Mathematica Pannonica
12/2 (2001), 201 – 215

EXISTENCE OF SOLUTIONS FOR FUNCTIONAL ANTIPERIODIC BOUNDARY VALUE PROBLEMS

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Received: February 2000

MSC 2000: 34 K 10

Keywords: Antiperiodic boundary value problem, functional differential equation.

Abstract: This paper contains several comparison results which are useful to show the existence of solutions to antiperiodic boundary value problems of functional differential equations.

1. Introduction

Let us consider the following antiperiodic boundary value problem for functional differential equations of the form

$$(1)\qquad \begin{cases} x'(t) = f(t, x_t), & t \in J = [0, T], \quad T > 0, \\ x(s) = x(0) = -x(T), & s \in J_0, \end{cases}$$

where $f \in C(J \times C_0, \mathbb{R})$, $C_0 = C(J_0, \mathbb{R})$ with $J_0 = [-\tau, 0]$ for $\tau > 0$, and for any $t \in J$, $x_t \in C_0$ is defined by $x_t(s) = x(t+s)$ for $s \in J_0$.

Note that the differential equation from problem (1) is a very general type. It includes, for example, as special cases, ordinary differential equations if $\tau = 0$, differential–difference equations, and integro–differential equations too (see, for example [3]).

The method of lower and upper solutions is useful to obtain approximate solutions to differential equations (for details, see, for example [4], [5]). Recently, this method has been extended so as to be applicable to a much larger class of nonlinear problems (see, for example [2], [4]–[8]). The purpose of this paper is to show that it can be applied successfully to antiperiodic boundary value problems of functional differential equations. Under some assumptions on f it is shown that linear monotone iterations converge to the unique solution of our problem. A particular case of (1), [namely when $f(t, x_t) = f(t, x(t))$], is investigated in [8], see also [5].

2. Assumptions

Put $C_1 = C(\bar{J}, \mathbb{R}) \cap C^1(J, \mathbb{R})$ with $\bar{J} = [-\tau, T]$. Two functions $u, v \in C_1$ are called weakly coupled (w.c.) lower and upper solutions of problem (1) if

$$\begin{cases} u'(t) \le f(t, u_t), & t \in J, \quad u(s) = u(0) \le -v(T), \quad s \in J_0, \\ v'(t) \ge f(t, v_t), & t \in J, \quad v(s) = v(0) \ge -u(T), \quad s \in J_0. \end{cases}$$

Now, we list the following assumptions for later use.

H_1 $f \in C(J \times C_0, \mathbb{R})$,

H_2 $y_0, z_0 \in C_1$ are w.c. lower and upper solutions of (1) and $y_0(t) \le$ $\le z_0(t)$ on J,

H_3 there exists $N > 0$ such that for $u, v \in C_0$, $y_{0,t} \le u \le v \le z_{0,t}$, $t \in J$, function f satisfies the one-sided Lipschitz condition of the form

$$f(t, v) - f(t, u) \ge N \int_{-\tau}^{0} [v(s) - u(s)]ds, \quad \tau > 0,$$

H_4 there exists $L > 0$ such that for $u, v \in C_0$ we have

$$|f(t, u) - f(t, v)| \le L \max_{0 \le r \le t} |u(r) - v(r)|.$$

3. Some lemmas

We need some comparison results.

Lemma 1. *Let Assumptions H_1 and H_4 hold. Then the initial problem*

$$(2)\ \ w'(t) = f(t, w_t) + N \int_{-\tau}^{0} w(t+r)dr, \quad t \in J, \quad w(s) = w(0), \quad s \in J_0$$

has a unique solution.

Proof. Put $|u|_* = \max_{t\in J}[e^{-Kt}|u(t)|]$ with $K > L + |N|\tau$. Integrating (2) we obtain

$$w(t) = w(0) + \int_0^t \left[f(s, w_s) + N \int_{-\tau}^{0} w(s+r)dr \right] ds \equiv Aw(t), \quad t \in J.$$

Then

$$|Aw - A\bar{w}|_* =$$

$$= \max_{t\in J} e^{-Kt} \left| \int_0^t \left\{ f(s, w_s) - f(s, \bar{w}_s) + N \int_{-\tau}^{0} [w(r+s) - \bar{w}(r+s)]dr \right\} ds \right| \leq$$

$$\leq (L + |N|\tau) \max_{t\in J} e^{-Kt} \int_0^t \max_{0\leq u\leq s} \left[|w(u) - \bar{w}(u)| e^{-Ku} e^{Ku} \right] ds \leq$$

$$\leq \frac{L + |N|\tau}{K} |w - \bar{w}|_* \left[1 - e^{-KT} \right].$$

By Banach fixed point theorem, problem (2) has a unique solution since $1 - e^{-KT} < 1$ and $K > L + |N|\tau$.

