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ON PIECEWISE CONFLUENT MAPPINGS

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Abstract: In the Houston Problem Book W. T. Ingram posed a problem connecting the confluence of two special partial mappings to the confluence of the original one. We solve this problem and answer one closely related question.

Introduction

All spaces considered in the paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected

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space. Given a space X and its subset S, we denote by $\operatorname{cl} S$ the closure of S in X.

A mapping $f : X \to Y$ is said to be *confluent* provided that for each subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped under f onto Q.

We pose the following general problem.

Problem 1. *Let continua X and Y be given such that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed subspaces of Y, and let a mapping $f : X \to Y$ be such that the two partial mappings $f|f^{-1}(Y_1)$ and $f|f^{-1}(Y_2)$ are confluent. Under what conditions concerning the structure of Y_1, Y_2 and $Y_1 \cap Y_2$ the mapping f is confluent?*

One can also pose a similarly formulated problem, where a decomposition of the domain is considered in place of the decomposition of the range space.

Problem 2. *Let continua X and Y be given such that $X = X_1 \cup X_2$, where X_1 and X_2 are closed subspaces (or subcontinua) of X, and let a mapping $f : X \to Y$ be such that the two partial mappings $f|X_1$ and $f|X_2$ are confluent. Under what conditions concerning the structure of X_1, X_2 and $X_1 \cap X_2$ the mapping f is confluent?*

To see a (rather big) difference in approaching to the union results for confluent mappings between considering decompositions of the domain and of the range, let us discuss the following simple example.

Example 3. In the plane put

$$a = (0, 1), \quad b = (0, -1), \quad c = (0, 0), \quad d = (-1, 0). \tag{3.1}$$

Thus the straight line segments $X_1 = ab$ and $X_2 = ad$ have a as the only common point. Define $X = X_1 \cup X_2$. Let a mapping $f : X \to Y = X_1$ be defined as the identity $f|X_1$ on X_1 and as a natural projection $f|X_2 : X_2 \to ac$ (that assigns to any point $(x, y) \in X_2$ the point $(0, y) \in X_1$ having the same second coordinate). Then the partial mappings $f|X_1$ and $f|X_2$ are both homeomorphisms (thus confluent), while f is not confluent. Observe however, that if we define $Y_i = f(X_i)$ for $i \in \{1, 2\}$, then every X_i is a proper subset of $f^{-1}(Y_i)$, and the restriction of f, being confluent on X_1, is not confluent on $f^{-1}(Y_1)$.

This example indicates that, when considering assumptions on confluence of partial mappings, we should accept rather approach as in Problem 1 than that in Problem 2.

The research related to these problems has various partial results which are dispersed in the literature. A large information on confluent

and related mappings is contained in T. Maćkowiak's paper [9]. We recall here an interesting result obtained by A. Lelek in [7, p. 58–59].
Theorem 4 (A. Lelek). *Suppose X and Y are compact metric spaces, $f : X \to Y$ is a continuous mapping of X onto Y, and*

$$Y = Y_0 \cup Y_1 \cup Y_2 \cup \cdots$$

is a decomposition of Y into closed subsets Y_i such that the following three conditions are satisfied:
(4.1) *$f|f^{-1}(Y_i)$ is a confluent mapping of $f^{-1}(Y_i)$ onto Y_i for $i \in \in \{0, 1, 2, \ldots\}$;*
(4.2) *$Y_i \cap Y_j \subset Y_0$ for $i \neq j$ and $i, j \in \{0, 1, 2, \ldots\}$;*
(4.3) *for each subcontinuum K of Y the intersection $K \cap Y_0$ has only finite number of components.*
Then f is confluent.

Ingram's problem

W. T. Ingram posed in [5, Problem 35, p. 373] the following problem.
(5) *Suppose f is a continuous mapping of a continuum X onto a continuum Y, $Y = H \cup K$ is a decomposition of Y into subcontinua H and K, $f|f^{-1}(H)$ and $f|f^{-1}(K)$ are confluent, and $H \cap K$ is a continuum which does not cut Y and is an end continuum of both H and K. Is f confluent?* (W. T. Ingram, 10/11/72)

Thus the continuum Y considered in (5) is represented as the union $Y = H \cup K$ of two continua H and K such that
(5.1) the intersection $H \cap K$ is a continuum;
(5.2) $H \cap K$ is an end continuum of H;
(5.3) $H \cap K$ is an end continuum of K;
(5.4) $H \cap K$ does not cut Y.

Recall that a subset A of a space X *cuts* the space if the complement $X \setminus A$ contains two points x, y with the property that every subcontinuum of X containing x and y meets A.

