Mathematica Pannonica
10/2 (1999), 193–196

A SIMPLE PROOF OF A THEOREM BY TUTTE

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Dedicated to Professor Ferenc Schipp on his 60th birthday

Received: November 1998

MSC 1991: 05 C 70

Keywords: The factorization of graphs, 1-factor of graphs, Tutte's theorem.

Abstract: In this paper we provide a simple proof of a famous Tutte's theorem concerning the 1-factors of graphs.

We provide a simple proof of a famous theorem by Tutte concerning the factors of degree 1 of graphs. The main idea of the proof is a variant of Th. 3 in Belck's paper [3]. The path followed, however, is not identical with Belck's, but essentially shorter. (Belck does not consider the case of degree 1 factors separately, but he deals with f-factors instantly. For this purpose he needs to use a more complicated machinery and more involved theorems.) The structure of our paper is related to [6]. (Lemmas 1 and 2 – with a different proof – essentially occur in Tutte's paper already.)

The graphs in this paper have n (a finite number) vertices, contain no loops, nor multiple edges. The vertices of the graph are denoted by lower case letters a, b, ... etc., the edge connecting a with b is denoted by ab. A *factor of degree* 1 of the graph G (or a 1-*factor* of G) is a subgraph of G that contains all vertices and in which every vertex is incident to exactly one edge (of G). We call a graph not containing a 1-factor a *prime graph*. A 1-factor determines a decomposition of the vertex set into disjoint pairs (each pair consisting of two vertices connected by an edge of the 1-factor). Hence every graph with an odd number of vertices is prime. We denote by $G+xy$ the graph obtained by adding the edge xy to a graph G. (Either x or y, or even both may have already been vertices of the graph G.) Given a subset of vertices A, we denote by $G-A$ the graph obtained from G by removing all vertices belonging to A, and all edges incident to any of these vertices. The number of odd connected components of A (i.e., the number connected components having an odd number of vertices) will be denoted by $p(A)$.

Theorem [8, Tutte]. *A graph is prime if and only if it contains a subset of vertices A satisfying $p(A) > |A|$.*

When n is odd, the theorem holds with $A = \emptyset$. Hence in the following we will only consider the case when n is even. Let G be a prime graph on n vertices and let x and y be vertices not connected by an edge. If for any such pair of vertices the graph $G+xy$ already contains a factor of degree 1, then we call the graph G a *hyperprime* graph. If G is prime but not hyperprime let us replace G with a graph $G+xy$ with more edges. The resulting graph is either hyperprime or we may still add an edge to obtain a prime graph. The iteration of this procedure yields a hyperprime graph, since the complete graph (in which any pair of vertices is connected by an edge) does contain a 1-factor. (For example the pairs $(1,2),(3,4),\ldots,(n-1,n)$ constitute a 1-factor.) We call a vertex x of the graph G *singular* if it is connected by edges to all other vertices of G.

Lemma 1. *If G is a hyperprime graph on an even number of vertices, and the set of its singular vertices is S, then every connected component of the graph $G-S$ is a complete graph.*

Proof. Let C be a connected component of the graph $G-S$. If C is not a complete graph, then it contains vertices a and x which are not connected by edge. Since C is connected, one may give a sequence of vertices $a,b,c,\ldots,x$ of minimum length in which the consecutive elements are connected by an edge. (The vertex c may already be equal to x.) By minimality, ac is not an edge of G. Since b is a vertex of $G-S$ and so b is not singular, there is a vertex d in G which is not connected to b. Thus the hyperprime graph G contains four vertices: a,b,c,d such that ab and bc are edges of the graph but ac and bd are not. Since G is hyperprime, the graph $G+ac$ contains a 1-factor F_1. Similarly, the graph $G+bd$ contains a 1-factor F_2. Let us color blue those edges of F_1 which do not belong to F_2 and let us color red those edges of F_2 which do not belong to F_1. (Hence ac becomes blue, bd becomes red.) The collection of all blue and red edges with their endpoints form a graph in which every vertex belongs to exactly one red and one blue edge. Every connected component of this graph is a circuit with an even number of edges which are red and blue alternately. If the edges ac and bd belong to different circuits, then the red edges of the circuit containing ac, the blue edges of the other circuits, and all common edges of F_1 and F_2 constitute a 1-factor of the graph G, in contradiction to the assumption of G being prime. Now assume that the edges ac and bd belong to the same circuit. Add to this circuit the edges ab and bc of G and omit the edge ac and the two edges that are incident to b. (One of these edges is bd.) The circuit is transformed into a path on an even number of vertices, which contains a 1-factor. Add to this partial 1-factor the blue edges of all other circuit and the common edges of F_1 and F_2. This way we obtain again a 1-factor of G, in contradiction to the assumption of G being prime. ◊