It ends the proof. ◊

Remark 1. Note that the problem

$$w'(t) = N \int_{-\tau}^{0} w(t+r)dr, \quad t \in J, \quad w(s) = w(0), \quad s \in J_0$$

has still a unique solution, and if $w(0) = 0$, then $w(t) = 0,\ t \in J$ is this unique solution.

Lemma 2. *Let $N > 0$ and $m \in C(J, \mathbb{R})$. Then the problem*

(3)
$$\begin{cases} \alpha'(t) = N\int_{-\tau}^{0} \alpha(t+s) + m(t), \ t \in J, \ \alpha(s) = \alpha(0) = -\beta(T), \ s \in J_0, \\ \beta'(t) = N\int_{-\tau}^{0} \beta(t+s) - m(t), \ t \in J, \ \beta(s) = \beta(0) = -\alpha(T), \ s \in J_0 \end{cases}$$

has at most one solution.

Proof. Put $p = \alpha + \beta$, so

$$(4) \quad p'(t) = N\int_{-\tau}^{0} p(t+s)ds, \quad t \in J, \quad p(s) = p(0) = -p(T), \quad s \in J_0.$$

Note that $p(t) = 0$, $t \in J$ is a solution of (4). We need to show that $p(t) = 0$, $t \in J$ is the unique solution of (4). Suppose that (4) has another solution $w \in C_1$. Let $A = \{t_k \in J : w(t_k) = 0\}$. Assume that $t_0 \in A$. If $t_0 = 0$ or $t_0 = T$, then $w(0) = 0$. Hence $w(t) = 0$, $t \in J$, by Remark 1. It is a contradiction. If $0 < t_0 < T$, then $w(t_0) = 0$ showing that $w(t) = 0$, $t \in [t_0, T]$. Thus $-w(T) = w(0) = 0$, so $w(t) = 0$, $t \in J$. It is a contradiction again. If we assume that $w(t) > 0$, $t \in J$, then $w(T) < 0$ since $w(0) = -w(T)$. It is a contradiction too. Similarly, if we assume $w(t) < 0$, $t \in J$, then $w(T) > 0$ which is also a contradiction. It proves that $p(t) = 0$, $t \in J$ is the unique solution of problem (4). It means that $\alpha(t) = -\beta(t)$, $t \in J$, so

$$(5) \quad \alpha'(t) = N\int_{-\tau}^{0} \alpha(t+s)ds + m(t), \ t \in J, \ \alpha(s) = \alpha(0) = \alpha(T), \ s \in J_0,$$

$$(6) \quad \beta'(t) = N\int_{-\tau}^{0} \beta(t+s) - m(t), \ t \in J, \ \beta(s) = \beta(0) = \beta(T), \ s \in J_0.$$

Suppose that (5) has two solutions γ_1 and γ_2. Let $B = \{t_k \in J : : \gamma_1(t_k) = \gamma_2(t_k)\}$. If $t_0 = 0$ or $t_0 = T$, then $\gamma_1(0) = \gamma_2(0)$. Hence $\gamma_1(t) = \gamma_2(t)$, $t \in J$, by Remark 1. It is a contradiction. If $0 < t_0 < < T$, then $\gamma_1(t_0) = \gamma_2(t_0)$ showing that $\gamma_1(t) = \gamma_2(t)$, $t \in [t_0, T]$. Thus $\gamma_1(T) = \gamma_2(T)$, so $\gamma_1(0) = \gamma_2(0)$ proving that $\gamma_1(t) = \gamma_2(t)$, $t \in J$. It is a contradiction again. We assume that $\gamma_1(t) < \gamma_2(t)$, $t \in J$. Put $\delta =$

$= \gamma_2 - \gamma_1$, so $\delta(s) = \delta(0) = \delta(T)$, $s \in J_0$. Now integrating the equation for δ, we obtain

$$\delta(t) = \delta(0) + N \int_0^t \int_{-\tau}^0 \delta(r+s)drds > \delta(0), \quad t \in J.$$

Hence $\delta(T) > \delta(0)$. It is a contradiction. Same argument holds if we assume $\gamma_1(t) > \gamma_2(t)$, $t \in J$. It proves that problem (5) has at most one solution, so problem (6) has also at most one solution.

