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SOME LARGE AND SMALL SETS IN TOPOLOGICAL GROUPS

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Abstract: Large, small and medium subsets of a group are studied, namely for the integers, for totally bounded groups, for locally compact and σ-compact groups.

In this paper all groups are abelian and infinite. In [1] some concepts of size for subsets of a group are provided. Some of them are connected with similar notions already used in the literature under

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different names (big [4], discretely syndetic [6], [7], relatively dense [8]). We recall the definitions of [1] for Abelian groups.

A subset L of a group G is said to be large if there exists a finite subset F such that $L + F = G$. A subset S of G is said to be small if for every finite subset F the complement of $L + F$ is large. A set which is neither large nor small is said to be medium. Obviously, every finite subset is small. A different notion of smallness is given in [5].

A set of the form $g + F$, where $g \in G$ and $F \subseteq G$, is said to be a circle of radius F. Circles are useful in the following criterium [1].

Proposition 0.1. *The set $G \setminus A$ is not large in G iff A contains circles of any finite radius.*

The purpose of Section 1 is studying the above concepts in the group of the integers. In particular, we prove that the set of prime integers is small in $\mathbb{Z}$. In view of this result it is enough to study large and small subsets in the natural numbers.

In Section 2 we answer in the negative to the question whether there exists a topological group in which a set is large iff it has non-empty interior [1]. We also prove that in totally bounded groups closed subsets cannot be medium; furthermore, every countable unbounded group has a closed discrete subset which is medium.

In Section 3 we discuss some properties connected to category and measure of sets. In particular, every locally compact σ-compact group has a small dense subset which is meager and every compact group has a small closed subset of arbitrarily large measure.

1. The naturals

In this section we are going to study large and small subsets of $\mathbb{N}$.

A subset L of $\mathbb{N}$ is said to be large in $\mathbb{N}$ if there exists a finite subset K of $\mathbb{N}$ such that $L \pm K \supseteq \mathbb{N}$.

A subset S is said to be small in $\mathbb{N}$ if for every finite subset K of $\mathbb{N}$ we have that the complement of $S \pm K$ is large in $\mathbb{N}$.

If $X \subseteq \mathbb{N}$, we shall write $X = \{x_0, x_1, \dots, x_n, \dots\}$, where the sequence x_n is increasing.

Proposition 1.1. *An infinite subset $X \subseteq \mathbb{N}$ is large iff $\delta = \sup\{x_{n+1} - x_n\} < +\infty$.*

Proof. Suppose $\delta < +\infty$ and let $m = \max\{\delta, x_0\}$. Then $X \pm \{0, 1, \dots, m\} \supseteq \mathbb{N}$. Conversely, let $F = \{0, 1, \dots, m\}$ such that $X \pm F \supseteq \mathbb{N}$. Then $\delta \leq 2m + 1$. ◊

The previous proposition says that X is large iff $\limsup(x_{n+1} - x_n) < +\infty$. For small subsets, only a sufficient condition is available.

Proposition 1.2. *If* $\lim(x_{n+1} - x_n) = +\infty$ *then* X *is small.*

Proof. For every finite subset $K = \{0, 1, \dots, d\}$, put $(X \pm K) \cap \mathbb{N} = Y$. Choose $\bar{n}$ such that $x_{n+1} - x_n > 2d + 1$ for $n > \bar{n}$. Then $Y \subseteq Z = [0, x_{\bar{n}} + d] \cup (\bigcup_{n>\bar{n}}[x_n - d, x_n + d])$. If a and b are consecutive elements of $\mathbb{N} - Z$, then $b - a \leq 2d + 2$ and thus $\mathbb{N} - Z$ is large by Prop. 1.1. ◊

Corollary 1.3. *For every integer* $q > 1$ *the set* $T = \{q^k : k \in \mathbb{N}\}$ *is small.*

Corollary 1.4. *If* X *is a medium subset of* $\mathbb{N}$, *then* $x_{n+1} - x_n$ *has no limit.*

In the following examples we show that if $\liminf(x_{n+1} - x_n) < +\infty$ and $\limsup(x_{n+1} - x_n) = +\infty$, the set X may be either medium or small (therefore the Prop. 1.2 cannot be improved).