Let us try to analyze the above set of the four conditions the continuum Y has to satisfy. The understanding of the Ingram problem (and thus its possible solution) depends heavily on the meaning of the term "end continuum". Unfortunately, neither W. T. Ingram himself, nor the editors of the Houston Problem Book explain which definition of the end point is used in (5).

One of the commonly used meanings of the concept is the following one, quoted here after D. E. Bennett and J. B. Fugate, [2, Def. 1.9, p. 8].

Definition A. A proper subcontinuum K of a continuum X is said to be an *end continuum* of X provided X is not the union of two proper subcontinua each intersecting K.

For a deeper discussion of the subject we recall some auxiliary concepts. Let X be a continuum and $A \subset X$. Then X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. A continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$ (then X is said to be *irreducible between* p *and* q). The following characterization of end continua is due to R. H. Rosen ([10, p. 118]; see also D. E. Bennett and J. B. Fugate [2, Th. 1.16, p. 10]).

Theorem 6 (D. E. Bennet, J. B. Fugate, R. H. Rosen). *A subcontinuum K is an end continuum of a continuum X if and only if there is a point $p \in X \setminus K$ such that X is irreducible between p and any point of K.*

Observe first that, according to Def. A, if a continuum E is an end continuum of a continuum C, then E is a *proper* subcontinuum of C. Consequently, conditions (5.2) and (5.3) above imply that

(7) *$H \cap K$ is a proper subcontinuum of H and of K.*

Second, condition (5.4) implies that

(8) *either $H \subset K$ or $K \subset H$.*

Indeed, suppose on the contrary that both two sets $H \setminus K$ and $K \setminus H$ are nonempty. Observe that these two sets $H \setminus K$ and $K \setminus H$ are mutually separated in $Y \setminus (H \cap K)$. Hence $H \cap K$ separates Y. Evidently, every set which separates a space Y cuts the space (see [6, 6.S.12, p. 317]). This contradicts the condition that $H \cap K$ does not cut Y. Therefore (8) is proved.

Without loss of generality we can assume that $H \subset K$. Then $H \cap K = H$, which contradicts statement (7). This shows that conditions (5.2), (5.3) and (5.4) cannot be satisfied simultaneously, or — in other words — that the continuum Y which satisfies all the conditions required in the formulation of problem (5) does not exist.

It follows that, if Def. A is accepted, the problem (**5**) is not correctly formulated. Therefore either we use another definition of the concept of an and continuum, or we modify (**5**) by changing conditions (5.1)–(5.4).

Looking for a more suitable meaning of the term "end continuum" recall that R. H. Bing in [3, condition (B), p. 660], considered a condition which concerns a point p of a continuum X and is one of the three equivalent forms of the definition of the concept of an end point in an arc-like continuum (see [3, Th. 13, p. 661]).

(9) If each of two subcontinua of X contains p, one of the subcontinua contains the other.

This definition of an end point p in an arc-like continuum X can easily be extended to the one of an *end continuum* P in an *arbitrary* continuum X as follows.

Definition B. A subcontinuum P of a continuum X is said to be an *end continuum* in X provided that for every two subcontinua K and L of X the condition $P \subset K \cap L$ implies that either $K \subset L$ or $L \subset K$.

This formulation of the concept of an end continuum can be found e.g. in [4, p. 385]. The difference between the two definitions is evident. Namely with notation as in (3.1) put $M = ab \cup cd$ and note that ab is an end continuum in M in the sense of Def. B, while it is not in the sense of Def. A (because the continuum M is not irreducible). In particular, Def. B does not imply that an end continuum P is a proper subcontinuum of M. Accepting Def. B we get the following easy solution of (5).

Proposition 10. *If the concept of an end continuum is understood in the sense of* Def. B, *then the answer to problem* (**5**) *is affirmative.*

Proof. Let $Y = H \cup K$ be a decomposition of a continuum Y into subcontinua H and K as in (**5**). In the light of (8) we may assume that $Y = H$. Hence $X = f^{-1}(H)$, and f is confluent. $\Diamond$

Thus let us come back to Def. A and try to change problem (5) to eliminate the previously indicated controversy. The simplest way is to omit condition (5.4). Then the modified problem has a negative answer in Ex. 12, i.e., conditions (5.1)–(5.3) do not guarantee the confluence of f. The following example is a modification of [8, Ex. 5.6, p. 110]. To describe it, we recall the Def.s of some concepts used in the proof. If C is a dense subspace of a compact space Z, then Z is called a *compactification* of C, and $Z \setminus C$ is called the *remainder* of C in Z (see e.g. [1, p. 34]). It is known that if C is a locally compact, noncompact, separable metric space, then each continuum is a remainder of C in some compactification of C, [1, Theorem, p. 35]. A *ray* means a one-to-one image of the closed half-line $[0, +\infty)$, and the image of 0 is called the *end point of the ray.*

Taking as C a ray we conclude the following statement, which will be used in Ex. 12.

