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NOTE ON A FUNCTIONAL EQUATION RELATED TO THE POWER MEANS

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Abstract: M. E. Kuczma in [1] considered analytic solutions of the functional equation

$$x + g(y + f(x)) = y + g(x + f(y))$$

on the real line. In [2] solutions in the class of twice differentiable functions were presented. In the present paper we deal with additive solutions and with solutions that have the form of a weighted quasi-arithmetic mean with an exponential generator.

In this article we present some remarks on solutions of a functional equation arising from considerations concerning a problem of compatibility of means. The background for this problem is to be found in Marcin E. Kuczma's paper [1], where the functional equation

$$(1) \qquad x + g(y + f(x)) = y + g(x + f(y)), \quad x, y \in \mathbb{R}$$

was studied. There equation (1) is solved in the class of analytic functions. Regardless of the problem of compatibility of means it seems that equation (1) itself deserves further investigations. In [2] all twice differentiable solutions of (1) have been described.

Following the earlier papers we will assume that $f(0) = g(0) = 0$ since we have

Remark 1. If a pair of functions $f, g : \mathbb{R} \to \mathbb{R}$ is a solution of (1), then the functions $f_1, g_1 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) := f(x) - f(0), \quad g_1(x) := g(x + f(0)) - g(f(0)),$$

yield also a solution of (1) and $f_1(0) = g_1(0) = 0$.

Conversely, if two functions $f_1, g_1 : \mathbb{R} \to \mathbb{R}$ give a solution of (1) and $f_1(0) = g_1(0) = 0$ then the pair of functions given by

$$f(x) := f_1(x) + b, \quad g(x) := g_1(x - b) + c,$$

is a solution of (1), for all $b, c \in \mathbb{R}$, as well.

The main result of [2] reads as follows

Theorem A. *The general solution of equation* (1) *in the class of twice differentiable functions* $f, g : \mathbb{R} \to \mathbb{R}$ *vanishing at zero is given by the following formulas:*

$$f(x) = ax, \quad g(x) = \frac{1}{1-a}\, x,$$

where $a \in \mathbb{R} \setminus \{1\}$ *is arbitrarily fixed;*

$$f(x) = -x, \quad g(x) = \frac{1}{2}\, x + p(x),$$

where $p : \mathbb{R} \to \mathbb{R}$ *is an arbitrary even, twice differentiable function such that* $p(0) = 0$*;*

$$f(x) = -\frac{1}{\gamma} \ln\left(a\mathrm{e}^{\gamma x} + (1-a)\right), \quad g(x) = -\frac{1}{\gamma} \ln\left(\frac{1}{1+a}\mathrm{e}^{-\gamma x} + \frac{a}{1+a}\right),$$

where $\gamma \in \mathbb{R} \setminus \{0\}$ *and* $a \in [0,1]$ *are arbitrarily fixed.*

Conversely, each of the three pairs of functions listed above yields a solution of (1) *in the class of twice differentiable functions.*

In particular, looking at the form of the third family of solutions, but also at the first one, one can detect the form of a weighted quasi-arithmetic mean there. We will check that all functions of that form with an arbitrary exponential or additive generator are solutions of (1).

We start with some notations. Let $M_\alpha(x, y) := m^{-1}(\alpha m(x) + (1-\alpha)m(y))$, $\alpha \in [0,1]$, be a weighted quasi-arithmetic mean with a bijective generator m which will be described below. It is obvious that $M_\alpha(x, y) = M_{1-\alpha}(y, x)$ for all $\alpha \in [0, 1]$. Using this notation we state the first result.

Proposition 1. *Let* $m : \mathbb{R} \to (0, \infty)$ *be an exponential bijection (or* $m : \mathbb{R} \to \mathbb{R}$ *be an additive bijection). For any* $\alpha \in [0, 1]$ *the pair of functions*

$$(2)\qquad f(x) = M_\alpha(-x,0), \quad g(x) = M_{\frac{1}{1+\alpha}}(x,0)$$

satisfies equation (1).

Proof. We check that a pair of functions (f,g) defined for an exponential generator $m:\mathbb{R}\to(0,\infty)$ by

$$f(x) := M_\alpha(-x,0) = m^{-1}(\alpha m(-x) + (1-\alpha)),$$
$$g(x) := M_{\frac{1}{1+\alpha}}(x,0) = m^{-1}\left(\frac{1}{1+\alpha}m(x) + \frac{\alpha}{1+\alpha}\right),$$

is a solution of (1).

