

A COLLECTION OF DENDROIDS

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Abstract: An uncountable collection of uniformly arcwise connected plane Suslinian dendroids is constructed which shows that the negations of the following properties: local connectedness, smoothness, pointwise smoothness, hereditary contractibility, and selectibility are not countable in this class of curves.

Let \mathcal{S} be a class of spaces and let \mathcal{P} be a property. We say that the property \mathcal{P} is finite (countable) in \mathcal{S} provided that there is a finite (countable, respectively) set $\mathcal{F} \subset \mathcal{S}$ such that a member of \mathcal{S} has the

property \mathcal{P} if and only if it contains a homeomorphic copy of a member of \mathcal{F} . For example, the property of being nonembeddable in the 2-sphere S^2 is finite in the classes of graphs and of local dendrites by the classical result of Kuratowski [10], and in the class of locally connected continua by the well-known result of Claytor [7]. Nothing similar is true for nonplanability of curves that are not locally connected. Namely it was shown that the property of being nonplanable is not countable in the classes of smooth dendroids [6] and of (nonsmooth) fans [14].

Let us note that the following proposition (which is a consequence of the definition) was used in the above results.

1. Proposition. *Let a class S of spaces and a property \mathcal{P} of members of S be given. If there exists an infinite (uncountable) collection $\mathcal{C} \subset S$ such that*

(2) *every member of \mathcal{C} has the property \mathcal{P} , and*

(3) *for every two distinct members C_1 and C_2 of \mathcal{C} if a member Y of S is topologically contained in both C_1 and C_2 , then Y does not have the property \mathcal{P} ,*

then the property \mathcal{P} is not finite (countable, respectively) in the class S .

The same method, exploiting Prop. 1, was used in [19] to show that the property of not being contractible is not countable in the class of (semi-smooth) fans.

A property \mathcal{P} of continua is said to be *hereditary* provided that if a continuum X has \mathcal{P} , then each subcontinuum of X has \mathcal{P} , too.

The following statement is a consequence of the definitions.

4. Statement. *If a property non- \mathcal{P} is either finite or countable in a class S of continua, then the property \mathcal{P} is hereditary.*

The aim of this paper is to show that the negations of the following properties: smoothness, pointwise smoothness, hereditary contractibility and selectibility are not countable in the class of uniformly arcwise connected plane Suslinian dendroids. This will be done using one uncountable collection of dendroids each having all the needed properties.

We recall the necessary definition first. By a *continuum* we mean a compact connected metric space. A one-dimensional continuum is called a *curve*. A *dendrite* means a locally connected continuum containing no simple closed curve. A *dendroid* is defined as an arcwise connected and hereditarily unicoherent continuum. Thus each dendrite is a dendroid, and each dendroid is a curve ([1, (48), p. 239]). An arc A with end points p and q contained in a space X is said to be *free* if $A \setminus \{p, q\}$ is an open subset of the space. A continuum X is said to be

uniformly arcwise connected provided that it is arcwise connected and that for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that for every arc $A \subset X$ there are points $a_0, a_1, \dots, a_k \in A$ such that

$$A = \bigcup \{a_i a_{i+1} : i \in \{0, 1, \dots, k-1\}\}$$

and $\text{diam } a_i a_{i+1} < \varepsilon$ for each $i \in \{0, 1, \dots, k-1\}$ (i.e., if each arc in X contains $k+1$ points which cut the arc into subarcs of diameter less than ε). Note that uniform arcwise connectedness is a hereditary property. It is shown in [2, Th. 19, p. 194] that uniform arcwise connectedness is an invariant under continuous mappings between dendroids:

(5) *If a dendroid Y is the image of a uniformly arcwise connected dendroid X under a continuous mapping, then Y is also uniformly arcwise connected.*

We shall use the notion of order of a point in the classical sense (i.e., the number of arcs emanating from the point and mutually disjoint out of it, see [1, p. 229]), and we denote by $\text{ord}(p, X)$ order of the space X at a point $p \in X$. Points of order 1 in a space X are called *end points* of X ; the set of all end points of X is denoted by $E(X)$. Points of order at least 3 are called *ramification points* of X ; the set of all ramification points of X is denoted by $R(X)$. A dendroid that has exactly one ramification point is called a *fan*. A fan is called a *simple triod* (a *simple 4-od*) if its ramification point is of order three (of order four, respectively).

