

ON ESSENTIAL LEFT IDEALS OF ASSOCIATIVE RINGS

D.I.C. Mendes

Universidade da Beira Interior, 6200 Covilhã, Portugal

R. Wiegandt

A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1364 Budapest, P.O.Box 127, Hungary

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Abstract: The structure of rings without proper essential left ideals is described, as well as that of rings having an essential minimal left ideal.

In the recent radical theoretical paper [4] Puczyłowski and Zand introduced and used the notion of essential left ideals. The purpose of the present note is to investigate the effect of the presence and absence of essential left ideals to the structure of rings. Analogous results to some known ones [3] concerning two-sided essential ideals are established; namely, we study some properties of rings having no non-trivial essential left ideals, or having an essential minimal left ideal. Since conditions on essential left ideals are stronger than those imposed on essential two-sided ideals, we get more precise information on the structure of rings. These results are then used to give some characterizations

E-mail addresses: dmendes@noe.ubi.pt, wiegandt@renyi.hu

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of UD -semisimple artinian rings, where UD denotes the upper radical class determined by the class of all division rings.

Throughout this paper all rings are associative and R will always denote a ring. We write $I \triangleleft R$ (respectively, $I \triangleleft_\ell R$) to denote that I is a two-sided (respectively, left) ideal of R . A left ideal L of R is said to be a direct summand of R if there exists an ideal I of R such that $R = L \oplus I$ and L is called essential in R if, for every $0 \neq J \triangleleft R$, we have $L \cap J \neq 0$ and, in this case, we write, $L \triangleleft_\ell^\bullet R$. If $L \triangleleft_\ell R$, then $\ell(R, L) = \{r \in R : rL = 0\}$, $r(R, L) = \{r \in R : Lr = 0\}$ and $\text{ann}(R, L) = \{r \in R : Lr = rL = 0\}$. For the fundamentals of radical theory we refer to [5].

Lemma 1. *Let $0 \neq L \triangleleft_\ell R$ and suppose that L has a unity. Then $L = R$ if and only if $\ell(R, L) = 0$.*

Proof. Let e be the unity of L . Suppose first that $\ell(R, L) = 0$. For each $x \in R$ we can write $x = xe + x - xe$. Let $K = \{x - xe : x \in R\}$. Now, $xe = e(xe)$ so that $(x - ex)eL = 0$. So, $(x - ex)L = 0$ and since $\ell(R, L) = 0$, it follows that $x = ex$. It is now easy to show that $K \triangleleft R$. Furthermore, $L \cap K = 0$. Indeed, if $a \in L \cap K$, then $a = x - xe$ for some $x \in R$ and $a \in L$. So, $a = ae = (x - xe)e = 0$. But, since $\ell(R, L) = 0$ implies that L is an essential left ideal of R , we conclude that $K = 0$. Consequently, $x = xe$ for any $x \in R$ and so $L = R$. The converse follows easily. \diamond

In particular, we can state the following

Corollary 2. *If $0 \neq L \triangleleft_\ell R$ and L has a unity, then $L \triangleleft_\ell^\bullet R$ if and only if $L = R$.*

Proof. Suppose that $L \triangleleft_\ell^\bullet R$. Since L has a unity and L is essential in R , it follows that $\ell(R, L) = 0$. Hence $L = R$ by Lemma 1. The converse is trivial. \diamond

Lemma 3 (see [3], Lemma 1). *If $0 \neq L \triangleleft_\ell R$ ($L \neq R$), then either L is a direct summand of R or there is a proper essential left ideal of R containing L .*

Proof. Let $0 \neq L \triangleleft_\ell R$ ($L \neq R$) and suppose that L is not a direct summand of R . Let $\mathcal{M} = \{I \triangleleft R : L \cap I = 0\}$. By Zorn's lemma, there exists a maximal element $M \in \mathcal{M}$. Now $L + M$ is an essential left ideal of R . Indeed, if $L \cap I \neq 0$, then it is clear that $(L + M) \cap I \neq 0$. So, let us assume that $L \cap I = 0$. Then $I \in \mathcal{M}$. Now, if $I \subseteq M$, we have $(L + M) \cap I \neq 0$, as desired. If I is not contained in M , then $(I + M) \cap L \neq 0$ since $M \subseteq I + M$ and M is maximal with respect

to $L \cap M = 0$. Thus, there exists $a + m \in I + M$ such that $a + m = l \in L$. Then $a \in L + M$ and hence $(L + M) \cap I \neq 0$, if $a \neq 0$. If $a = 0$, then $m = l$ so that $L \cap M \neq 0$. Moreover, since L is not a direct summand of R , $L + M$ is a proper essential left ideal of R and, obviously, $L \subseteq L + M$. \diamond

The proof of the next lemma is straightforward.

