ON LAGRANGE INTERPOLATION FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract: The study of Lagrange interpolation based on equidistant nodes has not been a popular subject in approximation theory. This is due to some famous divergence results discovered by C. Runge and S.N. Bernstein. However, in 1942, P. Szász established the surprising result that if a function f is of bounded variation on $[-1, 1]$ and continuous at zero then the sequence of the equidistant Lagrange interpolation polynomials converges at 0 to $f(0)$. In this note we prove that only the local behavior of f around the zero point contributes to this positive convergence phenomenon.

As is well known equidistant Lagrange interpolation polynomials need not provide a good tool in approximating a given (continuous) function on a certain interval. For a survey of contributions in this direction see, for example, Bernstein [1], Faber [2] and Runck [6].

To begin with, let us consider the equidistant interpolatory matrix

$$E = \left\{ x_j^{(n)} := -1 + \frac{2j}{n}; j = 0, 1, \dots, n; n \in \mathbb{N} \right\}$$

and let f be an arbitrarily defined function on $[-1, 1]$. Then, to each

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such f there corresponds a unique interpolating polynomial $L_n(f,.)$ of degree at most n coinciding with f at the nodes $x_j^{(n)}$, that is

$$L_n\left(f, x_j^{(n)}\right) = f\left(x_j^{(n)}\right), \quad j = 0, 1, \dots, n; n \in \mathbb{N}.$$

We refer to $L_n(f,.)$ as the Lagrange interpolating polynomial of order n corresponding to f. In 1942, P. Szász [7] established the following surprising result:

Theorem 1. *Let f be of bounded variation on $[-1, 1]$ and continuous at 0, then*

$$L_n(f, 0) \longrightarrow f(0), \quad as\ n \to \infty.$$

Note that the point 0 may not be replaced by any other point $x_0 \in (-1, 1), x_0 \neq 0$ in order to get a corresponding result, since then we have the following well known Bernstein example [1]:

Theorem 2. *Let $f(x) = |x|$ on $[-1, 1]$. Then*

(1) $$\varlimsup_{n \to \infty} |L_n(f, x_0)| = \infty \quad \forall x_0 \in (-1, 1), x_0 \neq 0.$$

In other words (1) exhibits a particularly simple function for which the interpolating polynomials diverge throughout $[-1, 1]$, except at a few points. On the other hand, it is important to see, that continuity of f (even on the whole interval) is generally not strong enough in order to establish convergence at any fixed point in the interval. This is easily to be seen from the fact that the sequence of operator norms of the linear functionals

$$L_n(., x_0): C[-1, 1] \to \mathbb{R}, \qquad f \longmapsto L_n(f, x_0)$$

is unbounded for every fixed $-1 < x_0 < 1$. Here, $C[-1, 1]$ denotes the Banach space of continuous functions equipped with the usual uniform norm $\|.\|_\infty$. This fact has the somewhat disappointing consequence that there exists a continuous function f, such that the sequence $(L_n(f, 0))_{n \geq 1}$ is unbounded itself and hence cannot converge to $f(0)$. An interesting exposition of this topic is given, for instance, in Gál [3].

Now, keeping this facts in mind, one should be inclined to think that bounded variation on $[-1, 1]$ as well as continuity at 0 is a minimal assumption in order to develope positive convergence, i.e., $L_n(f, 0) \longrightarrow$ $\longrightarrow f(0)$, as $n \to \infty$.

But surprisingly again, this is not the case. We shall prove the following result which generalizes the result of P. Szász:

Theorem 3. *Let f be of bounded variation in an arbitrary ε-neighborhood around the zero point and let f to be bounded on the rest of the*

interval $[-1, 1]$ *as well as to be continuous at* 0 *then*

$$L_n(f, 0) \longrightarrow f(0), \quad as \ n \to \infty.$$

Before we are investigating into the proof we mention that similar investigations have been performed in case of Hermite-Fejér (HFI) interpolation based on equidistant nodes. We refer the interested reader to Mills and Smith [5] and the citations therein.

