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A NOTE ON THE TOLERANCE LATTICE OF ATOMISTIC ALGEBRAIC LATTICES

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Abstract: We prove that any tolerance of an algebraic atomistic lattice L with a regular kernel is a congruence. Moreover the lattices Tol$L$ and Con$L$ satisfy the same identities involving the pseudocomplementation.

1. Introduction

It is a known fact that the tolerance lattice (Tol$L$) and the congruence lattice (Con$L$) of a finite atomistic lattice coincide. In general this assertion is not true in the infinite case, however we show that if L is an atomistic algebraic lattice then any tolerance of L, whose kernel is a principal ideal, is a congruence.

It is also well-known [1] that Tol$L$ and Con$L$ are pseudocomplemented lattices. Let T^* (or θ^*) stands for the pseudocomplement of a

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$T \in \operatorname{Tol} L$ (or of a $\theta \in \operatorname{Con} L$). Bounded pseudocomplemented lattices considered as universal algebras $(L, \wedge, \vee, ^*, 0, 1)$ satisfying the identity

$$(L_n:)\qquad \begin{gathered}(x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \\ \vee (x_1^* \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge x_2 \wedge \ldots \wedge x_n^*)^* = 1\end{gathered}$$

form an equational class for any $n \in \mathbb{N}$. Any equational class of distributive bounded pseudocomplemented lattices coincides with one of these classes (see [7]). Our principal result is the following:

Main Theorem. *If L is an atomistic algebraic lattice, then* $\operatorname{Con} L$ *satisfies the identity (L_n) for some $n \in \mathbb{N}$ if and only if* $\operatorname{Tol} L$ *satisfies the same identity.*

2. Preliminaries

The lattice L with 0 element is called atomistic if any $x \in L$ is the join of atoms below x. The set of all atoms of L is denoted by $A(L)$ and for $x \in L$ let $A(x) = \{a \in A(L) \mid a \leq x\}$. Clearly, $A(x \wedge y) = A(x) \cap A(y)$ and $A(x) \cup A(y) \subseteqq A(x \vee y)$ for all $x, y \in L$. Let $(x]$ stand for the principal ideal generated by $x \in L$.

An element $a \in L$ is called standard if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ holds for all $x, y \in L$. The standard elements of L form a sublattice denoted by $S(L)$ (see [4]).

Remark 2.1. The following simple facts can be found e.g. in [4] or [5].

(i) s is a standard element of L iff the relation $\theta(s) = \{(x, y) \in L^2 \mid x \vee y = (x \wedge y) \vee a, \text{ for some } a \leq s\}$ is a congruence of L.

(ii) For any $s_1, s_2 \in S(L)$ we have $\theta(s_1 \vee s_2) = \theta(s_1) \vee \theta(s_2)$, $\theta(s_1 \wedge s_2) = \theta(s_1) \wedge \theta(s_2)$ and $\theta(s_1) \circ \theta(s_2) = \theta(s_2) \circ \theta(s_1)$.

Let $T(a, b)$ stand for the principal tolerance generated by the pair $(a, b) \in L^2$, $a \neq b$. For the identical and the unit relations on L, we write $\triangle$ and $\triangledown$, respectively. If $T \in \operatorname{Tol} L$, then for any $a, b \in L$ we have $(a, b) \in T \Leftrightarrow (a \wedge b, a \vee b) \in T$ and $(a, b) \in T$, $a \leqq x \leqq b \Rightarrow (a, x), (x, b) \in T$ (see [2]). Clearly, $\operatorname{Con} L \subseteqq \operatorname{Tol} L$ and for any $\theta_1, \theta_2 \in \operatorname{Con} L$ their greatest lower bound in $\operatorname{Tol} L$ is the same as the "meet" $\theta_1 \wedge \theta_2 \in \operatorname{Con} L$. However this is not true for the "join" operation; denoting by $T_1 \sqcup T_2$ the least upper bound of $T_1, T_2 \in \operatorname{Tol} L$ in $\operatorname{Tol} L$, we have $\theta_1 \sqcup \theta_2 \leqq \theta_1 \vee \theta_2$ for all $\theta_1, \theta_2 \in \operatorname{Con} L$. If $\operatorname{Tol} L = \operatorname{Con} L$, then L is called tolerance-trivial.

Lemma 2.2. *Let L be a lattice with 0 and $s \in S(L)$. Then*

(i) $\theta(s) = T(0, s)$,

(ii) *For every $s_1, s_2 \in S(L)$ we have $\theta(s_1) \sqcup \theta(s_2) = \theta(s_1) \vee \theta(s_2)$.*

Proof. (i) Obviously, $\theta(s) \in \mathrm{Tol}\, L$ and $(0, s) \in \theta(s)$. Take any $T \in \in \mathrm{Tol}\, L$ with $(0, s) \in T$. Then we have $(0, a) \in T$ for each $a \leq s$, whence we obtain $\theta(s) = \{(x, y) \in L^2 \mid x \vee y = (x \wedge y) \vee a$, for some $a \leq s\} \subseteqq \{(x, y) \in L^2 \mid (x \wedge y, x \vee y) \in T\} = T$. Hence $\theta(s) = T(0, s)$.