A continuum X is said to be *Suslinian* provided that each collection of pairwise disjoint nondegenerate subcontinua of X is countable, [13]. Suslinian continua are hereditarily decomposable, and thus one-dimensional.

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for every sequence of points x_n of X converging to a point x the sequence of arcs px_n converges to the arc px . A dendroid X is said to be *smooth* provided that there exists a point $p \in X$ at which X is smooth. The following fact is known.

(6) *Every dendrite is smooth.*

A dendroid X is said to be *pointwise smooth* provided that for each point $x \in X$ there is a point $p(x) \in X$ such that for every convergent sequence of points x_n of X the condition $\lim x_n = x$ implies that the sequence of arcs $p(x)x_n$ is convergent, and $\text{Lim } p(x)x_n = p(x)x$.

A continuum X is said to be *contractible* provided that there are a homotopy $h : X \times [0, 1] \rightarrow X$ and a point $p \in X$ such that for every

point $x \in X$ we have $h(x, 0) = x$ and $h(x, 1) = p$. For example a disk is contractible, while a simple closed curve is not. The following results concerning contractibility of curves are well-known (see e.g. [3]).

(7) *Every contractible curve is a dendroid.*

The inverse is not true, and the main problem related to contractibility of curves is to find a structural characterization of contractible dendroids.

(8) *Every contractible dendroid is uniformly arcwise connected.*

(9) *A locally connected curve is contractible if and only if it is a dendrite.*

It is known that contractibility (of dendroids) is not a hereditary property, even in the class of plane Suslinian fans (see e.g. [5, Prop. 12, p. 234]). Hence the following is a consequence of Statement 4.

(10) *Noncontractibility is not a countable property in the class of plane Suslinian fans.*

Recall that the above result was proved in [19] using a more complicated argument, viz. Prop. 1. Hereditary contractibility of dendroids implies pointwise smoothness, while the opposite implication remains an open question, [8].

Given a metric space X with a metric d , we denote by 2^X the space of all nonempty compact subsets of X equipped with the Hausdorff distance H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

Further, $C(X)$ means a subspace of 2^X composed of all (nonempty compact) connected subsets of X . If X is a continuum, $C(X)$ is called the *hyperspace of subcontinua* of X . A *continuous selection* for a family $\mathcal{F} \subset 2^X$ is defined as a mapping $\sigma : \mathcal{F} \rightarrow X$ such that $\sigma(A) \in A$ for each $A \in \mathcal{F}$. Answering a question of Michael [16] Kuratowski, Nadler and Young characterized [12] locally compact separable metric spaces X for which there exists a continuous selection for 2^X . In particular, they proved that if a continuum X admits such a selection, then X is an arc. Since each arc admits such a selection (taking $\sigma(A) = \min A$, for example), we infer the following assertion.

(11) *A continuum X admits a continuous selection for 2^X if and only if X is an arc.*

So, the problem of finding a structural characterization of continua that admit a continuous selection for a given family $\mathcal{F} \subset 2^X$ is solved in the case when $\mathcal{F} = 2^X$.

Another very interesting case is when $\mathcal{F} = C(X)$. A continuum X is said to be *selectible* provided that it admits a continuous selection for $C(X)$. The following important results in this area were obtained by Nadler and Ward [17].

(12) *Every selectible continuum is a dendroid.*

(13) *A locally connected continuum is selectible if and only if it is a dendrite.*

(14) *Each selectible dendroid is a continuous image of the Cantor fan, so it is uniformly arcwise connected.*

On the other hand, there are uniformly arcwise connected dendroids which are not selectible, as one pictured in [17, Fig. 1, p. 372], and again the main problem related to selectibility of curves is to find a structural characterization of selectible dendroids.

15. Remark. Note that since selectibility for $C(X)$ implies that the continuum X is a dendroid (12), we can restrict our considerations concerning selectibility to dendroids only. In this case nonselectibility for 2^X is a finite property, namely the collection \mathcal{C} in matter consists of the simple triod only.