Proof of the theorem. First we introduce some definitions and notation. For f to be a function of bounded variation on the interval $[-\varepsilon, \varepsilon], \varepsilon > 0$ we write $f \in BV[-\varepsilon, \varepsilon]$. Further, if $f : [-1, 1] \to \mathbb{R}$ is bounded, and $0 \le \delta < 1$, define

$$\omega_0(f; \delta) = \sup_{|t| \le \delta} |f(t) - f(0)|.$$

Note that f is continuous at 0 if and only if $\lim_{\delta \to 0} \omega_0(f; \delta) = 0$. By $\lceil x \rceil$ we denote the usual upper integer part of x. Next, we employ Lagrange's interpolation formula. Let $m \in \mathbb{N}$, $n = 2m - 1$ and $-1 \le \le x \le 1$ then

$$(2) \qquad L_{2m-1}(f, x) = \sum_{j=0}^{2m-1} f\left(x_j^{(2m-1)}\right) l_j^{(2m-1)}(x),$$

where the $l_j^{(n)}$ are the fundamental polynomials of exact degree n defined by

$$l_j^{(n)}(x) = \frac{w(x)}{\left(x - x_j^{(n)}\right) w'\left(x_j^{(n)}\right)}, \qquad (j = 0, 1, \dots, n),$$

$$w(x) = \left(x - x_0^{(n)}\right) \cdots \left(x - x_n^{(n)}\right).$$

An easy computation reveals that

$$(3) \qquad w'\left(x_j^{(2m-1)}\right) = (-1)^{j+1} \frac{(2m-1)!}{\left(m - \frac{1}{2}\right)^{2m-1} \binom{2m-1}{j}},$$

$$(j = 0, 1, \dots, 2m - 1).$$

By a more technical but also standard calculation (for the general method see Runck [6], pp. 56-57) we establish

$$w(x) = (-1)^m \frac{\cos\left[\pi\left(m-\frac{1}{2}\right)x\right]}{\pi} \cdot$$

(4)

$$\cdot \frac{\Gamma\left[1+\left(m-\frac{1}{2}\right)(1+x)\right]\Gamma\left[1+\left(m-\frac{1}{2}\right)(1-x)\right]}{\left(m-\frac{1}{2}\right)^{2m}}.$$

Combining (3) and (4) together with the well known duplication formula for the gamma function (see [4], p. 946)

$$\Gamma(2m) = (2\pi)^{-\frac{1}{2}} 2^{2m-\frac{1}{2}} \Gamma(m)\Gamma\left(m+\frac{1}{2}\right),$$

we calculate the fundamental polynomials (evaluated at 0) to the following expression. For all $j = 0, 1, \ldots, 2m-1$ we have

(5) $\qquad l_j^{(2m-1)}(0) = \dfrac{(-1)^m}{\sqrt{\pi}} \dfrac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)} \dfrac{(-1)^{j+1}}{4^{m-1}} \dfrac{\binom{2m-1}{j}}{2m-1-2j}.$

Next, without loss of generality, we may assume $f \in BV[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, f continuous at 0, f to be bounded on $[-1, 1]$, even and $f(0) = 0$. To see this, consider the function $f_1(x) = \frac{1}{2}[f(x) + f(-x)] - f(0)$ and note that by (2) and (5) we have $L_{2m-1}(f_1, 0) = L_{2m-1}(f, 0) - f(0)$ for all $m \in \mathbb{N}$. Now, since $f \in BV[-\varepsilon, \varepsilon]$, f continuous at 0 and $f(0) = 0$ recall that f can be represented on the interval $[-\varepsilon, \varepsilon]$ by the expression

(6) $\qquad\qquad\qquad\qquad f = g - h,$

where g, h are both increasing, continuous at 0 and $g(0) = h(0) = = 0$. This special representation will be important later. From (2), (5) and the fact that we may assume f to be an even function an easy computation reveals that

(7)

$$L_{2m-1}(f, 0) = \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)} \frac{1}{4^m} \sum_{j=0}^{m-1} f\left(x_{m+j}^{(2m-1)}\right) \frac{(-1)^j \binom{2m-1}{m+j}}{1+2j}.$$