Example 1.5. Let $S = T \cup (T + \{1\})$, where $T = \{2^k : k \in \mathbb{N}\}$. Obviously S is small and $1 = \liminf(x_{n+1} - x_n) < \limsup\{x_{n+1} - x_n) = +\infty$.

Example 1.6. Let $M = \bigcup_{n\in\mathbb{N}}[2^n - n, 2^n + n]$. By Prop. 1.1, neither M nor $\mathbb{N} - M$ are large. Therefore M is medium.

An interesting question is to investigate whether the set of prime numbers is small in $\mathbb{N}$. The answer is positive. We need some preliminaries. Let us denote by P_r the subset of all natural numbers which have not more than r prime divisors, not necessarily different from each other. Thus $P_0 = \{1\}$ and P_1 is the set of prime numbers.

Proposition 1.7. *Let* $Q = \{q_1, q_2, \dots, q_l\}$ *be a set of* l *consecutive natural numbers such that* $Q \cap P_r = \emptyset$. *Let* M *be the least common multiple of these numbers and put* $q'_i = q_i + M$. *The set of consecutive numbers* $Q' = \{q'_i : i = 1, \dots, l\}$ *is disjoint from* P_{r+1}.

Proof. It is enough to observe that $q'_i = q_i\left(1 + \frac{M}{q_i}\right)$. ◊

Lemma 1.8. *Let* $l \geq 1$ *be a natural number. For every* $r \geq 0$ *there exists a set* Q_r *consisting of* l *consecutive numbers such that* $Q_r \cap P_r = \emptyset$.

Proof. By using Prop. 1.7, we can find Q_r inductively by putting $Q_0 = \{2, 3, \dots, l + 1\}$, $Q_1 = Q'_0$, $\dots$ $Q_r = Q'_{r-1}$. ◊

Theorem 1.9. *The set* P_r *is small for every* $r \in \mathbb{N}$.

Proof. We shall prove that for every subset $K = \{0, 1, \dots, k\}$ the set $\mathbb{N} \setminus (P_r \pm K)$ is large. Let $l = 2k + 1$ and choose subsets Q_s as in Lemma 1.8. If $Q_{r-1} = \{q_1, q_2, \dots, q_l\}$, then simply put $W = Q_r = \{M + q_1, \dots, M + q_l\}$, where M is the least common multiple of q_i's.

For each $m \geq 1$ let $W_m = \{mM + q_1, \dots, mM + q_{k+1}, \dots, mM + q_l\}$. Again $W_m \cap P_r = \emptyset$. Since $mM + q_{k+1} \pm K \subseteq W_m$, we have that $mM + q_{k+1} \notin P_r \pm K$. Thus the set $B = \{mM + q_{k+1} : m = 1, 2, \dots\}$ is contained in $\mathbb{N} \setminus (P_r \pm K)$. By Prop. 1.1, the set B is large in $\mathbb{N}$. $\Diamond$

Corollary 1.10. *For every $r \in \mathbb{N}$ the set $P_r \cup (-P_r)$ is small in $\mathbb{Z}$.*

Corollary 1.11. *The set of prime numbers is small in $\mathbb{N}$. The set of prime integers is small in $\mathbb{Z}$.*

If X is a subset of $\mathbb{N}$, we denote by $X \cap n$ the set $\{x \in X : x_n < n\}$ and by $|X|$ the power of the set X.

Definition 1.12. The *asymptotic density* of a set X is $d(X) = \lim_{n \to +\infty} \frac{|X \cap n|}{n}$, if it exists.

Proposition 1.13. *If a large subset X has asymptotic density $d(X)$, then $d(X) > 0$.*

Proof. Prop. 1.1 implies $d(X) > \frac{1}{d}$. $\Diamond$

The asymptotic density does not necessarily exist for large, small and medium subsets (examples may be easily found). The Ex. 1.6 is a medium set with asymptotic density 0. On the other hand, small subsets may have asymptotic density arbitrarily close to 1.