It ends the proof. $\diamondsuit$

Remark 2. If $m(t) = 0$, $t \in J$, then problem (3) has exactly zero solution, so $\alpha(t) = \beta(t) = 0$, $t \in J$.

Lemma 3. *Let Assumptions H_1, H_2 and H_3 hold. Let $u, v \in \Omega$ be w.c. lower and upper solutions of (1), and $u \le v$; $\Omega = \{\phi \in C_0 : y_{0,t} \le \phi \le \le z_{0,t}\}$. Then the antiperiodic boundary value system of the form*

$$\begin{cases} p'(t) = f(t, u_t) + N \int_{-\tau}^0 [p(t+s) - u(t+s)]ds, \\ \qquad t \in J,\ p(s) = p(0) = -q(T),\ s \in J_0, \\ q'(t) = f(t, v_t) + N \int_{-\tau}^0 [q(t+s) - v(t+s)]ds, \\ \qquad t \in J,\ q(s) = q(0) = -p(T),\ s \in J_0. \end{cases}$$

has a unique solution (p, q) in the segment $[u, v]$, and $p \le q$.

Proof. Let $M \ne 0$. Put

$$U(s, v, p) = Mp(s) + N \int_{-\tau}^0 [p(s+r) - v(s+r)]dr + f(s, v_s).$$

Note that

$$\begin{cases} p'(t) = -Mp(t) + U(t, u, p), & t \in J, \\ q'(t) = -Mq(t) + U(t, v, q), & t \in J, \end{cases}$$

and hence

$$\begin{cases} p(t) = e^{-Mt}\left(p(0) + \int_0^t e^{Ms}U(s,u,p)ds\right), & t \in J, \\ q(t) = e^{-Mt}\left(q(0) + \int_0^t e^{Ms}U(s,v,q)ds\right), & t \in J. \end{cases}$$

Using the boundary conditions $p(0) = -q(T)$, $q(0) = -p(T)$, we see that (p,q) is the solution of the following system

$$\text{(7)} \qquad \begin{cases} p(t) = A(t,p,q), & t \in J, \\ q(t) = B(t,p,q), & t \in J \end{cases}$$

with

$$\begin{cases} A(t,p,q) = \\ = \dfrac{e^{-Mt}}{e^{2MT}-1}\left\{\int_0^T G(t,s)e^{Ms}U(s,u,p)ds - e^{MT}\int_0^T e^{Ms}U(s,v,q)ds\right\}, \\ B(t,p,q) = \\ = \dfrac{e^{-Mt}}{e^{2MT}-1}\left\{\int_0^T G(t,s)e^{Ms}U(s,v,q)ds - e^{MT}\int_0^T e^{Ms}U(s,u,p)ds\right\}, \end{cases}$$

$$G(t,s) = \begin{cases} e^{2MT} & \text{if } \ 0 \le s \le t, \\ 1 & \text{if } \ t < s \le T. \end{cases}$$

Let us construct the sequences $\{p_{n+1}, q_{n+1}\}$ by the following relations

$$\text{(8)} \qquad \begin{cases} p_{n+1}(t) = A(t,p_n,q_n), & p_0(t) = u(t),\ t \in J, \\ q_{n+1}(t) = B(t,p_n,q_n), & q_0(t) = v(t),\ t \in J, \end{cases}$$

and $p_0(s) = u(0)$, $q_0(s) = v(0)$, $p_{n+1}(s) = p_{n+1}(0)$, $q_{n+1}(s) = q_{n+1}(0)$ on J_0.