Some conditions are known that imply nonselectibility of dendroids. A dendroid X is of *type N between its points p and q* if there are two sequences of arcs $p_n p'_n$ and $q_n q'_n$ in X and points $p''_n \in q_n q'_n \setminus \{q_n, q'_n\}$ and $q''_n \in p_n p'_n \setminus \{p_n, p'_n\}$ with

$$pq = \text{Lim } p_n p'_n = \text{Lim } q_n q'_n,$$

$$p = \lim p_n = \lim p'_n = \lim p''_n \quad \text{and} \quad q = \lim q_n = \lim q'_n = \lim q''_n.$$

The above concept is due to L. G. Oversteegen [18] and is related to the following condition of B. G. Graham [9]. A dendroid X is said to *contain a zigzag between its points p and q* provided there exist in X a sequence of arcs $p_n q_n$ and two sequences of points p'_n and q'_n situated in these arcs in such a manner that $p_n < q'_n < p'_n < q_n$ (where $<$ denotes the natural order on $p_n q_n$ from p_n to q_n), for which the following conditions hold:

$$pq = \text{Lim } p_n q_n, \quad p = \lim p_n = \lim p'_n, \quad q = \lim q_n = \lim q'_n.$$

Note that if a dendroid contains a zigzag between its points p and q , then it is of type N (between these points) but not conversely, even for fans. The next fact follows from the definitions.

(16) *If a dendroid is of type N (if it contains a zigzag, in particular), then it is not smooth and is not pointwise smooth.*

The following statement is shown in [18, Th. 2.1, p. 838].

(17) *If a dendroid is of type N (if it contains a zigzag, in particular), then it is not contractible.*

Maćkowiak introduced [15] the following concept. Let a continuum X and two its subcontinua A and B with $B \subset A$ be given. Then B is called a *bend set* of A provided that there are two sequences $\{A_n\}$ and $\{A'_n\}$ of subcontinua of X such that $A_n \cap A'_n \neq \emptyset$ for every $n \in \mathbb{N}$, and

$$A = \text{Lim } A_n = \text{Lim } A'_n \quad \text{and} \quad B = \text{Lim } (A_n \cap A'_n).$$

We say that X has the *bend intersection property* if for each subcontinuum A of X the intersection of all bend sets of A is nonempty. The following assertion is proved in [15, Cor., p. 548].

(18) *Each selectable dendroid has the bend intersection property.*

Note that if a dendroid X is of type N between some points $p, q \in X$, then the singletons $\{p\}$ and $\{q\}$ are bend sets of $\text{Lim } p_n p'_n = \text{Lim } q_n q'_n$. Thus, by (18), the following holds.

(19) *If a dendroid is of type N (if it contains a zigzag, in particular), then it does not have the bend intersection property, so it is not selectable.*

Let a sequence of mutually disjoint continua $\{X_n : n \in \mathbb{N}\}$ (lying e.g. in the Hilbert cube) be given, which is tending to a point p . For each $n \in \mathbb{N}$ choose two points a_n and b_n in X_n , and consider a sequence of mutually disjoint arcs $\{b_n a_{n+1} : n \in \mathbb{N}\}$, also having the point p as the only point of its topological limit, and such that

$$X_m \cap b_n a_{n+1} = \begin{cases} \emptyset & \text{if } n \neq m \neq n + 1, \\ \{b_n\} & \text{if } n = m, \\ \{a_{n+1}\} & \text{if } m = n + 1. \end{cases}$$

Then the set defined by

$$(20) \quad X = \{p\} \cup \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\}$$

is a continuum, which is called a *string of continua* X_n . Each continuum X_n is called a *bead* of the string, the points a_n and b_n are called *extreme points* of the bead X_n , and the point p is called the *final point* of X .

Observe that each arc $b_n a_{n+1}$ is a free arc in the string X , that the final point p is an end point of X , and that

$$\text{Lim } X_n = \text{Lim } b_n a_{n+1} = \{p\},$$

whence it follows that $\lim \text{diam } X_n = \lim \text{diam } b_n a_{n+1} = 0$.

We shall use the concept of a string exclusively in the case when each bead X_n is a dendrite and when the extreme points a_n and b_n are end points of X_n . It is easy to verify that then the string X constructed according to (20) is a dendrite. In such a case we will say that X is a *string of dendrites* (see [4, Chapter 6, p. 27 ff], where this concept is introduced and studied).

