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CHAIN OF DENDRITES WITHOUT OPEN SUPREMUM

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Abstract: We consider an ordering with respect to open mappings on the class of all dendrites and construct a bounded chain of dendrites which does not have a supremum answering a question of its existence.

A chain of n-ods

All spaces considered in this paper are assumed to be metric. A *continuum* means a nonempty compact connected space. A *simple closed curve* is any space which is homeomorphic to the unit circle. A *dendrite* means a locally connected continuum containing no simple closed curve.

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A *mapping* means a continuous function. A surjective mapping $f : X \to Y$ is said to be *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [2, §51, I, p. 274], and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$ or just $\operatorname{ord}(p)$ if there is no risk of confusion. Points of order 1 in a continuum X are called *end points* of X. Points of order 2 are called *ordinary points* of X. Points of order at least 3 are called *ramification points* of X; the set of all ramification points of X is denoted by $R(X)$.

Let S be a connected space, and let $p \in S$. If $S \setminus \{p\}$ is not connected, then p is called a *cut point of S*. An *arc* is any space which is homeomorphic to $[0; 1]$. We denote by pq an arc with end points p and q.

In [1, p. 7] J. J. Charatonik, W. J. Charatonik and J. R. Prajs introduced a quasiordering $\leq_O$ on the class of all dendrites. We recall the definition. If X, Y are dendrites, then $Y \leq_O X$ if there is a surjective open mapping $f : X \to Y$. This quasiordering is not an ordering (i.e. $X \leq_O Y$ & $Y \leq_O X$ does not imply X is homeomorphic to Y). To guarantee this they said that X and Y are $\mathbb{O}$-*equivalent* iff $X \leq_O Y$ and $Y \leq_O X$ and then considered the quotient of the class of all dendrites. The quasiordering $\leq_O$ induces the ordering $\leq_{\mathbb{O}}$ on the quotient.

J. J. Charatonik, W. J. Charatonik and J. R. Prajs posed a question if every chain bounded with respect to this ordering has a supremum (see [1, §7, Q4(b)$\mathbb{O}$, p. 51]). In Theorem we give a negative answer to this question by proving that the sequence of n-ods has no supremum (for $n \in \mathbb{N}$, $n \geq 3$, denote by S_n the n-od, i.e. the dendrite where the only ramification point is of order n).

Fact. *Let X be a dendrite.*

(a) *Each point of X is either a cut point or an end point.*
(b) *For each point $x \in X$ we have* $\operatorname{ord}(x) = c(x)$, *where $c(x)$ is the number of components of $X \setminus \{x\}$, whenever either of these is finite.*
(c) *For every subcontinuum $Z \subset X$ every component of $X \setminus Z$ is open in X.*
(d) *Let $f : X \to Y$ be a surjective open mapping. Then Y is a dendrite.*
(e) *The order of a point is never increased under an open mapping.*

See [3, Chapter 10, Th. 10.7, p. 168], [3, Chapter 10, Th. 10.13, p. 170], [3, 5.22(a), p. 83], [3, Cor. 13.41, p. 297] and [4, Chapter 8, (7.31), p. 147].

Observation. *Let D be a dendrite where ramification points* (a) *form a finite set (say of cardinality k) and* (b) *are of a finite order. If some ramification point r of D is of order $m \geq 3$, then $S_n \leq_O D$ for each $n \leq m$.*

Proof. Easy by induction on k. If $k = 1$ we identify some arcs starting from r and obtain a dendrite S_n (via an open mapping). If $k > 1$ we find a ramification point $p \neq r$ such that there is exactly one component P of $D \setminus \{p\}$ containing some ramification point of D. We identify the other components (arcs) starting from p and obtain (via an open mapping) a dendrite D' with $k-1$ ramification points (the ramification point p of D has turned into an ordinary point of D'). $\lozenge$

Theorem. *The chain $S_3 \leq_O S_4 \leq_O \cdots \leq_O S_n \leq_O \cdots$ is bounded and has no supremum.*

Proof. We finish the proof in 22 steps.

(1) For $n \in \mathbb{N}$, $n \geq 3$, $1 \leq k \leq n$, $k \in \mathbb{N}$ we set $a_n^k = (0, 2^{-n} + 2^{-2nk})$, $b_n = (2^{-n}, 0)$, $c_n = (2^{-n-1}, 2^{-n-1})$, $a = (0,0)$, $b = (1,0)$,

$$B_n = \bigcup_{k=1}^{n} a_n^k b_n, \quad B = ab \cup \bigcup_{n=3}^{\infty} B_n,$$

$$C_n = b_n c_n \cup \bigcup_{k=1}^{n} a_n^k c_n, \quad C = ab \cup \bigcup_{n=3}^{\infty} C_n.$$

Both B and C are dendrites (see Fig. 1-2).