It is easy to see that

$$\begin{cases} U(t,u,u) = Mu(t) + f(t,u_t) \ge Mu(t) + u'(t), & t \in J, \\ U(t,v,v) = Mv(t) + f(t,v_t) \le Mv(t) + v'(t), & t \in J. \end{cases}$$

Using the above properties we try to prove the following relation

$$\text{(9)} \qquad p_0(t) \le p_1(t) \le q_1(t) \le q_0(t), \quad t \in J.$$

Indeed, putting $\theta = \frac{e^{-Mt}}{e^{2MT}-1}$, for the sake of brevity we have

$$p_1(t) = A(t, p_0, q_0) = A(t, u, v) =$$

$$=\theta \left\{ \int_0^T G(t,s)e^{Ms}U(s,u,u)ds - e^{MT}\int_0^T e^{Ms}U(s,v,v)ds \right\}$$

$$\geq \theta \left\{ \int_0^T G(t,s)e^{Ms}[Mu(s)+u'(s)]ds - e^{MT}\int_0^T e^{Ms}[Mv(s)+v'(s)]ds \right\}$$

$$=\theta\left\{ \left[e^{2MT}-1\right] e^{Mt}u(t) - e^{2MT}[u(0)+v(T)] + e^{MT}[v(0)+u(T)] \right\} \geq u(t)$$

since $u(0)+v(T) \leq 0,\ v(0)+u(T) \geq 0$. Similarly, we have

$$q_1(t) = B(t, p_0, q_0) = B(t, u, v)$$

$$=\theta \left\{ \int_0^T G(t,s)e^{Ms}U(s,v,v)ds - e^{MT}\int_0^T e^{Ms}U(s,u,u)ds \right\}$$

$$\leq \theta \left\{ \int_0^T G(t,s)e^{Ms}[Mv(s)+v'(s)]ds - e^{MT}\int_0^T e^{Ms}[Mu(s)+u'(s)]ds \right\}$$

$$=\theta\left\{ \left[e^{2MT}-1\right] e^{Mt}v(t) - e^{2MT}[v(0)+u(T)] + e^{MT}[v(T)+u(0)] \right\} \leq v(t).$$

Moreover

$$p_1(t) = \theta \left\{ \int_0^T G(t,s)e^{Ms}U(s,u,u)ds - e^{MT}\int_0^T e^{Ms}U(s,v,v)ds \right\}$$

$$\leq \theta \left\{ \int_0^T G(t,s)e^{Ms}U(s,v,v)ds - e^{MT}\int_0^T e^{Ms}U(s,u,u)ds \right\} = q_1(t)$$

because $U(t,u,u) \leq U(t,v,v)$. It means that (9) holds.

Let us assume that

$$p_0(t) \leq \cdots \leq p_{k-1}(t) \leq p_k(t) \leq q_k(t) \leq q_{k-1}(t) \leq \cdots \leq q_0(t), \quad t \in J$$

for some $k \geq 1$. Obviously,

$$p_{k+1}(t) = A(t, p_k, q_k) \geq A(t, p_{k-1}, q_{k-1}) = p_k(t),$$
$$q_{k+1}(t) = B(t, p_k, q_k) \leq B(t, p_{k-1}, q_{k-1}) = q_k(t),$$
$$p_{k+1}(t) = A(t, p_k, q_k) \leq B(t, p_k, q_k) = q_{k+1}(t)$$

for $t \in J$. It proves that

$$p_k(t) \le p_{k+1}(t) \le q_{k+1}(t) \le q_k(t), \quad t \in J.$$

Hence, by induction, we have

$$p_0(t) \le p_1(t) \le \cdots \le p_n(t) \le q_n(t) \le \cdots \le q_1(t) \le q_0(t), \quad t \in J$$

for all n. Employing standard techniques, it can be shown that the sequences $\{p_n\}, \{q_n\}$ converge uniformly and monotonically to the solution (p,q) of (7), so $p_n \to p$, $q_n \to q$ and $u(t) \le p(t) \le q(t) \le v(t)$ on J. Now we are going to show that problem (7) has a unique solution. Assume that it has two solutions (x,y) and (z,w). Put $\alpha = x - z$, $\beta = y - w$. Then,

$$\begin{cases} \alpha'(t) = N \int_{-\tau}^{0} \alpha(t+s)ds, \ t \in J, & \alpha(s) = \alpha(0) = -\beta(T), \quad s \in J_0, \\ \beta'(t) = N \int_{-\tau}^{0} \beta(t+s)ds, \ t \in J, & \beta(s) = \beta(0) = -\alpha(T), \quad s \in J_0. \end{cases}$$

Remark 2 yields $\alpha(t) = \beta(t) = 0$ on J showing that $x(t) = z(t)$, $y(t) = w(t)$ on J. It proves that problem (7) has a unique solution. The assertion of Lemma 3 holds.