Let \mathbf{S} be the set of all zero-one sequences. We define an equivalence relation \simeq on \mathbf{S} as follows. If $r, s \in \mathbf{S}$ with $r = (r_1, r_2, \dots)$ and $s = (s_1, s_2, \dots)$, then $r \simeq s$ if there exist positive integers i and j such that

$$(21) \quad r_{i+n} = s_{j+n} \text{ for each } n \in \mathbb{N}.$$

Thus \simeq is an equivalence relation on \mathbf{S} and the equivalence classes are countable. Hence, the set

$$(22) \quad \mathbf{T} = \mathbf{S} / \simeq$$

of equivalence classes of \mathbf{S} is uncountable. We shall construct, for each $t \in \mathbf{T}$, a uniformly arcwise connected plane Suslinian dendroid $X(t)$. We take as \mathcal{P} the negation of any of the following: local connectedness, smoothness, pointwise smoothness, hereditary contractibility, and selectibility. It will be shown that the collection \mathcal{C} of the constructed dendroids satisfies conditions (2) and (3) of Prop. 1, and therefore the main result of the paper will be proved.

To this aim put (in the Cartesian coordinates in the plane) $p = (0, 0)$, $a_1 = (1, 0)$, and take in the closed unit interval from p to a_1 of the x -axis two sequences of points $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ such that

$$(23) \quad p = \lim a_n = \lim b_n$$

and that, in the natural ordering $<$ of the segment pa_1 ,

$$(24) \quad p < \dots < b_{n+1} < a_{n+1} < b_n < a_n < \dots < b_1 < a_1.$$

For each $n \in \mathbb{N}$ let c_n stand for the center of the segment $b_n a_n$. Denote by $B_n(0)$ the simple triod $c_n a_n \cup c_n b_n \cup c_n d_n$ and by $B_n(1)$ the simple 4-od $c_n a_n \cup c_n b_n \cup c_n d_n \cup c_n e_n$ such that all their arms are straight line segments of equal length, and all are situated in the upper half plane.

Next, for each element $t \in \mathbf{T}$ select a representative $r(t) \in \mathbf{S}$ such that $[r(t)] = t$. Let $r_n(t) \in \{0, 1\}$ be the n -th term of the representative $r(t) \in \mathbf{S}$. Consider, for each $t \in \mathbf{T}$, a string $D(t)$ of dendrites $X_n(t)$ such that

$$(25) \quad X_n(t) = B_n(r_n(t)) \text{ for each } n \in \mathbb{N}.$$

For each $t \in \mathbf{T}$ we put, according to (20),

$$(26) \quad D(t) = \{p\} \cup \bigcup \{X_n(t) \cup b_n a_{n+1} : n \in \mathbb{N}\}.$$

We construct, for each $t \in \mathbf{T}$, the dendroid $X(t)$ by adding to the dendrite $D(t)$ a sequence of arcs $\{A(t, k) : k \in \mathbb{N}\}$ which approximate $D(t)$ in a special way. Let an element $t \in \mathbf{T}$ be fixed. To define the arc $A(t, k)$ we perform the following construction.

We start with defining a continuous surjective mapping $f_0 : pa_1 \rightarrow D(t)$. Put $f_0(p) = p$, and for each $n \in \mathbb{N}$ let $f_0(a_n) = a_n$ and $f_0(b_n) = b_n$. We divide each straight line segment $a_{n+1}b_n$ into three equal parts by points b^*_n and a^*_{n+1} such that

$$(27) \quad a_{n+1} < b^*_n < a^*_{n+1} < b_n$$

(compare (24)), and we put $f_0(b^*_n) = b_n$ and $f_0(a^*_{n+1}) = a_{n+1}$. Next consider two cases. First, if $r_n(t) = 0$, we divide each straight line segment $b_n a_n$ into four equal parts by points c^*_n , c_n and c^{**}_n such that $b_n < c^*_n < c_n < c^{**}_n < a_n$ (compare (24)), and we put

$$f_0(c^*_n) = c_n, \quad f_0(c_n) = d_n, \quad \text{and} \quad f_0(c^{**}_n) = c_n.$$

Second, if $r_n(t) = 1$, we divide each straight line segment $b_n a_n$ into six equal parts by points c^*_n , d^*_n , c_n , e^*_n and c^{**}_n such that

$$b_n < c^*_n < d^*_n < c_n < e^*_n < c^{**}_n < a_n$$

(compare (24)), and we put

$$f_0(c^*_n) = c_n, \quad f_0(d^*_n) = d_n, \quad f_0(c_n) = c_n, \quad f_0(e^*_n) = e_n \quad \text{and} \quad f_0(c^{**}_n) = c_n.$$