Lemma 4. *If every left ideal of R is a direct summand of R , then for any ideal I of R , we have:*

- (i) *If $L \triangleleft_{\ell} I$, then $L \triangleleft_{\ell} R$.*
- (ii) *Any left ideal L of I is a direct summand of I and also of R .*

The following theorem and its proof are also analogous to ([3] Th. 3).

Theorem 5. *The following conditions are equivalent for any ring R :*

- (i) *R has no proper essential left ideals.*
- (ii) *Each left ideal of R is a direct summand of R .*
- (iii) *R is a direct sum of simple rings having only trivial left ideals.*

Proof. If R has no proper essential left ideals, then by Lemma 3, each left ideal of R is a direct summand of R and hence (i) implies (ii). Clearly, (ii) implies (i). We shall now prove that (iii) implies (ii). Let R be a direct sum of rings I_i ($i \in \Lambda$) where each I_i is a simple ring having only trivial left ideals. Let $0 \neq L \triangleleft_{\ell} R$. For each subset $J \subseteq \Lambda$, let $I_J = \sum_{j \in J} I_j$. For any i , $L \cap I_i = 0$ or $L \cap I_i = I_i$. If $L \cap I_i = I_i$, for all i , then $L = R$. This implies $|\Lambda| = 1$ and in this case condition (ii) has been proved. Suppose that there exists j such that $I_j \cap L = 0$. Applying Zorn's lemma, we can choose a set J which is maximal with respect to $L \cap I_J = 0$. If $j \in J$, then $I_j \subseteq I_J + L$. If j does not belong to J , then $I_j \not\subseteq I_J + L$ so that $(I_j + I_J) \cap L \neq 0$ and so $I_j \cap (I_J + L) \neq 0$. Thus $I_j \cap (I_J + L) = I_j$. Therefore, $I_j \subseteq I_J + L$ and hence $R = I_J \oplus L$.

We shall now prove that (ii) implies (iii). First, we show that every non-zero ideal of R contains a non-zero simple ideal having only trivial left ideals. Let $0 \neq I \triangleleft R$. If $0 \neq a \in I$, let $\mathcal{M} = \{L \triangleleft_{\ell} I : a \text{ does not belong to } L\}$. Then, according to Zorn's lemma, there exists $T \triangleleft_{\ell} I$ which is maximal with respect to $a \notin T$. By Lemma 4, $I = T \oplus U$ for some $U \triangleleft I$. Now U is simple having only trivial left ideals. Indeed, if there exists $0 \neq V \triangleleft_{\ell} U$ then, by the previous lemma, $U = V \oplus W$ for some $W \triangleleft U$. Then $I = T \oplus V \oplus W$. If $W \neq 0$, then we have that $T \subsetneq T \oplus V$ and $T \subsetneq T \oplus W$ so that $a \in T \oplus V$ and $a \in T \oplus W$. Hence,

since $I = T \oplus V \oplus W$, we have that $t + v + 0 = a = t_1 + 0 + w$ for some $t, t_1 \in T, w \in W$ and $v \in V$. Then $v = 0 = w$ and $a = t_1 = t \in T$, which is a contradiction. Thus, $W = 0$ and so $V = U$. Therefore U is simple having only trivial left ideals. We notice also that U is an ideal of R . Next we show that if

$$S = \sum (\text{all simple ideals of } R \text{ having only trivial left ideals}),$$

then $S = R$. Indeed, if $S \neq R$, then $R = S \oplus K$ for some $0 \neq K \triangleleft R$. But, by the previous step, we can find a simple ideal of R containing only trivial left ideals, which is not contained in S . This is a contradiction. Finally, if

$$R = \sum (\text{all simple ideal of } R \text{ having only trivial left ideals}),$$

then R is a direct sum of simple rings having only trivial left ideals. In fact, if $\{I_i\}_{i \in \Lambda}$ is the collection of all simple ideals of R having only trivial left ideals, then, by Zorn's lemma, there exists $J \subseteq \Lambda$ which is maximal with respect to $\sum_{j \in J} I_j$ being a direct sum. Since we have shown that every non-zero ideal of R contains a non-zero simple ideal having only trivial left ideals, we conclude that $R = \bigoplus_{j \in J} I_j$. \diamond

