a height sequence $(\ell_1, \ell_2, \dots)$ of type $t(X)$ with $\ell_i \leq h_i$ for all i. But then we have

$$y \in \langle p_i^{-\ell_i} y : i = 1, 2, 3, \dots \rangle \subseteq Y$$

where the indicated subgroup has type $t(X)$. ◊

Corollary 1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a set of torsion-free groups of rank one, Γ a subset of Λ such that $\{t(X_\gamma) : \gamma \in \Gamma\}$ is a cofinal subset of*

$\{t(X_\lambda) : \lambda \in \Lambda\}$. *Then*

$$\mathrm{Gen}\left(\bigoplus_{\gamma\in\Gamma} X_\gamma\right) = \mathrm{Gen}\left(\bigoplus_{\lambda\in\Lambda} X_\lambda\right).$$

It follows from this that no loss of generality ensues if we assume the following in the sequel.

(*) If τ is a type such that $t(X_\lambda) \leq \tau$ for some $\lambda \in \Lambda$, then there exists $\mu \in \Lambda$ for which $t(X_\mu) = \tau$.

Thus until further notice, $\{X_\lambda : \lambda \in \Lambda\}$ is a set of torsion-free groups of rank one satisfying (*).

The following notation will be useful. If τ is a type represented by a height sequence $(h_1, h_2, \dots)$ we let τ_0 be the type of $(\ell_1, \ell_2, \dots)$ where $\ell_i = h_i$ if $h_i = \infty$, and $\ell_i = 0$ otherwise. (Note that $\tau_0 = \tau : \tau$ in the sense of [3] Vol.I, p.111.)

Lemma 2. *If* $\bigoplus_{\lambda\in\Lambda} X_\lambda$ *is self-pseudoprojective and* $t(X_\lambda) = \tau$ *for some* λ, *then* $t(X_\mu) = \tau_0$ *for some* μ.

Proof. We make use of a group of rank two similar to that constructed in [4] (cf. Mutzbauer [10] for other similar groups). Let $(h_1, h_2, \dots)$ be a height sequence of type τ and let $M = \{p_n : h_n = \infty\}$. Let $\{k_1, k_2, \dots\} = \{h_i : 0 < h_i < \infty\}$ (a set we may clearly assume to be infinite) and let $\{q_1, q_2, \dots\}$ be the corresponding set of primes. Let $\{x, y\}$ be a basis for a two-dimensional $\mathbb{Q}$-vector space and let

$$G = \langle p^{-\infty}x,\, p^{-\infty}y,\, q_i^{-k_i}x,\, q_i^{-k_i}\left(q_i^{-k_i}x + y\right) : p \in M,\ i = 1, 2, \dots\rangle.$$

A routine argument, using the linear independence of x and y, shows that $t(x) = \tau$ and $t(y) = \tau_0$. Since $G(\tau)$ can't have rank two, we have $G(\tau) = \langle x\rangle_*$. Also

$$G/\langle x\rangle_* = \langle p^{-\infty}\,\overline{y},\, q_i^{-k_i}\,\overline{y} : p \in M,\ i = 1, 2, \dots\rangle,$$

where $\overline{y} = y + \langle x\rangle_*$, and this has rank one and type τ. If $a \in G\backslash\langle x\rangle_*$, then $t(a) \leq t(a + \langle x\rangle_*) = \tau$, while as both $\langle x\rangle_*$ and $G/\langle x\rangle_*$ are M-divisible, so is G, whence $t(a) \geq \tau_0$. It is not possible to have $\tau_0 < t(a) < \tau$, as G has rank two. Thus we conclude that $t(a) = \tau_0$ for all $a \in G\backslash\langle x\rangle_*$.

As G is an extension of X_λ by X_λ, G is in $\mathrm{Gen}\left(\bigoplus_{\lambda\in\Lambda} X_\lambda\right)$. In particular, there is a homomorphism from a direct sum of copies of

$\underset{\lambda\in\Lambda}{\oplus} X_\lambda$ to G whose image is not contained in $G\backslash\langle x\rangle_*$, so some X_ρ is mapped into a rank-one subgroups of type τ_0. But then $t(X_\rho) \le \tau_0$ and so $t(X_\mu) = \tau_0$ for some $\mu \in \Lambda$. ◊