Statement 11. *Each nondegenerate continuum B is a remainder of a ray C in some compactification of C, and then $Y = B \cup C$ is a continuum having C as an arc-component with $B = \operatorname{cl} C \setminus C$.*

Example 12. There are continua X and Y, a mapping $f : X \to Y$ and a decomposition of Y into subcontinua H and K satisfying conditions (5.1)–(5.3) (where the term "end continuum" is used in the sense of Def. A), such that $f|f^{-1}(H)$ and $f|f^{-1}(K)$ are confluent, and f is not confluent.

Proof. If p and q are points of the plane, we denote by pq the straight line segment with end points p and q. Let $p_0 = (1,0)$, $q_0 = (0,1)$ and $p_n = (1 + 1/n, 0)$ for $n \in \mathbb{N}$. Define

$$M = p_0q_0 \cup \bigcup\{p_nq_0 : n \in \mathbb{N}\}.$$

Thus M is a continuum. Now let M_1 and M_2 be the continua

$$M_1 = M \cup \bigcup\{p_{2n-1}p_{2n} : n \in \mathbb{N}\} \quad \text{and} \quad M_2 = M \cup \bigcup\{p_{2n}p_{2n+1} : n \in \mathbb{N}\}.$$

Let φ be the symmetry of the plane defined by $\varphi((x,y)) = (-x, y)$. Let C_1 and C_2 be two mutually disjoint rays each of which is disjoint from $M_1 \cup M_2 \cup \varphi(M_1 \cup M_2)$ such that for $i \in \{1,2\}$ the nondegenerate continuum $M_i \cup \varphi(M_i)$ is a remainder of a ray C_i in some compactification of C_i. Put $X_i = M_i \cup \varphi(M_i) \cup C_i$. Then X_i is a continuum having C_i as an arc-component with $M_i \cup \varphi(M_i) = \operatorname{cl} C_i \setminus C_i$ as the remainder (see Statement 11). For $i \in \{1,2\}$ denote by c_i the end point of the ray C_i. Put $X = X_1 \cup X_2$. Clearly X is a continuum and $X_1 \cap X_2 = M \cup \varphi(M)$.

Now, let $\mathcal{R}$ be the equivalence relation in X defined by the formula

$$\mathcal{R} = \{(p, \varphi(p)) : p \in p_0q_0\} \cup \{(\varphi(p), p) : p \in p_0q_0\} \cup \{(p,p) : p \in X\}.$$

Define f as the natural projection from X onto the quotient space $Y = X/\mathcal{R}$. Clearly, Y is a continuum again. The mapping f identifies the opposite points p and $\varphi(p)$ on p_0q_0 and $\varphi(p_0q_0)$.

Observe the following properties.

(12.1) $Y_i = f(X_i)$ is a nondegenerate proper subcontinuum of Y, for $i \in \{1,2\}$;

(12.2) $L = Y_1 \cap Y_2 = f(M \cup \varphi(M))$ is a nondegenerate subcontinuum of Y;

(12.3) L is an end continuum of Y_i for $i \in \{1,2\}$;
(12.4) $f|f^{-1}(Y_i)$ is confluent for $i \in \{1,2\}$;
(12.5) f is not confluent.

Properties (12.1)–(12.2) are easy observations. To prove (12.3)–(12.4) fix $i \in \{1,2\}$. Each Y_i is irreducible between the end point c_i of the ray C_i and any point $y \in L$. Hence, Theorem 6 applies and property (12.3) holds.

To prove (12.4) we consider four cases, in which Q means a subcontinuum of $f^{-1}(Y_i)$, for any fixed $i \in \{1,2\}$.

(α) $Q \subset f(C_i)$;

(β) Q meets both $f(C_i)$ and $f(M_i \cup \varphi(M_i))$;

(γ) Q does not meet $f(C_i)$ and Q contains $f(q_0)$;

(δ) Q does not meet $f(C_i)$ and Q does not contain $f(q_0)$.

In any case it follows from the construction that each component of the set $(f|f^{-1}(Y_i))^{-1}(Q)$ is mapped onto Q. Thus $f|f^{-1}(Y_i)$ is confluent as needed. To see (12.5) put $Q = f(p_0p_1) \cup f(\varphi(p_0p_1))$. Clearly, Q is a subcontinuum of Y. Then $f^{-1}(Q)$ has just two components in X, namely p_0p_1 and $\varphi(p_0p_1)$. These components are not mapped onto Q. Finally, put $H = Y_1$ and $K = Y_2$. $\Diamond$

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