(ii) In view of the above (i) we get $\theta(s_1) \vee \theta(s_2) = \theta(s_1 \vee s_2) = = T(0, s_1 \vee s_2)$. On the other hand, $(0, s_1) \in \theta(s_1)$ and $(0, s_2) \in \theta(s_2)$ implies $(0, s_1 \vee s_2) \in \theta(s_1) \sqcup \theta(s_2)$, whence we obtain $T(0, s_1 \vee s_2) \subseteqq \subseteqq \theta(s_1) \sqcup \theta(s_2) \subseteqq \theta(s_1) \vee \theta(s_2) = T(0, s_1 \vee s_2)$, and this proves (ii). ◊

Remark 2.3. An easy consequence of the above (ii) is that

$$\theta(s_1) \sqcup \ldots \sqcup \theta(s_n) = \theta(s_1) \vee \ldots \vee \theta(s_n), \text{ for all} s_1, \ldots, s_n \in S(L).$$

Let $\overline{T}$ denote the transitive closure of a $T \in \mathrm{Tol}\, L$, it is well-known that $\overline{T} \in \mathrm{Con}\, L$ and that $T(0) = \{x \in L \mid (0, x) \in L\}$ is an ideal of L (see [8]). For an atomistic algebraic lattice L we shall make use of the following notations: $A(T) = A(L) \cap T(0) = \{a \in A(L) \mid (0, a) \in T\}$ and $w_T = \bigvee\{a \mid a \in A(T)\}$.

Remark 2.4. It is easy to check that for any family $T_i \in \mathrm{Tol}\, L$, $i \in I$, we have $A\left(\bigwedge_{i \in I} T_i\right) = \bigcap_{i \in I} A(T_i)$.

The following result will play an important role in our development:

Proposition A ([9], Prop. 1.11). *For each $T \in \mathrm{Tol}\, L$ we have $A(T) = = A(w_T)$ and w_T is a standard element of L.*

A lattice L with 0 is called a pseudocomplemented lattice if for each $x \in L$ there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow \Leftrightarrow y \leqq x^*$. If $x^* \vee x^{**} = 1$ for all $x \in L$, then L is called a Stone lattice. A lattice L is said to be a 0-modular if, for any $a, b, c \in L$, $a \leqq c$ and $b \wedge c = 0$ imply $(a \vee b) \wedge c = a$ (as given in [10]). According to [1], Tol$L$ is a pseudocomplemented 0-modular lattice for any lattice L.

3. Tolerances with a regular kernel

Definition 3.1. We say that a tolerance $T \in \mathrm{Tol}\, L$ has a *regular kernel* if $T(0)$ is a principal ideal of L.

Clearly, if L satisfies the ascending chain condition then any tolerance of it has a regular kernel. (Indeed, in this case any ideal of L is a principal ideal [3].) It is also obvious that any congruence $\theta(s)$ induced by a standard element $s \in L$ is a tolerance with a regular kernel.

Proposition 3.2. *If L is an atomistic algebraic lattice, then any tolerance T with a regular kernel is a congruence of L induced by a standard element of L.*

Proof. Let $T \in \mathrm{Tol}\, L$ be a tolerance with a regular kernel. As $T(0) = (u]$ for some $u \in L$, Prop. A gives $A(w_T) = A(T) = A(u)$ implying $u = w_T$. Thus $(0, w_T) \in T$, whence by Lemma 2.2(i) we get $\theta(w_T) = T(0, w_T) \leqq T$.

Conversely, take $(x, y) \in T$. Then $(x \wedge y, x \vee y) \in T$ and for each $a \in A(x \vee y) \setminus A(x \wedge y)$ we get that $(0, a) = ((x \wedge y) \wedge a, (x \vee y) \wedge a) \in T$, i.e. $a \in A(T)$. Since $A(T) = A(w_T)$, we have $a \leqq w_T$. Hence we obtain $q = \bigvee\{a \mid a \in A(x \vee y) \setminus A(x \wedge y)\} \leqq w_T$. Now the relations $(0, q) \in \theta(w_T)$ and $x \vee y = (x \wedge y) \vee q$ imply $(x \vee y, x \wedge y) \in \theta(w_T)$, that is $(x, y) \in \theta(w_T)$. Hence $T \leqq \theta(w_T)$, and this proves $T = \theta(w_T)$. $\Diamond$

A tolerance (congruence) φ is called complete, if for any $(a_i, b_i) \in \varphi$, $i \in I$, we have

$$\left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right) \in \varphi \quad \text{and} \quad \left(\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} b_i\right) \in \varphi$$

(see e.g., [11]).