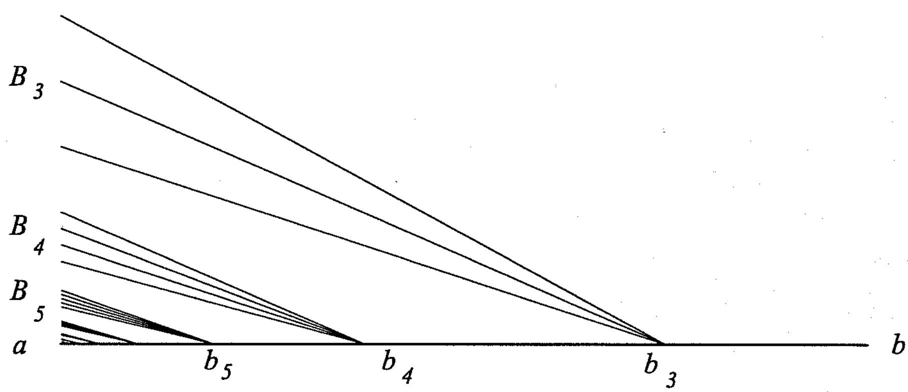

Figure 1 – dendrite B

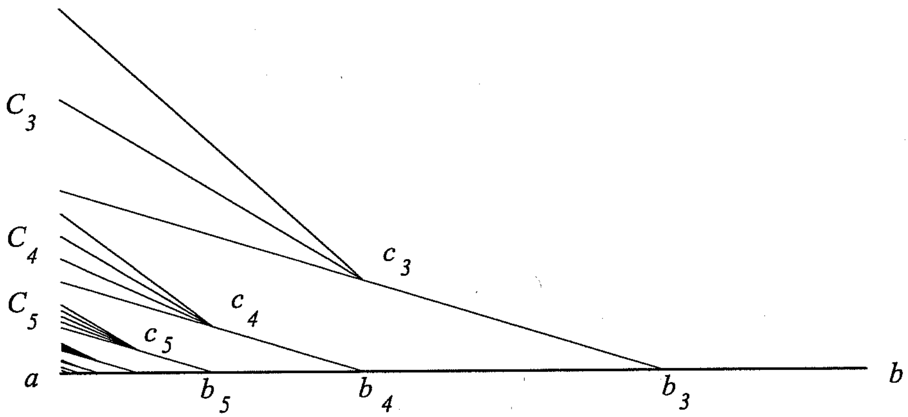

Figure 2 – dendrite C

(2) We see that both a and b are end points of both B and C.

(3) Ramification points of B are exactly the points b_n, $n \in \mathbb{N}$, $n \geq 3$, $R(B)$ is infinite, the only cluster point of $R(B)$ is a, $\operatorname{ord} p$ is finite for each $p \in R(B)$.

(4) Ramification points of C are exactly points b_n and c_n, $n \in \mathbb{N}$, $n \geq 3$, $R(C)$ is infinite, the only cluster point of $R(C)$ is a, $\operatorname{ord} p$ is finite for each $p \in R(C)$.

(5) $S_n \leq_O B$ for $n \in \mathbb{N}$, $n \geq 3$.

(Proof. Using the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(x, 0)$, $x, y \in \mathbb{R}$, we map $\cup_{k=n+1}^{\infty} B_k$ onto ab_{n+1}; then using the identity elsewhere in B we obtain an open image $\mathcal{B}_n = ab \cup \bigcup_{k=3}^{n} B_k$ of the dendrite B. The dendrite $\mathcal{B}_n$ has finitely many ramification points, all of them of finite order and at least one of order $n+2$, hence Observation applies and $S_n \leq_O \mathcal{B}_n \leq_O B$.)

(6) $S_n \leq_O C$ for $n \in \mathbb{N}$, $n \geq 3$.

(Proof. The projection π maps $\cup_{k=n+1}^{\infty} C_k$ onto ab_{n+1}; then using the identity elsewhere in C we obtain an open image $\mathcal{C}_n = ab \cup \cup \bigcup_{k=3}^{n} C_k$ of the dendrite C. The dendrite $\mathcal{C}_n$ has finitely many ramification points, all of them of finite order and at least one of order $n+1$, hence Obs. applies and $S_n \leq_O \mathcal{C}_n \leq_O C$.)

(7) Suppose that there exists a dendrite D such that the following three conditions hold:

(i) $D \leq_O B$ (with a surjective open mapping $f : B \to D$),

(ii) $D \leq_O C$ (with a surjective open mapping $g : C \to D$),

(iii) $S_n \leq_O D$ for each $n \in \mathbb{N}$, $n \geq 3$.