Proposition 1.14. *For every $\varepsilon > 0$ there exists a small subset $S \subseteq \mathbb{N}$ such that $d(S) > 1 - \varepsilon$.*

Proof. Take $m \in \mathbb{N}$ such that $\sum_{j=1}^{+\infty} \frac{2j-1}{m^j} < \varepsilon$ and for every $j \geq 1$ let $T_j = \bigcup \{rm^j \pm \{0, 1, \dots, j-1\} : r = 1, 2, \dots\}$. It is easy to check that $d(T_j) = \frac{2j-1}{m^j}$. Put $T = \cup_j T_j$ and $S = \mathbb{N} \setminus T$. Since $d(T) \leq \sum_{j=1}^{+\infty} d(T_j) < \varepsilon$, we have $d(S) > 1 - \varepsilon$.

For every finite subset $K = \{0, 1, \dots, k\}$ there exists j such that $rm^j \notin S \pm K$ for each $r > 0$. Therefore the complement of $S \pm K$ is large for every finite subset K. $\Diamond$

2. Bounded and unbounded groups

We say that a subset B of a topological group G is bounded if for every neighborhood V of the neutral element there exists a finite subset F of G such that $F + V \supseteq B$ (the set F may be chosen to be a subset of B). In the converse case, we say that B is unbounded. If G itself is bounded, then it is said to be a totally bounded topological group.

Proposition 2.1. *Let G be a topological group. The following are equivalent:*

(i) *G is totally bounded.*
(ii) *Every non-empty open subset is large.*
(iii) *Every neighborhood of the neutral element is large.*

Since a nowhere dense set cannot be large, we have immediately the following proposition.

Proposition 2.2. *Let G be a totally bounded topological group. A closed subset is small iff it is nowhere dense. Otherwise it is large, i.e. there are no closed medium subsets.*

We have the following strengthening of the Prop. 2.1.

Proposition 2.3. *A topological group is unbounded iff there exists a neighborhood of 0 which is small.*

Proof. The sufficiency is obvious by Prop. 2.1. Conversely, let V be a closed neighborhood of 0 such that $F+V \neq G$ for every finite subset F. Let W be a closed neighborhood of 0 such that $W \pm W \subseteq V$. We must prove that for every finite subset F the open set $G \setminus (F+W)$ is large. Take $h \in G \setminus (F+V)$. It is easy to check that $(h+W) \cap (F+W) = \emptyset$. Therefore, by traslating $G \setminus (F+W)$ with the elements of the finite subset $-h+F$, we cover $F+W$. $\Diamond$

In [1] it is asked whether there exists a topological group in which large subsets are exactly subsets with non-empty interior. By Prop. 2.3, such an example must be a totally bounded group. This group cannot be compact, because in [1] it is proved that every compact group has a dense small subset. In the next theorem we prove that the answer is negative by showing that every totally bounded group has a large subset which is codense.

Theorem 2.4. *Every totally bounded group has a large subset with empty interior.*

Proof. Let G be a totally bounded group. If G is countable, by [2] there exists a subset S which is dense and small. Otherwise, let H be a countable subgroup of G and let S be a subset of H which is small and dense in H. The quotient group G/H is infinite. Choose an element g_λ in every coset and put $E(S) = \bigcup_\lambda (g_\lambda + S)$. The subset $E(S)$ is dense. It remains to prove that $G \setminus E(S)$ is large.

We have $G \setminus E(S) = \bigcup_\lambda (g_\lambda + (H \setminus S))$. Since $H \setminus S$ is large in H, there exists a finite subset F of H such that $F + (H \setminus S) \supset S$.

Therefore:

$$F + (G \setminus E(S)) = F + \bigcup_\lambda (g_\lambda + (H \setminus S)) = \\ = \bigcup_\lambda (g_\lambda + F + (H \setminus S)) \supset \bigcup_\lambda (g_\lambda + S) = E(S). \Diamond$$

By Prop. 2.3, in unbounded topological groups there exists a base of neighborhoods of 0 which are closed and small. The situation is curious since there exist closed nowhere dense subsets which fail to be small (hence they are medium).