So, the mapping f_0 is now defined either at the points

$$(28) \quad p < \dots < b_n < c^*_n < c_n < c^{**}_n < a_n < \dots < a_1$$

(if $r_n(t) = 0$), or at the points

$$(29) \quad p < \dots < b_n < c^*_n < d^*_n < c_n < e^*_n < c^{**}_n < a_n < \dots < a_1$$

(if $r_n(t) = 1$). For each straight line subsegment uv of the segment pa_1 whose end points u and v are two consecutive points listed in either (28) or (29) we extend f_0 linearly, i.e., we define $f_0|_{uv} : uv \rightarrow f_0(u)f_0(v) \subset D(t)$ as a linear mapping. Since each point of $pa_1 \setminus \{p\}$ belongs to some uv , and since the singleton $\{f_0(p)\}$ together with the images $f_0(u)f_0(v)$ (running over all segments uv considered above) cover $D(t)$, we see that the mapping $f_0 : pa_1 \rightarrow D(t)$ is a well-defined continuous surjection.

Consider the harmonic fan F_H defined as the cone with the vertex $p = (0, 0)$ over the set $\{(1, 1/k) : k \in \mathbb{N}\}$. Thus, if $q_k = (1, 1/k)$, and if pq means the straight line segment with end points p and q in the plane, we have

$$F_H = pa_1 \cup \bigcup \{pq_k : k \in \mathbb{N}\}.$$

The mapping $f_0 : pa_1 \rightarrow D(t)$ induces a mapping f of F_H into the upper half plane $\{(x, y) : y \geq 0\}$ such that $f|pa_1 = f_0$ and that the partial mapping $f| \bigcup \{pq_k : k \in \mathbb{N}\}$ is one-to-one. Consequently, for each $k \in \mathbb{N}$, the partial mapping $f|pq_k$ is a homeomorphism of the straight line segment pq_k onto an arc $A(t, k)$ from p to a point $q(t, k) = f(q_k)$ such that $\lim_{k \rightarrow \infty} q(t, k) = a_1$. We can assume that the arc $A(t, k)$ is the union of the singleton $\{p\}$ and of countably many straight line segments described as follows. For each $k \in \mathbb{N}$ and for each straight line subsegment uv of the segment pa_1 considered above (i.e., u and v are two consecutive points listed in either (27) or (28)) we denote by $u(t, k)v(t, k)$ the subsegment of pq_k that projects onto uv ; then $f|u(t, k)v(t, k)$ is a linear mapping of $u(t, k)v(t, k)$ onto a straight line segment $f(u(t, k))f(v(t, k))$ in the upper half plane which is close to the segment $f_0(u)f_0(v) \subset D(t)$ in the sense that

$$(30) \quad \text{Lim}_{k \rightarrow \infty} f(u(t, k))f(v(t, k)) = f_0(u)f_0(v) \subset D(t).$$

Now $A(t, k)$ is the union of $\{p\}$ and of countably many straight line segments of the form $f(u(t, k))f(v(t, k))$.

Further, we can assume that

$$(31) \quad A(t, k) \cap D(t) = \{p\},$$

$$(32) \quad \text{if } k_1 \neq k_2, \text{ then } A(t, k_1) \cap A(t, k_2) = \{p\}.$$

By continuity of f we have

$$(33) \quad D(t) = \text{Lim}_{k \rightarrow \infty} A(t, k).$$

Finally we define

$$(34) \quad X(t) = f(F_H).$$

Thus $X(t)$ is a plane continuum. By the construction above we see that

$$(35) \quad X(t) = D(t) \cup \bigcup \{A(t, k) : k \in \mathbb{N}\}.$$

Conditions (31) and (32) guarantee arcwise connectedness and hereditary unicoherence of $X(t)$, so $X(t)$ is a dendroid. It follows from (33) and (35) that it is Suslinian. Uniform arcwise connectedness of the

harmonic fan F_H implies that of $X(t)$ by its definition (34) according to (5).

Let us come back to the definition of the mapping f , and consider, for a fixed $k \in \mathbb{N}$ and for an arbitrary $n \in \mathbb{N}$, the subsegment

$$(36) \quad \begin{aligned} & a_{n+1}(t, k)b_n(t, k) = \\ & = a_{n+1}(t, k)b_n^*(t, k) \cup b_n^*(t, k)a_{n+1}^*(t, k) \cup a_{n+1}^*(t, k)b_n(t, k) \end{aligned}$$

of the segment pq_k that projects onto $a_{n+1}b_n$ (see (27)). Recall that the partial mappings of f restricted to each of the three segments in the right member of equality (36) are linear, so that we have a broken line from $f(a_{n+1}(t, k))$ to $f(b_n(t, k))$ consisting of three straight line segments. By the limit condition (30) in which we substitute for the pair (u, v) pairs of consecutive points of (27) we conclude that the sequence of arcs $\{f(a_{n+1}(t, k))f(b_n(t, k)) : k \in \mathbb{N}\}$ together with its limit arc $a_{n+1}b_n$ form a zigzag in $X(t)$ between a_{n+1} and b_n .