It is well known [5] that a simple ring having only trivial left ideals, is either a zero ring of prime order or a division ring. Hence, we can state the following

Corollary 6. *For a semi-prime (respectively, nil) ring R , the following conditions are equivalent:*

- (i) *R is a direct sum of division rings (respectively, zero-rings of prime order).*
- (ii) *Every left ideal of R is a direct summand of R .*
- (iii) *R has no proper essential left ideals.*

Theorem 7. *The following conditions are equivalent for a ring R :*

- (i) *R is UD -semisimple artinian (where UD denotes the upper radical determined by the class \mathcal{D} of all division rings).*
- (ii) *R has no proper essential left ideals and R has a right unity.*
- (iii) *Every left ideal L of R is a direct summand of R and R has a right unity.*
- (iv) *R is a finite direct sum of division rings.*
- (v) *Every left ideal of R has a unity.*

Proof. Since UD -semisimple rings are semi-prime, and the simple UD -semisimple rings are division rings, the equivalence of (i) and (iv) is obvious. That condition (ii) is equivalent to condition (iii) follows immediately from the previous theorem. We now prove that (iii) implies (iv). Suppose that every left ideal L of R is a direct summand of R and that R has a right unity. Then, from the previous theorem, R is a direct sum of division rings. Since R has a right unity, this direct sum must be finite. Then the right unity e of R is in a finite sum of these division rings:

$$e \in I_1 \oplus I_2 \oplus \dots \oplus I_r.$$

If I_j is any further component of R , then

$$I_j = I_j e \subseteq I_1 \oplus I_2 \oplus \dots \oplus I_r,$$

which is a contradiction. To show that (iii) implies (v), let $0 \neq L \triangleleft \triangleleft R$. Then, from (iii), there exists $M \triangleleft R$ such that $R = L \oplus M$. So, $L \cong R/M$. Now, since (iii) implies (iv) and we have shown that (iv) is equivalent to (i), we know that R has a unity. Thus L also has a unity. Finally, we show that (v) implies (ii). Suppose that every left ideal of R has a unity and that $L \triangleleft \overset{\circ}{\triangleleft} R$. Then, from Cor. 2, it follows that $L = R$. \diamond

Rings which have an essential minimal left ideal are described in the following theorem:

Theorem 8. *A ring R has an essential minimal left ideal L if and only if R is subdirectly irreducible with heart $H = L + LR$ and*

- (i) *either $L^2 \neq 0$, L is a minimal left ideal in R and then H is a simple prime ring with minimal left ideal L ,*
- (ii) *or $L^2 = RL = 0$, H is a zero-ring on an elementary p -group, and the additive group of L is a cyclic group of prime order p ,*
- (iii) *or $RL \neq 0$, $L^2 = H^2 = 0$ and L is a minimal left ideal of R .*

Proof. Since L is minimal and essential in R , L is contained in every non-zero ideal of R . Thus R is subdirectly irreducible with heart $H = L + LR$.

We consider case (i). Since $L^2 \neq 0$, by the minimality of L , there exists $a \in L$ such that $L = Ra$. Obviously,

$$H = L + LR = L + (Ra)R = L + \sum_{x \in R} (Ra)x.$$

Each left ideal Rax is an R -homomorphic image of Ra , so, by the

minimality of L , either $L \cong_R Rax$ or $Rax = 0$. This proves that H is the sum of all minimal left ideals of R which are R -isomorphic to L . Moreover, by the essentiality of L and H in R , each minimal left ideal of R is R -isomorphic to L . Let $0 \neq K \triangleleft_\ell H$ such that $K \subseteq L$. If $LK = 0$, then $K \subseteq r(H, L)$ and $r(H, L) \triangleleft H$. Since by $0 \neq L^2 \subseteq H^2$, the heart H is a simple prime ring, we have $r(H, L) = H$ and, consequently, $L^2 \subseteq LH = 0$; a contradiction. Hence $LK \neq 0$ and there exists $k \in K$ such that $Lk \neq 0$. Since $Lk \triangleleft_\ell R$ and $Lk \subseteq L$, necessarily, $L = Lk$. Thus, $L = Lk \subseteq LK \subseteq HK \subseteq K$. This proves that L is a minimal left ideal also in H .