Theorem 2.5. *In $\mathbb{R}^n$ there exists a closed nowhere dense subset which is medium; furthermore, every closed discrete subset is small.*

Proof. It is enough to construct a dense open subset which is not large. Let $D = \{v_j\}$ be a countable dense subset of $\mathbb{R}^n$ and consider the set $A = \bigcup B_j$, where B_j is the open ball of radius $\frac{1}{2^j}$ centered at v_j. For every finite subset F of $\mathbb{R}^n$, the Lebesgue measure of $A + F$ is finite, hence $A + F$ cannot coincide with $\mathbb{R}^n$. The second assertion holds because every closed discrete subset is countable. $\Diamond$

One may ask whether the group $\mathbb{Q}$ of rational numbers have a nowhere dense subset which is medium. A consequence of the next theorem is that every countable unbounded topological group has a closed discrete subset which is medium.

Lemma 2.6. *Let G be an unbounded topological group and let U be a symmetric neighborhood of 0 such that $K + U + U \neq G$ for every finite subset K. Let A and B be subsets of G which are contained in a finite union of translations of U. Then there exists $z \in G$ such that $A \cap (B + z) = \emptyset$.*

Proof. Routine. $\Diamond$

Theorem 2.7. *In any unbounded topological group G with a countable network there exists a discrete family consisting of translations of network elements whose union is medium.*

Proof. Let $\mathcal{N} = \{N_i : i \in \omega\}$ be a countable network. Since the group is unbounded, there exists a symmetric neighborhood V of 0 such that $(V + V + V + V) + K \neq G$ for each finite subset K of G. It is not restrictive to assume that for each $N \in \mathcal{N}$ there exists $x_N \in G$ such that $N \in x_N + V$.

Let $P_n = \bigcup\{N_i : i < n\}$. Let $\{w_i : i \in \omega\}$ be a numeration of all finite discrete families of network elements and let $W_i = \cup w_i$. We are going to construct the desired family by induction.

Let us assume that we have found some $z_i, g_i \in G$ for $i < n$ and let $M_n = \bigcup\{W_i + z_i : i < n\}$. It is clear that $M_n + P_n$ is contained in the union of finitely many translations of $V+V$; therefore $M_n + P_n \neq G$. Let g_n be an element of $G \setminus (M_n + P_n)$. Since the sets $(M_n + V) \cup \{g_i : i \leq n\}$ and $W_n + P_n$ are contained in the union of finitely many translations of $V+V$, by Lemma 2.6 we can choose an element z_n such that

$$\big((M_n + V) \cup \{g_i : i \leq n\}\big) \cap (W_n + P_n + z_n) = \emptyset.$$

After finishing our induction, let us put $M = \bigcup\{M_n : n \in \omega\}$. By the properties of our construction, it is clear that M is the union of a discrete family of translations of network elements.

Let us prove that M is medium. Let K be a finite subset of G. Then K is a subset of P_n for some $n \in \omega$. Thus $M + K \subseteq M + P_n \subseteq (M_n + P_n) \cup (\bigcup_{j>n} W_j + P_j + z_j)$, and therefore $g_n \notin M + K$; consequently M is not large. On the other side, K is contained in W_m for some $m \in \omega$ and hence $K + z_m \subseteq W_m + z_m \subseteq M$; consequently M is not small because it contains circles of every finite radius. ◊

Corollary 2.8. *Every countable unbounded group has a discrete closed medium subset.*

3. Small sets and measure

In this section, groups are not required to be commutative.

Let G be a locally compact topological group and let λ be a left Haar measure on G.

In [1] it is observed that a compact group has a dense subset which is small. We are going to prove that every locally compact σ-compact group has a small dense subset which is meager. First we need to describe the behaviour of large and small subsets with regard to surjective homomorphisms (the proof is straightforward, see e.g. [9]).

Proposition 3.1. *Let $f : G \longrightarrow H$ be a surjective homomorphism.*

1. *If E is large in G, then $f(E)$ is large in H.*
2. *If F is large in H, then $f^{-1}(F)$ is large in G.*
   If $f(E)$ is small in H, then E is small in G.
3. *If $f^{-1}(F)$ is small in G, then F is small in H.*

The following proposition is useful in the sequel.