Equation (1) may equivalently be rewritten in the form

$$(3)\qquad m(x)m(g(y+f(x))) = m(y)m(g(x+f(y))), \quad x,y\in\mathbb{R},$$

which after suitable substitutions (for the given form of f and g) yields the desired assertion. ◊

Similar computations show that we have our claim with an additive generator $m:\mathbb{R}\to\mathbb{R}$ and $f(x) := M_\alpha(-x,0) = m^{-1}(\alpha m(-x))$, $g(x) := M_{\frac{1}{1+\alpha}}(x,0) = m^{-1}\left(\frac{1}{1+\alpha}m(x)\right)$, $x\in\mathbb{R}$.

The next result shows that in the case of an exponential bijection, in general, it is enough to assume that only one of the functions f and g is of the form (2). Namely, we have the following

Theorem 1. *Let a pair (f,g) be a solution of* (1). *If the function $g:\mathbb{R}\to\mathbb{R}$ is of the form*

$$(4)\qquad g(x) = M_\beta(x,0)$$

with an exponential generator m and some $\beta\in[\frac{1}{2},1]$, then necessarily

$$f(x) = M_{\frac{1-\beta}{\beta}}(-x,0), \quad x\in\mathbb{R}.$$

Likewise, if the function $f:\mathbb{R}\to\mathbb{R}$ is of the form

$$(5)\qquad f(x) = M_\alpha(-x,0)$$

for some continuous exponential bijection m, $\alpha\in[0,1)$ and if g is continuous at zero, then necessarily

$$g(x) = M_{\frac{1}{1+\alpha}}(x,0), \quad x\in\mathbb{R}.$$

Proof. Let $g(x) = m^{-1}(\beta m(x) + (1-\beta))$, $x\in\mathbb{R}$. Substituting this and $y:=0$ into (3) we get

$$\beta m(x)m(f(x)) = (1-\beta) + (2\beta-1)m(x),$$

whence

$$f(x) = m^{-1}\left(\frac{1-\beta}{\beta}m(-x) + \frac{2\beta-1}{\beta}\right),$$

so that f is of the desired form.

Now, assume that $f(x) = m^{-1}(\alpha m(-x) + (1-\alpha))$ for all $x \in \mathbb{R}$, with some $\alpha \in [0,1)$ and with a continuous exponential bijection $m : \mathbb{R} \to (0,\infty)$. By induction we prove that for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ the n-th iterate of f is given by

$$\begin{aligned} f^n(x) &= m^{-1}\left(\alpha\,\frac{1-(-\alpha)^n}{1+\alpha}\,m(-x) + \frac{1-(-\alpha)^{n+1}}{1+\alpha}\right) - \\ &\quad - m^{-1}\left(\alpha\,\frac{1-(-\alpha)^{n-1}}{1+\alpha}\,m(-x) + \frac{1-(-\alpha)^n}{1+\alpha}\right) = \\ &= m^{-1}\left(\frac{\alpha\left(1-(-\alpha)^n\right)m(-x) + 1 - (-\alpha)^{n+1}}{\alpha\left(1-(-\alpha)^{n-1}\right)m(-x) + 1 - (-\alpha)^n}\right). \end{aligned} \tag{6}$$

Equation (1) with $y := 0$ gives

$$x + g(f(x)) = g(x), \quad x \in \mathbb{R}, \tag{7}$$

which leads easily to

$$x + f(x) + f^2(x) + \cdots + f^n(x) + g(f^{n+1}(x)) = g(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{8}$$

Applying (6) we rewrite (8) as

$$\begin{aligned} &x + m^{-1}\left(\alpha\,\frac{1-(-\alpha)^n}{1+\alpha}\,m(-x) + \frac{1-(-\alpha)^{n+1}}{1+\alpha}\right) + \\ &\quad + g\left(m^{-1}\left(\frac{\alpha(1-(-\alpha)^{n+1})\,m(-x) + 1 - (-\alpha)^{n+2}}{\alpha(1-(-\alpha)^n)\,m(-x) + 1 - (-\alpha)^{n+1}}\right)\right) = g(x), \end{aligned}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Using the assumption that m is continuous and g is continuous at the origin, $\alpha \in [0,1[$ and letting n tend to infinity we obtain

$$g(x) = x + m^{-1}\left(\frac{\alpha}{1+\alpha}\,m(-x) + \frac{1}{1+\alpha}\right), \quad x \in \mathbb{R},$$

whence

$$\begin{aligned} g(x) &= m^{-1}(m(x)) + m^{-1}\left(\frac{\alpha}{1+\alpha}\,m(-x) + \frac{1}{1+\alpha}\right) = \\ &= m^{-1}\left(\frac{\alpha}{1+\alpha} + \frac{1}{1+\alpha}\,m(x)\right) = M_{\frac{1}{1+\alpha}}(x,0), \end{aligned}$$