The proof is complete. ◊

Lemma 4. *Assume that* $N, \tau > 0$. *Let*

$$\begin{cases} \alpha'(t) \le N \int_{-\tau}^{0} \alpha(t+s)ds, & t \in J, \ \alpha(s) = \alpha(0) = \beta(T), \ s \in J_0, \\ \beta'(t) \le N \int_{-\tau}^{0} \beta(t+s)ds, & t \in J, \ \beta(s) = \beta(0) = \alpha(T), \ s \in J_0. \end{cases}$$

Then $\alpha(t) \le 0$ *and* $\beta(t) \le 0$ *on* J.

Proof. Put $p(t) = \alpha(t)e^{-Mt}$, $q(t) = \beta(t)e^{-Mt}$ with $M > N(T+\tau)$. Then

$$p'(t) = -Mp(t) + \alpha'(t)e^{-Mt} \le -Mp(t) + Ne^{-Mt} \int_{-\tau}^{0} e^{M(t+s)}p(t+s)ds$$

$$= -Mp(t) + Ne^{-Mt} \int_{t-\tau}^{t} e^{Ms}p(s)ds, \quad t \in J,$$

$$q'(t) \le -Mq(t) + Ne^{-Mt} \int_{t-\tau}^{t} e^{Ms}q(s)ds, \quad t \in J.$$

Assume that the conclusion of Lemma 4 is false. We shall distinguish three cases.

Case 1. Assume that $\alpha(t) \leq 0$ on J. Hence $q(0) = \beta(0) = \alpha(T) \leq \leq 0$, $q(T) = \beta(T)e^{-MT} = \alpha(0)e^{-MT} \leq 0$. We need to show that $q(t) \leq 0$ on $(0,T)$. Assume that it is not true. Then there exists $t_0 \in (0,T)$ such that $q(t_0) = \epsilon > 0$ and $q(t) \leq \epsilon$, $t \in (0,T)$, so $q(t_0) - - q(t_0 - h) \geq 0$ for small $h > 0$. It yields

$$0 \leq q'(t_0) \leq -Mq(t_0)+Ne^{-Mt_0} \int_{t_0-\tau}^{t_0} e^{Ms}q(s)ds \leq -M\epsilon+N(T+\tau)\epsilon < 0$$

since $M > N(T+\tau)$. It is a contradiction proving that $q(t) \leq 0$ on J, so $\beta(t) \leq 0$ on J.

Case 2. Use the proof from Case 1 to show that $\alpha(t) \leq 0$, $t \in J$ when $\beta(t) \leq 0$ on J.

Case 3. There exist $t_1, t_2 \in J$ such that $p(t_1) = \epsilon_1 > 0$, $q(t_2) = = \epsilon_2 > 0$ and $p(t) \leq \epsilon_1$, $q(t) \leq \epsilon_2$ on J. Let $t_1 \in (0,T]$. Then

$$0 \leq p'(t_1) \leq -Mp(t_1) + Ne^{-Mt_1} \int_{t_1-\tau}^{t_1} e^{Ms}p(s)ds \leq$$
$$\leq -M\epsilon_1 + N(T+\tau)\epsilon_1 < 0$$

which is a contradiction. Let $t_2 \in (0,T]$. Similarly as before we obtain

$$0 \leq q'(t_2) \leq -Mq(t_2) + Ne^{-Mt_2} \int_{t_2-\tau}^{t_2} e^{Ms}q(s)ds \leq$$
$$\leq -M\epsilon_2 + N(T+\tau)\epsilon_2 < 0$$

which is also a contradiction. If $t_1 = t_2 = 0$, then $\alpha(0) = \beta(T) < < \beta(0) = \alpha(T) < \alpha(0)$ which is a contradiction too. It proves that the assertion of Lemma 4 holds. It ends the proof. ◊

Lemma 5. *Let Assumptions H_1 and H_3 hold. Let*

$$\begin{cases} y'(t) = f(t, y_t), & t \in J, \quad y(s) = y(0) = -z(T), \quad s \in J_0, \\ z'(t) = f(t, z_t), & t \in J, \quad z(s) = z(0) = -y(T), \quad s \in J_0, \end{cases}$$

and $y_{0,t} \leq y_t, z_t \leq z_{0,t}$, $t \in J$.

Then $y(t) = z(t)$ on J, so y and z are solutions of problem (1).