Moreover, note that if a subcontinuum Y of $X(t)$ is not locally connected, then the set of points of non local connectedness of Y contains all the dendrites $X_n(t) \cup b_n a_{n+1}$ from some sufficiently great n on. Hence

(37) *for each subcontinuum Y of $X(t)$ that is not locally connected there exists $n_0 \in \mathbb{N}$ such that Y contains the union $\bigcup\{X_n(t) \cup b_n a_{n+1} : n \in \mathbb{N} \text{ and } n \geq n_0\}$, and Y intersects infinitely many arcs $A(t, k)$,*

whence it follows from the construction of $X(t)$ that

(38) *each non locally connected subcontinuum of $X(t)$ contains a zigzag,*

and therefore, by (16), (17) and (19), we infer that

(39) *each subcontinuum Y of $X(t)$ that is not locally connected is not: smooth, pointwise smooth, (hereditarily) contractible, and selectable.*

40. Proposition. *Let the set \mathbf{T} be as in (22), and for each $t \in \mathbf{T}$ let $X(t)$ be defined by (34). Take the collection*

$$C = \{X(t) : t \in \mathbf{T}\}.$$

If a continuum Y is topologically contained in some two members $C_1 = X(t_1)$ and $C_2 = X(t_2)$ of C with $t_1 \neq t_2$, then Y is a dendrite.

Proof. In fact, if not, then Y is a subcontinuum of both $X(t_1)$ and of $X(t_2)$ which is not locally connected. Then by (37) there are n_1 and n_2 in \mathbb{N} such that Y contains all the beads $X_n(t_1)$ of the string $D(t_1)$ for

every $n \geq n_1$ and all the beads $X_n(t_2)$ of the string $D(t_2)$, and these beads appear in both $D(t_1)$ and $D(t_2)$ in the the same order. Thus it follows from (25) that if $i = j = \max\{n_1, n_2\}$, then for each $n \in \mathbb{N}$ we have $r_{i+n}(t_1) = r_{j+n}(t_2)$, whence, according to the definition of the relation \simeq , we infer that $t_1 = t_2$, a contradiction. \diamond

Now let \mathcal{S} be the class of dendroids such that

$$(41) \quad \mathcal{S} \text{ contains the collection } \mathcal{C} = \{X(t) : t \in \mathbf{T}\}.$$

For example, one can take as \mathcal{S} the class of uniformly arcwise connected dendroids, or the class of Suslinian dendroids, or of plane ones, or any intersection of these classes. Further, take as \mathcal{P} any property such that

$$(42) \quad \text{no dendrite has the property } \mathcal{P}.$$

Let a continuum Y be as in Prop. 40. Thus Y is a dendrite, and by (42) it does not have the property \mathcal{P} , and so condition (3) of Prop. 1 is satisfied. Thereby the following result is established.

43. Theorem. *Let \mathcal{S} be a class of dendroids satisfying (41), and let a property \mathcal{P} be such that no dendrite has the property \mathcal{P} (see (42)). If there exists an uncountable collection \mathcal{C} of members of \mathcal{S} such that every member of \mathcal{C} has the property \mathcal{P} (see (2)), then the property \mathcal{P} is not countable in the class \mathcal{S} .*

Taking as \mathcal{P} the negation of one of the following: local connectedness, smoothness, pointwise smoothness, hereditary contractibility, and selectibility, we see that (42) holds (compare in particular (6), (9) and (13)). Thus taking $\mathcal{C} = \{X(t) : t \in \mathbf{T}\}$ we see by (39) that (2) is satisfied, and therefore all the assumptions of Th. 43 hold true. Thus we have a corollary.

44. Corollary. *The negation of each of the following properties: local connectedness, smoothness, pointwise smoothness, hereditary contractibility, and selectibility, is not countable in any class \mathcal{S} of dendroids satisfying condition (41).*

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