for all $x \in \mathbb{R}$, which was to be proved. ◊

Remark 2. The restriction of β to the interval $[\frac{1}{2},1]$ in the first part of the theorem guarantees that $\frac{1-\beta}{\beta}$ belongs to the interval $[0,1]$. The restriction of α to the interval $[0, 1)$ in the second part of the theorem is needed in order to get the existence of the limit $\lim_{n\to\infty} f^n(x)$ in (6). On the other hand for $\alpha = 1$ it is shown in the Th. A that we do not have the uniqueness of g for a function f defined as $f(x) := := M_\alpha(-x, 0)$, $x \in \mathbb{R}$.

We get similar results in the case where $m : \mathbb{R} \to \mathbb{R}$ is an additive bijection, but as a matter of fact we need not restrict α and β to special intervals. It is easy to see that only for $\alpha = -1$ and $\beta = 0$ the corresponding functions $f(x) = x$, $g(x) = 0$, $x \in \mathbb{R}$, fail to satisfy equation (1). Moreover, since functions f and g defined by (5) and (4), respectively, are additive, further on we will deal with arbitrary (invertible) additive functions.

Proposition 2. *For any invertible additive function $a : \mathbb{R} \to \mathbb{R}$ the pair $(\mathrm{id} - a^{-1}, a)$ yields a solution to* (1). *Moreover, if a pair (f, g) of functions from $\mathbb{R}$ to $\mathbb{R}$ is a solution of* (1) *and g is an invertible additive function then $f = \mathrm{id} - g^{-1}$.*

Proof. The first part follows from the direct substitution into (1). The second part is obtained from (7). ◊

Now, suppose that f is an arbitrary additive function, not necessarily invertible. Then we have the following

Theorem 2. *Let f, g be real functions on $\mathbb{R}$ and let f be an additive function. Suppose that at least one of the functions f and g is continuous. If the pair (f, g) is a solution of* (1) *then there exists an $a \in \mathbb{R}\setminus\{1\}$ such that*

$$f(x) = ax, \quad g(x) = \frac{x}{1-a}, \tag{9}$$

or there exists an even function $p : \mathbb{R} \to \mathbb{R}$ with $p(0) = 0$, continuous whenever g is continuous, such that

$$f(x) = -x, \quad g(x) = \frac{1}{2}x + p(x). \tag{10}$$

Conversely, each pair (f, g) of functions of the form (9) *or* (10) *yields a solution of* (1).

Proof. Suppose first that f is continuous, so that $f(x) = ax$, $x \in \mathbb{R}$, for some real $a \neq 1$. Equation (1) assumes the form

$$x + g(y + ax) = y + g(x + ay), \tag{11}$$

for all $x, y \in \mathbb{R}$. Substituting here $y := -x$ we obtain

$$g((1-a)x) - g((a-1)x) = 2x, \tag{12}$$

for all $x \in \mathbb{R}$, respectively. Let g_o, g_e stand for the odd and even part of the function g. From (12) we get

$$g_o((1-a)x) - g_o((a-1)x) = 2x, \quad g_o((1-a)x) = x, \quad g_o(x) = \frac{x}{1-a},$$

for all $x \in \mathbb{R}$, whence, by substituting this into (11), we obtain

$$g_e(y + ax) = g_e(x + ay), \quad x \in \mathbb{R}.$$

Therefore, g_e is constant, so that $g_e = 0$, or $a = -1$ and then g_e can be an arbitrary even function. This gives the first part of the assertion of the theorem.

Assume now that g is continuous. Since the function f is additive, we have $f(w) = cw$ for all rational w with $c := f(1)$. Equation (1) admits the form

$$(13) \qquad w + g(y + cw) = y + g(w + f(y)), \quad w \in \mathbb{Q}, \ y \in \mathbb{R}.$$

Since g is continuous, we obtain from (13) that the equalities

$$x + g(y + cx) = y + g(x + f(y)) = x + g(y + f(x)),$$

hold for every $x, y \in \mathbb{R}$, so that

$$g(y + cx) = g(y + f(x)).$$

The last equality is true for all $y \in \mathbb{R}$, so also for $y := -cx$. Using this substitution we obtain

$$g(f(x) - cx) = 0, \quad x \in \mathbb{R}.$$

When f is continuous then the earlier considerations lead us to the assertion. If f were discontinuous then, since the set $\{f(x) - cx : x \in \in \mathbb{R}\}$ is dense in $\mathbb{R}$, g would vanish identically, which is impossible. This finishes the proof of the theorem. ◊

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