Next we consider case (ii). Since $L^2 = 0$, we have

$$H^2 = (L + LR)^2 = L^2 + L^2R + L(RL) + L(RL)R = L^2 = 0.$$

$RL = 0$ implies that $L = \mathbb{Z}a$ for any $0 \neq a \in L$. Hence, for $b \in L$ with $0 \neq b \neq a$, we have $b = na$ for some $n \in \mathbb{Z}$ and, by the same token also, $a = kb$ for some $k \in \mathbb{Z}$. Hence, $a = kna$ and $(kn - 1)a = 0$. If such an element b does not exist, then L is the two element group. Taking into account the minimality of L , L is a zero-ring on a cyclic group of prime order p .

Case (iii) is clear.

The converse statement is obvious. \diamond

Remark 1. In case (i), the heart H is isomorphic to a dense ring of linear transformations on a vector space over a division ring, and hence H contains nonzero idempotents, that is, R is Behrens-semisimple.

Remark 2. In case $RL \neq 0$ and $L^2 = H^2 = 0$ there is nothing to say about the structure of L .

For instance, consider the split-null extension $\mathbb{Q} * \mathbb{Q}$ of the rational numbers \mathbb{Q} by itself: On $\mathbb{Q} * \mathbb{Q}$ addition is defined componentwise and multiplication by the rule

$$(a, b)(c, d) = (ad + bc, bd)$$

for all a, b, c, d in \mathbb{Q} . Now $\mathbb{Q} * \mathbb{Q}$ is subdirectly irreducible with heart $H = (\mathbb{Q}, 0)$ and $L = H$ is the unique essential minimal left ideal in $\mathbb{Q} * \mathbb{Q}$ and $L^2 = 0$.

Corollary 9. *A ring R has finitely many essential minimal left ideals L_1, \dots, L_n if and only if R is subdirectly irreducible with heart $H = \sum L_i$ and*

- (i) *either $L_1^2 \neq 0$, $n = 1$ and $R = L_1$ is a division ring,*

- (ii) or $L_1^2 \neq 0$, $2 \leq k \leq n$, and $H = R$ is isomorphic to a matrix ring $M_k(F)$ over a finite field F ,
- (iii) or $RH = 0$ and H is a finite direct sum of zero-rings on the cyclic group of prime order p ,
- (iv) or $RH \neq 0$, $H^2 = 0$ and H is a finite direct sum of minimal left ideals of R .

Proof. For the proof of (ii) we notice that a matrix ring $M_k(D)$, $2 \leq k$, over an infinite division ring D has infinitely many minimal left ideals (cf. Lai [2]; this result can be deduced also from Beidar and Salavová [1]) and each of them is essential. The rest is obvious in view of Th. 8. \diamond

It is easy to show that the semisimple class σ of a left hereditary radical γ is left essentially closed (that is, $L \triangleleft_\ell R$ and $L \in \sigma$ imply $R \in \sigma$). Recall that a radical γ is left subhereditary if $0 \neq L \triangleleft_\ell R \in \gamma$ implies $\gamma(L) \neq 0$. We notice that a radical γ belongs to a left essentially closed semisimple class σ if and only if γ is left subhereditary but not necessarily left hereditary. An example of such a radical is the upper radical $\gamma = \mathcal{UC}$ of the class \mathcal{C} of all commutative reduced rings, which is also a special radical (see [4]).

Proposition 10. *The semisimple class $\sigma = \mathcal{SUC}$ is left hereditary, whence $\gamma = \mathcal{UC}$ is a left stable radical (that is, if $L \triangleleft_\ell R$ then $\gamma(L) \subseteq \gamma(R)$).*

Proof. Since \mathcal{UC} is a special radical, every ring $R \in \sigma$ is a subdirect sum of commutative reduced rings, that is, there exists a set $\{I_\lambda \triangleleft R : R/I_\lambda \in \mathcal{C}\}$ such that $\bigcap I_\lambda = 0$. If $L \triangleleft_\ell R$, then

$$L/(L \cap I_\lambda) \cong (L + I_\lambda)/I_\lambda \triangleleft_\ell R/I_\lambda,$$

and so also $L/(L \cap I_\lambda)$ is a commutative reduced ring. Obviously, L is a subdirect sum of the rings $L/(L \cap I_\lambda)$, whence $L \in \sigma$. \diamond

As is well-known, the left hereditary and left stable radicals are precisely the hereditary A -radicals. Hence \mathcal{UC} is not left hereditary, reproving a statement of [4].

We terminate with the following related problem:

Problem. Characterize the semisimple classes of left hereditary radical classes! This would yield also a characterization of the semisimple classes of hereditary A -radicals.

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