Lemma 3. *If $\underset{\lambda\in\Lambda}{\oplus} X_\lambda$ is self-pseudoprojective and there are incomparable non-empty sets A, B of primes and there exist $\lambda, \mu \in \Lambda$ with $X_\lambda \cong \cong \mathbb{Q}(A)$ and $X_\mu \cong \mathbb{Q}(B)$, then there exists $\rho \in \Lambda$ with $X_\rho \cong \mathbb{Q}(A \cap B)$.*

Proof. Let $\mathbb{Q}(A)^\omega$ (resp. $\mathbb{Q}(A)^{(\omega)}$) denote the direct product (resp. direct sum) of a countably infinite set of copies of $\mathbb{Q}(A)$. In $\mathbb{Q}(A)^\omega / /\mathbb{Q}(A)^{(\omega)}$ the pure subgroup generated by $(1!, 2!, 3!, \ldots) + \mathbb{Q}(A)^{(\omega)}$ is isomorphic to $\mathbb{Q}$ so $\mathbb{Q}(A)^\omega / \mathbb{Q}(A)^{(\omega)}$ has a subgroup $H/\mathbb{Q}(A)^{(\omega)} \cong \cong \mathbb{Q}(B)$. If $x \in H\backslash\mathbb{Q}(A)^{(\omega)}$ then (in H) $t(x) \le t\left(x + \mathbb{Q}(A)^{(\omega)}\right) = = t(\mathbb{Q}(B))$ and $t(x) \le t(\mathbb{Q}(A))$ = the type of x in $\mathbb{Q}(A)^\omega$. Thus $t(x) \le t(\mathbb{Q}(A)) \wedge t(\mathbb{Q}(B)) = t(\mathbb{Q}(A \cap B))$. But both $\mathbb{Q}(A)^{(\omega)}$ and $\mathbb{Q}(B)$ are $A \cap B$-divisible, so H is too. Hence $t(x) \ge t(\mathbb{Q}(A \cap B))$, so $t(x) = t(\mathbb{Q}(A \cap B))$. The rest of the proof is like that of Lemma 2. ◊

Proof of Theorem. Let $\underset{\lambda\in\Lambda}{\oplus} X_\lambda$ be self-pseudoprojective and left $\mathcal{F}$ be the class of torsion-free groups in Gen $\left(\underset{\lambda\in\Lambda}{\oplus} X_\lambda\right)$. Then $\mathcal{F}$ is closed under extensions. Let $T = \{t(x) : x \in F \in \mathcal{F}\}$. We continue to assume $(*)$.

If $\lambda, \mu \in \Lambda$, $t(X_\lambda) = \tau$ and $t(X_\mu) = \sigma$ then there exist $\alpha, \beta \in \Lambda$ such that $t(X_\alpha) = \tau_0$ and $t(X_\beta) = \sigma_0$, by Lemma 2. But then by Lemma 3, $t(X_\gamma) = \tau_0 \wedge \sigma_0 \le \tau \wedge \sigma$ for some $\gamma \in \Lambda$, so by $(*)$ there exists $\delta \in \Lambda$ such that $t(X_\delta) = \tau \wedge \sigma$. Thus $\{t(X_\lambda) : \lambda \in \Lambda\}$ is a filter in the lattice of types.

If $x \in F \in \mathcal{F}$, then x is in a homomorphic image of a finite direct sum $X_{\lambda_1} \oplus X_{\lambda_2} \oplus \ldots \oplus X_{\lambda_n}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$. Hence $t(x) \ge \ge t(X_1) \wedge t(X_{\lambda_2}) \wedge \ldots \wedge t(X_{\lambda_n})$, so $t(x) = t(X_\mu)$ for some $\mu \in \Lambda$. It follows that $T = \{t(X_\lambda) : \lambda \in \Lambda\}$ and that

$$\begin{aligned}\mathcal{F} &= \{F : x \in F \Rightarrow t(x) \in T\} = \\ &= \{F : x \in F \Rightarrow t(x) = t(X_\lambda) \text{ for some } \lambda \in \Lambda\}.\end{aligned}$$

Now Th. 2.12 of [5] asserts among other things that if Φ is a filter in the lattice of types and the class of torsion-free groups all of whose elements have types in Φ is closed under extensions, then Φ is the principal filter generated by an idempotent type. Using this result, we now see that there is a set S of primes for which

$$\{t(X_\lambda) = \lambda \in \Lambda\} = \{\tau : t(\mathbb{Q}(S)) \leq \tau\}.$$