We conclude a contradiction in (22).

(8) Points in $R(D)$ are of a finite order in D due to (3), (7):(i) and Fact (e); moreover $R(D)$ is infinite due to (7):(iii).

(9) We fix a neighborhood P of b in B such that $P \cap R(B) = \emptyset$ due to (3), and denote $\hat{P} = f(P)$. Then $\hat{P}$ is a neighborhood of $\hat{b} = f(b)$ due to (7):(i), hence $\hat{P} \cap R(D) = \emptyset$ due to Fact (e).

(10) Moreover, for each neighborhood Q of a in B the set $R(B) \setminus Q$ is finite due to (3). $R(D) \setminus f(Q)$ is finite due to (7):(i) and Fact (e). We conclude that $\hat{a} = f(a) \notin \hat{P}$ due to (9).

(11) We see that $\hat{a} \notin \hat{P}$, $\hat{b} \in \hat{P}$, hence $\hat{a} \neq \hat{b}$. Moreover $\hat{a}$ and $\hat{b}$ are end points of D due to (2), Fact (e) and (7):(i). Denote by J the arc $\hat{a}\hat{b}$.

(12) CLAIM: $R(D) \subset J$. Suppose not. We conclude a contradiction.

(13) Suppose that there exists $\hat{p} \in R(D) \setminus J$. Then we find $p \in \in R(B)$ such that $f(p) = \hat{p}$ due to (7):(i) and Fact (e).

(14) We find $\hat{r} \in R(D) \cap J$ such that $\hat{a}$, $\hat{b}$ and $\hat{p}$ are in different components of $D \setminus \{\hat{r}\}$ due to (11) and (13).

(15) $f(ap)$ is connected, hence meets $\hat{r}$ due to (14) and there exists $r_a \in R(B)$ such that $f(r_a) = \hat{r}$, $r_a \in ap \setminus \{a, p\}$ due to (7):(i), (14) and Fact (e).

(16) Similarly $f(bp)$ is connected, meets $\hat{r}$ due to (14); hence there exists $r_b \in R(B)$ such that $f(r_b) = \hat{r}$, $r_b \in bp \setminus \{b, p\}$ due to (7):(i), (14) and Fact (e).

(17) Denote by V the component of $B \setminus \{r_a\}$ containing a. Thus V is open due to Fact (c). Denote $U = B \setminus V$. Hence U is closed and contains only finitely many ramification points of B due to (3). Then $f(U)$ contains only finitely many ramification points of D due to (7):(i) and Fact (e). We conclude $f(U) \neq D$ due to (8).

(18) Observe that $f(U)$ is closed due to (7):(i) and (17); $U \setminus \{r_a\}$ is open due to (3) and Fact (c); $f(U \setminus \{r_a\})$ is open due to (7):(i). Moreover, $f(U \setminus \{r_a\}) = f(U)$ because $r_b \in U$ due to (17) and $f(r_a) = = f(r_b)$ due (15) and (16). Hence $f(U)$ is both open and closed in D. We conclude that $f(U) = D$ because D is connected due to (7).

(19) The conclusions in the above two items cannot hold simultaneously — a contradiction. Hence CLAIM holds.

(20) We choose three different ramification points $\hat{x}$, $\hat{y}$ and $\hat{z}$ in D of order at least 4 such that $\hat{x} \in \hat{a}\hat{y}$ and $\hat{y} \in \hat{a}\hat{z}$ due to (7):(i), (8) and (12).

(21) We find three different ramification points x, y and z of C such that $g(x) = \hat{x}$, $g(y) = \hat{y}$ and $g(z) = \hat{z}$ due to (7):(ii) and Fact (e). Obviously, x, y and z are of order at least 4 due to Fact (e). Hence $x = c_i$, $y = c_j$ and $z = c_k$ for some $i, j, k \in \mathbb{N}$ due to (1), $xz \setminus \{x, z\}$ contains ramification points (in C) of order 3 only due to (1) and (4). Hence $g(xz)$ contains at most two points of order greater than 3 due to Fact (e).

(22) $\{\hat{x}, \hat{y}, \hat{z}\} \subset g(xz)$ due to (20), hence $g(xz)$ contains at least three points of order greater than 3 due to (20).

The above two items cannot hold simultaneously. Hence (7):(i)-(iii) lead to a contradiction. We have proved that

$$S_3 \leq_O S_4 \leq_O S_5 \leq_O \cdots$$

is a bounded chain (bounded by both B and C due (5) and (6)) with no supremum due to (7). $\Diamond$

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