Lemma 2. *If G is a hyperprime graph on an even number of vertices, and S is the set of its singular vertices then $p(S) > |S|$.*

Proof. Assume that $p(S) \leq |S|$. Select a vertex from each odd component of $G - S$. Denote the selected vertices by $c_1, \dots, c_k$. (Here $k = p(S) \leq |S|$.) Let us also select k vertices from S, denoted by $d_1, \dots, d_k$. (If S is empty then the same holds for $\{c_1, \dots, c_k\}$ and for $\{d_1, \dots, d_k\}$.) Since every connected component of the graph $G - S - \{c_1, \dots, c_k\}$ is a complete graph on an even number of vertices, this graph has a 1-factor. Since the vertex set $S - \{d_1, \dots, d_k\}$ induces a complete subgraph of G on an even number of vertices, this graph also contains a 1-factor. These partial 1-factors, together with the edges $c_1d_1, \dots, c_kd_k$ constitute a 1-factor of G. This is not possible since G is prime. ◊

Proof of Theorem. If G is a prime graph on an even number of vertices, then by adjoining edges it may be completed to a hyperprime graph G'. (If G itself is hyperprime then let $G' = G$.) Let S be the set of singular vertices of this graph, and k the number of odd components of $G' - S$. When we remove now those edges of G' which do not belong to G, some components of $G' - S$ may fall apart into smaller components, but every odd components yields at least one odd component in $G - S$. Hence, by $k > |S|$ and by Lemma 2, the inequality $p(S) \geq k > |S|$ holds for G. Conversely, let us assume that G is a graph on an even number of vertices in which there is a subset of vertices A satisfying $p(A) > |A|$. We show that G is a prime graph. If G had a 1-factor and hence a decomposition of its vertices into disjoint pairs, then in every odd component C_i of $G - A$ $(i = 1, 2, \dots, p(A))$ there would be at least one vertex c_i that is matched up with a vertex from outside C_i, i.e., from A. The set A, however does not have enough elements for this purpose, since $p(A) > |A|$. We have reached a contradiction proving the theorem. ◊

At the light of Tutte's theorem for a hyperprime graph G on n (even) vertices and the set S of its singular vertices we have the following:

1. *Every connected component of* $G - S$ *is a complete graph on an odd number of vertices.* In fact, if we connect a vertex x of a connected component on an even number of vertices with a vertex y of a connected component on an odd number of vertices, the number of connected components having an odd number of vertices does not change. Hence $G+xy$ is prime, in contradiction to the assumption of G being hyperprime.◊

2. $p(S) = |S| + 2$. In fact, every new edge that is realizable in G connects two connected components of $G-S$. Adding a new edge replaces two odd connected components with one even connected component, that is, $p(S)$ decreases by two: $p(S) > |S| \geq p(S) - 2$. Here $p(S) = |S| + 1$ is not possible since the number of vertices of G is $n = |S| +$ the number of vertices in the components of $G - S$, that is, we have $n \equiv |S| + p(S) \pmod 2$. Hence in the case of $p(S) = |S| + 1$ we obtain $n \equiv 1 \pmod 2$, in contradiction to $n \equiv 0 \pmod 2$.

Remark. Tutte [6] proves his theorem using the theory of determinants. Gallai [2] arrives to Tutte's theorem with the help of critical graphs. (A graph will be called *critical* if removing any vertex of it the remaining graph has a 1-factor). Lovász [4] gives the following characterization of critical graphs: Every critical graph and only these graphs can be constructed from the one- point graph by the iteration of the following construction: we connect two (not necessarily different) vertices of an (already constructed) critical graph by a suspending arc of odd length. In our proof the role of the critical (subgraphs) of Gallai's paper are taken over by complete graphs on an odd number of vertices, that is, by critical graphs in which any pair of vertices is connected by an edge.

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