We embark now on our study of the properties of (7). To this end, we select an arbitrary but fixed number η with $0 < \eta < \frac{1}{2}$ and we denote by $M = \|f\|_\infty$. For each integer $m \in \mathbb{N}$ and every such η we define the sets

$$A_1^{(m,\eta)} = \left\{ j \in \{0, 1, \ldots, m-1\} : 0 \leq j \leq \left\lceil m^{\frac{1}{2}+\eta}\right\rceil \right\},$$

$$A_2^{(m,\eta)} = \{0, 1, \ldots, m-1\} \setminus A_1^{(m,\eta)}.$$

A routine argument shows that for all $\varepsilon > 0$ there exists $m_0(\varepsilon)$ such that

for all $m > m_0(\varepsilon)$ the set of nodes consisting of $\left\{ x_{m+j} : j \in A_1^{(m,n)} \right\}$ is contained inside the interval $[0, \varepsilon]$. We estimate (7) to the expression

(8)

$$
|L_{2m-1}(f,0)| \leq \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)} \frac{1}{4^m} \left(\left| \sum_{j \in A_1^{(m,n)}} \cdots \right| + \left| \sum_{j \in A_2^{(m,n)}} \cdots \right| \right) =
$$

$$
= \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)} \frac{1}{4^m} (S_1 + S_2).
$$

We begin with the sum S_1. Since in this case we may take advantage from the property (6) we estimate the latter quantity to

(9)

$$
S_1 \leq \left| \sum_{j \in A_1^{(m,n)}} g(x_{m+j})(-1)^j \frac{\binom{2m-1}{m+j}}{1+2j} \right| +
$$

$$
+ \left| \sum_{j \in A_1^{(m,n)}} h(x_{m+j})(-1)^j \frac{\binom{2m-1}{m+j}}{1+2j} \right|.
$$

Upon employing Abel's partial summation to both parts of (9) we find after some routine computations (with $N = \left\lceil m^{\frac{1}{2}+\eta} \right\rceil$) that

(10)

$$
S_1 \leq 3 \binom{2m-1}{m} [g(x_{m+N}) + h(x_{m+N})] \leq
$$

$$
\leq 3 \binom{2m-1}{m} \left[\omega_0 \left(g; 5m^{\eta-\frac{1}{2}} \right) + \omega_0 \left(h; 5m^{\eta-\frac{1}{2}} \right) \right].
$$

Now, we turn to the second sum S_2. From (8) it is easily to be seen that

(11)

$$
S_2 \leq M \sum_{j \in A_2^{(m,n)}} \binom{2m-1}{m+j} \frac{1}{1+2j} \leq
$$

$$
\leq \frac{M}{1+2\left\lceil m^{\frac{1}{2}+\eta} \right\rceil} \sum_{j=0}^{m-1} \binom{2m-1}{m+j} \leq \frac{M}{8} \frac{4^m}{m^{\frac{1}{2}+\eta}}.
$$

Employing the duplication formula for the gamma function to (10) and combining together with (8) and (11) one establishes the estimate

$$|L_{2m-1}(f,0)| \leq \frac{12}{\pi} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)\Gamma(m+1)} \cdot$$

(12)

$$\cdot \left[\omega_0\left(g; 5m^{\eta-\frac{1}{2}}\right) + \omega_0\left(h; 5m^{\eta-\frac{1}{2}}\right)\right] + \frac{M}{\sqrt{\pi}} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)\sqrt{m}} m^{-\eta}.$$

Next, we employ the asymptotic relation (see, for example, in [4], p. 945)

$$\lim_{n\to\infty} \frac{\Gamma(n+c)}{\Gamma(n)} n^{-c} = 1, \quad (c \in \mathbb{R}),$$

to conclude that

$$|L_{2m-1}(f,0) - f(0)| \leq$$

(13)

$$\leq O(1)\left[\omega_0\left(g; 5m^{\eta-\frac{1}{2}}\right) + \omega_0\left(h; 5m^{\eta-\frac{1}{2}}\right) + m^{-\eta}\right].$$

which holds for all $m > m_0(\varepsilon)$. Obviously, the $O(1)$ constant is independent of m. From this fact and since $0 < \eta < \frac{1}{2}$ it follows that the right-hand side of (13) tends to 0, as $m \to \infty$. $\Diamond$

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