Proof. Put $p = y - z$. Then

(10)
$$p'(t) = f(t, p_t + z_t) - f(t, z_t), \quad t \in J, \quad p(s) = p(0) = p(T), \quad s \in J_0.$$

Note that $p(0) = 0$, $t \in J$ is a solution of (10). We need to show that $p(t) = 0$ on J is the unique solution of (10). Assume that problem (10) has another solution w. Put $B = \{t_k \in J : w(t_k) = 0\}$. Let $t_0 \in B$. If $t_0 = 0$ or $t_0 = T$, then $w(0) = 0$. Hence $w(t) = 0$ on J since the problem

$$w'(t) = f(t, w_t + z_t) - f(t, z_t) \equiv \tilde{f}(t, w_t), \; t \in J, \; w(s) = w(0), \; s \in J_0$$

has the unique solution. If $0 < t_0 < T$, then $w(t) = 0$ on $[t_0, T]$. Since $w(T) = w(0)$, it proves that $w(t) = 0$ on J. Assume that $w(t) > 0$ on J. Integrating the equation for w, we obtain

$$w(t) = w(0) + \int_0^t [f(s, w_s + z_s) - f(s, z_s)]ds, \quad t \in J,$$
$$w(s) = w(0) = w(T), \quad s \in J_0$$

Note that f is nondecreasing with respect to the second argument. Hence $w(T) > w(0)$ which is a contradiction. If we assume that $w(t) < < 0$ on J, then $w(T) < w(0)$ which is a contradiction too. It proves that $p(t) = 0$ on J is the unique solution of (10), so $y(t) = z(t)$ on J. It means that $y(0) = -y(T)$ and $z(0) = -z(T)$ showing that y and z are solutions of problem (1). It ends the proof. ◊

4. Main results

Theorem 1. *Assume that Assumptions H_1 and H_4 are satisfied. Let $LT < 2$. Then problem (1) has exactly one solution.*

Proof. Integrating the equation in (1) we have

$$x(t) = x(0) + \int_0^t f(s, x_s)ds, \quad t \in J$$

Using the boundary condition $x(0) = -x(T)$, we see that problem (1) is equivalent to the following one

$$x(t) = \frac{1}{2} \int_0^T H(t, s) f(s, x_s)ds \equiv Ax(t), \quad t \in J \tag{11}$$

with $H(t, s) = 1$ if $0 \le s \le t$, and $H(t, s) = -1$ if $t < s \le T$. Then Assumption H_4 yields

$$|Ax - A\bar{x}|_1 \equiv \max_{t\in J} |Ax(t) - A\bar{x}(t)| \leq \frac{1}{2} \int_0^T |f(s,x_s) - f(s,\bar{x}_s)| ds \leq$$

$$\leq \frac{1}{2} LT |x - \bar{x}|_1.$$

Hence problem (11) has a unique solution since $LT < 2$. The proof is complete. ◊

Remark 3. Obviously, Th. 2 holds if Assumption H_4 is replaced by the following

$$|f(t,u) - f(t,v)| \leq L \max_{s\in J} |u(s) - v(s)|,\ L > 0$$

Example 1. Let

$$f(t,x_t) = g(t,x(t),x(t-\alpha_1(t)),\cdots,x(t-\alpha_r(t))), \quad t \in J = [0,T],$$

and $\alpha_i(t) \leq t$, $t \in J$, $i = 1,2,\cdots,r$. Let g, α_i be continuous. Assume that g satisfies the Lipschitz condition with respect to the last $r+1$ variables with a constant L. Then

$$|f(t,u) - f(t,v)| \leq L(r+1) \max_{0\leq s\leq t} |u(s) - v(s)|, \quad t \in J.$$

By Th. 1, problem (1) [with f as above] has a unique solution if $L(r+ +1)T < 2$.

Example 2. Consider the problem
(12)

$$\begin{cases} x'(t) = g(t, \int_0^T K(t,s,x(\alpha(s)))ds), & t \in J = [0,T], \\ x(s) = x(0) = -x(T), & s \in [-\tau, 0],\ \min_{t\in J} \alpha(t) = -\tau. \end{cases}$$

Assume that g, K are continuous and $\alpha \in C(J,[-\tau,T])$. Let

$$|g(t,u_1) - g(t,u_2)| \leq L_1|u_1 - u_2|, \quad |K(t,s,v_1) - K(t,s,v_2)| \leq L_2|v_1 - v_2|.$$

Obviously, in this case

$$|g(t,u) - g(t,v)| \leq L_1 L_2 T \max_{s\in J} |u(s) - v(s)|, \quad t \in J.$$

Then, by Th. 1 and Remark 3, problem (12) has a unique solution if $L_1 L_2 T^2 < 2$.