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a surjective open continuous map between two topological spaces. Then:*

(i) $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ *for every $S \subseteq Y$.*

(ii) *If M is a meager subset of Y, then $f^{-1}(M)$ is a meager subset of X.*

We give a reformulation of a well-known theorem.

Theorem 3.3 (Kakutani–Kodaira). *If G is a locally compact σ-compact group and Ω is a $\mathcal{G}_\delta$-set containing the identity, then there exists a compact normal subgroup $N \subseteq \Omega$ such that G/N is metrizable. Moreover, if G is not discrete, N may be chosen in such a way that G/N is not discrete.*

Proof. We prove only the latter statement (for the former one, see [10, 8.7]). Suppose that G is not discrete and G/N is discrete. Then N is open and we can choose an open neighborhood V_1 of the identity which is strictly contained in N. There exist a compact normal subgroup N_1 contained in V_1 such that G/N_1 is metrizable. If G/N_1 is not discrete, the proof is concluded. Otherwise we can proceed: if the process is infinite, we get a strictly monotone sequence of normal compact open subgroups N_k such G/N_k is discrete. Let $N_\infty = \cap N_k$. Since the $\mathcal{G}_\delta$-set N_∞ is the intersection of compact open subgroups, the quotient G/N_∞ is metrizable. Furthermore, N_∞ is not open because N is compact and N/N_∞ is infinite. $\Diamond$

Theorem 3.4. *Let G be a non-discrete group and suppose that G is locally compact and σ-compact. Then:*

1. *There exists a dense small subset which is meager.*
2. *There exists a residual set F with $\lambda(F) = 0$. Therefore F is medium.*
3. *There exists a meager set E such that $\lambda(E) = \lambda(G)$. Therefore E is medium.*
4. *If G is compact, then for every real number $r < \lambda(G)$ there exists a closed small set S such that $\lambda(S) > r$.*

Proof. 1. If G is metrizable, it contains a countable dense subset $D = \{b_n\}$. Clearly D is meager. Furthermore it is small since G has the power of continuum [3, Th. 3.9].

If G is not metrizable, take the quotient map $f : G \longrightarrow G/N$, where N is the subgroup of Th. 3.3 (choose N in such a way that $G/$ $/N$ is not discrete). The quotient map is open and continuous and G/N is metrizable, locally compact and σ-compact. Take a countable dense subset $D \subseteq G/N$. Then $f^{-1}(D)$ is small by Prop. 3.1 (3), is dense because f is open. Since G/N is not discrete, then N has empty interior and therefore $f^{-1}(D)$ is meager.

2. In the proof of 1, we have shown that the subgroup N of Th. 3.3 may be choosen in such a way that G/N is metrizable and non-discrete. Consider a countable dense subset $D = \{b_n\}$ of G/N. Let μ the Haar measure on G/N defined by $\mu(Y) = \lambda(f^{-1}(Y))$. For every $n \in \mathbb{N}$ and for each positive $\varepsilon \in \mathbb{Q}$, take an open neighborhood $V_n(\varepsilon)$ of b_n such that $\mu(V_n(\varepsilon)) < \frac{\varepsilon}{2^n}$. Consider the open set $A(\varepsilon) = \bigcup_{n>0} V_n(\varepsilon)$. Since $\mu(A_\varepsilon) < \varepsilon$, the dense $\mathcal{G}_\delta$ subset of G/N given by $\Omega = \bigcap_\varepsilon A_\varepsilon$ has measure 0. Therefore $M = f^{-1}(\Omega)$ is a residual subset of G with λ-measure equal to 0. M is not large because it is null; M is not small because $G \setminus M$ is meager and G is a Baire space (it is Čech-complete).

3. It is enough to take $E = G \setminus M$, where M is the residual subset constructed in the proof of 2. Notice that E is a meager $\mathcal{F}_\sigma$ set.

4. Let $r < \lambda(G)$ and choose E as in 3. Then $E = \bigcup_n F_n$, where F_n is an increasing sequence of closed nowhere dense subsets of G. Since $\lambda(E) = \sup_n \lambda(F_n)$, there exists $\overline{n}$ such that $\lambda(F_{\overline{n}}) > r$. By Prop. 2.2, $F_{\overline{n}}$ is small. $\diamondsuit$

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