If $(*)$ is not assumed, then for any type σ we have

$$(\exists\lambda \in \Lambda)(\sigma \geq \lambda) \Leftrightarrow \sigma \geq t(\mathbb{Q}(S)),$$

so the conclusion is the same. $\Diamond$

Now let W be a separable torsion-free group, i.e. a group such that every element is contained in a completely decomposable direct summand of finite rank, E the set of types of rank-one direct summands of W. Then we have

$$\mathrm{Gen}(W) = \mathrm{Gen}\left(\bigoplus_{\tau\in E} X_\tau\right)$$

where X_τ has rank one and type τ for all $\tau \in E$. Since for a prime p we have $pW = W$ if and only if $pX_\tau = X_\tau$ for all $\tau \in E$, the theorem has the following

Corollary 2. *A separable group W is self-pseudoprojective if and only if it has a direct summand isomorphic to $\mathbb{Q}(\{p : pW = W\})$.*

Our results provide us also with a small amount of information about self-pseudoprojectivity of a direct product $\prod_{\lambda\in\Lambda} X_\lambda$ of groups of rank one. First recall that a torsion-free group is *slender* if every homomorphism from $\mathbb{Z}^\omega$ to G takes all but finitely many copies of $\mathbb{Z}$ to 0. See [3], Vol.II pp.158-162 for properties of such groups.

Suppose $V = \prod_{\lambda\in\Lambda} X_\lambda$ is self-pseudoprojective. Then each $X_\lambda \in \mathrm{Gen}(V)$ so the corresponding group G of Lemma 2 is in $\mathrm{Gen}(V)$ also. But (except when $X_\lambda \cong \mathbb{Q}$) G is slender, so every homomorphic image of V in G is a homomorphic image of some finite direct sum $X_{\mu_1} \oplus X_{\mu_2} \oplus \ldots \oplus X_{\mu_n}$, $\mu_1, \mu_2, \ldots, \mu_n \in \Lambda$, so some $t(X_{\mu_i}) \leq \tau_0$. Let $S = \{p : pX_\lambda = X_\lambda\}$. Then $t(\mathbb{Q}(S)) = \tau_0$ and $\mathbb{Q}(S) \in \mathrm{Gen}(V)$. The group H of Lemma 3 is also slender so by an argument like that used for G, we have $\mathbb{Q}(A \cap B) \in \mathrm{Gen}(V)$ whenever $\mathbb{Q}(A), \mathbb{Q}(B) \in \mathrm{Gen}(V)$. Thus if $\lambda, \mu \in \Lambda$ then $\mathrm{Gen}(V)$ contains rank-one groups of types $t(X_\lambda)_0$, $t(X_\mu)_0$, $(t(X_\lambda) \wedge t(X_\mu))_0$ and hence for some $\rho \in \Lambda$ we have

$$t(X_\rho) \leq (t(X_\lambda) \wedge t(X_\mu))_0 \leq t(X_\lambda) \wedge t(X_\mu).$$

Proposition 1. *Let $V = \prod_{\lambda\in\Lambda} X_\lambda$ be a self-pseudoprojective direct product of torsion-free groups of rank one such that the set of types of the*

X_λ has at least one minimal member. Let $S = \{p : pV = V\}$. Then $X_\lambda \cong \mathbb{Q}(S)$ for some $\lambda \in \Lambda$, and so $\mathrm{Gen}(V) = \mathrm{Gen}(\mathbb{Q}(S))$.

Proof. Let X_α have minimal type σ. If τ is the type of some X_λ then some X_μ has type $\leq \sigma \wedge \tau \leq \sigma$ so this X_μ has type σ. But then $\sigma = t(X_\mu) \leq \sigma \wedge \tau \leq \tau$. Thus σ is the smallest type of any X_λ. As $\mathrm{Gen}(V)$ contains a group of rank one and type σ_0 it is clear that

$$\sigma = \sigma_0 = t(\mathbb{Q}(\{p : pX_\lambda = X_\lambda \forall \lambda \in \Lambda\})) = t(\mathbb{Q}(\{p : pV = V\})). \qquad \diamond$$

Note that the condition imposed on the type set in the proposition is much weaker than those required to make V separable. This is clear from [7], [9]; see [3] Vol.II, pp.170-171, even though by an example of Metelli [8], pp.219-220, the published descriptions of separable direct products are in some particulars incorrect. Note also that by a recent result of Giovannitti [6] there is no need to require the cardinality of Λ to be non-measurable.

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