Theorem 2. *Assume that Assumptions H_1, H_2, H_3 are satisfied. Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ such that $y_n \to y$, $z_n \to z$ as $n \to \infty$ uniformly and monotonically on J and $y = z$ is a unique solution of problem* (1).

Proof. Let $y_{n+1}(s) = y_{n+1}(0)$, $z_{n+1}(s) = z_{n+1}(0)$ on J_0 and

$$y'_{n+1}(t) = f(t, y_{n,t}) + N \int_{-\tau}^{0} [y_{n+1}(t+s) - y_n(t+s)]ds, \quad t \in J,$$

$$y_{n+1}(0) = -z_{n+1}(T),$$

$$z'_{n+1}(t) = f(t, z_{n,t}) + N \int_{-\tau}^{0} [z_{n+1}(t+s) - z_n(t+s)]ds, \quad t \in J,$$

$$z_{n+1}(0) = -y_{n+1}(T)$$

for $t \in J$, $n = 0, 1, \cdots$. Since y_0, z_0 are w.c. lower and upper solutions of problem (1) and $y_0(t) \le z_0(t)$ on J, it means, by Lemma 3, that y_1 and z_1 are well defined and

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J.$$

Now, we prove that y_1, z_1 are w.c. lower and upper solutions of problem (1). Obviously, $y_1(s) = y_1(0) = -z_1(T)$, $z_1(s) = z_1(0) = -y_1(T)$ on J_0. Using Assumption H_3, we get

$$y'_1(t) = f(t, y_{0,t}) + N \int_{-\tau}^{0} [y_1(t+s) - y_0(t+s)]ds - f(t, y_{1,t}) + f(t, y_{1,t})$$

$$\le f(t, y_{1,t}) - N \int_{-\tau}^{0} [y_1(t+s) - y_0(t+s)]ds +$$

$$+ N \int_{-\tau}^{0} [y_1(t+s) - y_0(t+s)]ds = f(t, y_{1,t}), \quad t \in J,$$

and

$$z'_1(t) = f(t, z_{0,t}) + N \int_{-\tau}^{0} [z_1(t+s) - z_0(t+s)]ds - f(t, z_{1,t}) + f(t, z_{1,t})$$

$$\ge f(t, z_{1,t}) + N \int_{-\tau}^{0} [z_0(t+s) - z_1(t+s)]ds +$$

$$+ N \int_{-\tau}^{0} [z_1(t+s) - z_0(t+s)]ds = f(t, z_{1,t}), \quad t \in J.$$

The above proves that y_1, z_1 are w.c. lower and upper solutions of (1).

Let us assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_{k-1}(t) \le y_k(t) \le$$
$$\le z_k(t) \le z_{k-1}(t) \le \cdots \le z_1(t) \le z_0(t), \quad t \in J,$$

and let y_k, z_k be w.c. lower and upper solutions of problem (1) for some $k \ge 1$. Then, by Lemma 3, the elements y_{k+1}, z_{k+1} are well defined, and

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), \quad t \in J.$$

Hence, by induction, we have

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \quad t \in J$$

for all n. Employing standard techniques, it can be shown that the sequences $\{y_n\}, \{z_n\}$ converge uniformly to the limit functions y, z, so $y_n \to y$, $z_n \to z$, and $y(t) \le z(t)$ on J. Indeed, y, z satisfy the system

$$\begin{cases} y'(t) = f(t, y_t), & t \in J, \quad y(s) = y(0) = -z(T), \quad s \in J_0, \\ z'(t) = f(t, z_t), & t \in J, \quad z(s) = z(0) = -y(T), \quad s \in J_0. \end{cases}$$

By Lemma 5, $y(t) = z(t)$ on J, so y and z are solutions of problem (1).

To prove that $y = z$ is a unique solution of (1) in $[y_0, z_0]$, we need to show that if w is any solution of (1) such that $y_0(t) \le w(t) \le z_0(t)$ on J, then

$$y_0(t) \le y(t) \le w(t) \le z(t) \le z_0(t), \quad t \in J.$$

To do this, suppose that for some k, $y_k(t) \le w(t) \le z_k(t)$ on J, and put $p = y_{k+1} - w, \quad q = w - z_{k+1}$. Then, Assumption H_3 yields

$$p'(t) = f(t, y_{k,t}) + N \int_{-\tau}^{0} [y_{k+1}(t+s) - y_k(t+s)]ds - f(t, w_t) \le$$

$$\le N \int_{-\tau}^{0} p(t+s)ds,$$

$$q'(t) = f(t, w_t) - f(t, z_{k,t}) - N \int_{-\tau}^{0} [z_{k+1}(t+s) - z_k(t+s)]ds \le$$

$$\le N \int_{-\tau}^{0} q(t+s)ds$$

for $t \in J$ with $p(0) = q(T)$, $q(0) = p(T)$. By Lemma 4, we obtain $p(t) \le$ ≤ 0, $q(t) \le 0$ on J showing that $y_{k+1}(t) \le w(t) \le z_{k+1}(t)$, $t \in J$. Since

$y_0(t) \le w(t) \le z_0(t)$ it proves, by induction, that $y_n(t) \le w(t) \le z_n(t)$ on J. Taking the limit as $n \to \infty$, we conclude that $y(t) \le w(t) \le \le z(t), \quad t \in J$. The proof is complete. $\Diamond$

Example 3. Let

$$f(t, x_t) = g(t, \int_{t-\tau}^{t} K(t, s, x(s))ds), \quad t \in J = [0, T], \quad \tau > 0.$$

Assume that g_2 and K_3 exist and $g_2(t, u)K_3(t, s, v) \ge L > 0$, [$g_2$ means the derivative of g with respect to the second variable, and K_3 the derivative of K with respect to the third one]. Then, using the mean value theorem, we see that assumptions H_3 holds, so

$$f(t, v_t) - f(t, u_t) \ge L \int_{t-\tau}^{t} [v(s) - u(s)]ds = L \int_{-\tau}^{0} [v(t+s) - u(t+s)]ds$$

for $v(t) \ge u(t)$ on J.

Theorem 3. *Assume that Assumption H_1 is satisfied. In addition, suppose there exist constants $K, L \ge 0$, $q, \alpha > 0$ and $(K + Lq^\alpha)T \le 2q$ such that the condition $|f(t, x)| \le K + L|x|^\alpha$ holds for $t \in J$, $x \in C_0$. Then problem (1) has a solution.*

Proof. Put $Q = \left\{ x \in C(\bar{J}, \mathbb{R}) : |x|_0 \equiv \max_{t \in \bar{J}} |x(t)| \le q \right\}$. It is obvious that the set Q is convex, closed and bounded. We prove that $AS \subset S$, where the operator A is defined as in (11). Let $x \in Q$. Then

$$|Ax|_0 = \max_{t \in J} \frac{1}{2} \left| \int_0^T H(t, s) f(s, x_s) ds \right| \le \frac{1}{2}(K + L|x|_0^\alpha)T \le$$
$$\le \frac{1}{2}\left(K + Lq^\alpha\right)T \le q$$

showing that operator A maps the set S in itself. Let us show that operator A is completely continuous. Note that the set $\{Ax\}$, $x \in Q$ is uniformly bounded. On the other hand, the functions x are in fact differentiable, and we have $\frac{d}{dt}Ax(t) = f(t, x_t)$. Hence, for $x \in Q$, we have

$$\max_{t \in \bar{J}} \left| \frac{d}{dt} Ax(t) \right| \le K + Lq.$$

Therefore the derivatives of functions Ax are uniformly bounded, which shows that these functions are equally continuous. The set $\{Ax\}$ will be relatively compact if $|x| \leq q$, hence the operator A is completely continuous. Therefore, operator A satisfies Schauder's theorem and has at least one fixed point. This completes the proof. ◊

Remark 4. If $\alpha = 1$, then $q \geq \dfrac{KT}{2 - LT}$ and we need to assume that $LT < 2$. In this case, Th. 3 holds under Assumption H_4 because we have

$$|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq K + L|x|_0 \text{ with } K = \max_{t \in J} |f(t,0)|.$$

Note that in this case problem (1) has a unique solution, by Th. 1.

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