Corollary 3.3. (i) *Any atomistic lattice L satisfying the ascending chain condition is tolerance-trivial.*

(ii) *Any complete tolerance of an atomistic algebraic lattice L is a congruence induced by a standard element of L.*

Proof. (i) As any lattice with 0 satisfying the ascending chain condition is an algebraic lattice (see e.g., [3]), L is an atomistic algebraic lattice. Since all $T \in \mathrm{Tol}\, L$ now have regular kernels, we get $\mathrm{Tol} L \subseteqq \mathrm{Con}\, L$, i.e. $\mathrm{Tol} L = \mathrm{Con}\, L$.

(ii) If $T \in \mathrm{Tol}\, L$ is a complete tolerance, then $w_T = \bigvee\{a \in A(L) \mid (0, a) \in T\} \in T(0)$, whence $T(0) = (w_T]$. Therefore T has a regular kernel, and so Prop. 3.2 gives (ii). $\Diamond$

4. Pseudocomplements in the tolerance lattice

In what follows let L denote an atomistic algebraic lattice. On the set $A(L)$ we define a relation R as follows: for $a, b \in A(L)$, $(a, b) \in R \Leftrightarrow \theta(0, a) \wedge \theta(0, b) \neq \triangle$. For any set $B \subseteqq A(L)$ let $R(B) = \{x \in A(L) \mid (b, x) \in R$ for some $b \in B\}$. In [9] we proved the following:

Proposition B ([9], Prop. 3.2). *For any $T \in \operatorname{Tol} L$, T^* is a congruence and we have $T^* = \theta(s)$ for $s = \bigvee\{a \mid a \in A(T^*)\} \in S(L)$ and $A(T^*) = A(L) \setminus R(A(T))$.*

Corollary 4.1. *For arbitrary $T_1, T_2 \in \operatorname{Tol} L$, $A(T_1) = A(T_2)$ implies $T_1^* = T_2^*$.*

Proof. $A(T_1) = A(T_2)$ implies $R(A(T_1)) = R(A(T_2))$, whence in view of Prop. B we get $A(T_1^*) = A(T_2^*)$ and this gives $T_1^* = T_2^*$. ◊

For any pseudocomplemented lattice $\mathcal{L}$ its Boolean part is defined as $B(\mathcal{L}) = \{x \in \mathcal{L} \mid x = x^{**}\}$. $(B(\mathcal{L}), \wedge, \veebar)$ is a Boolean algebra, where $a \veebar b$ is defined to be $(a^* \wedge b^*)^*$ (for more details, see [6]). Now we formulate the following

Corollary 4.2. *$B(\operatorname{Tol} L)$ and $B(\operatorname{Con} L)$ are the same Boolean algebras.*

Proof. Clearly, $B(\operatorname{Con} L) \subseteq B(\operatorname{Tol} L)$. Take any $T \in B(\operatorname{Tol} L)$. As $T = (T^*)^*$, Prop. B gives $T \in \operatorname{Con} L$ and $T = T^{**}$ implies $T \in B(\operatorname{Con} L)$. Hence we get $B(\operatorname{Tol} L) = B(\operatorname{Con} L)$ and obviously the corresponding Boolean algebras are the same. ◊

Proposition 4.3. *Let L be an atomistic algebraic lattice. If $\operatorname{Tol} L$ is a complemented lattice, then L is tolerance-trivial, $\operatorname{Con} L$ is a Boolean lattice and any $\varphi \in \operatorname{Con} L$ is a factor congruence of L.*

Proof. Take any $T \in \operatorname{Tol} L$. Denoting the complement of T by T', first we prove that $(T')^* = T$. Since $T \wedge T' = \triangle$ implies $T \leq (T')^*$ and since $\operatorname{Tol} L$ is 0-modular, we obtain $(T')^* = (T \sqcup T') \wedge (T')^* = T$. Now Prop. B gives $T = \theta(s) \in \operatorname{Con} L$ for some $s \in S(L)$, i.e. we get that $\operatorname{Tol} L = \operatorname{Con} L$. Since $\operatorname{Con} L$ is distributive and complemented, it is Boolean. Let $\varphi \in \operatorname{Con} L$, then φ has a complement φ', moreover $\varphi = \theta(s_1)$ and $\varphi' = \theta(s_2)$ for some $s_1, s_2 \in S(L)$. As in view of Remark 2.1(ii) we have $\theta(s_1) \circ \theta(s_2) = \theta(s_2) \circ \theta(s_1)$, φ and φ' permute, therefore they are factor congruences of L. ◊

5. The proof of the main result

To prove our Main Theorem we need the following

Lemma 5.1. ([9], Lemma 1.12). (i) *For each $T \in \operatorname{Tol} L$, $A(\overline{T}) = A(T)$.*

(ii) *For any $\alpha_i \in \operatorname{Con} L$, $i \in I$ we have*

$$A\left(\bigwedge_{i\in I}\alpha_i\right)=\bigcap_{i\in I}A(\alpha_i)\ \textit{and}\ A\left(\bigvee_{i\in I}\alpha_i\right)=\bigcup_{i\in I}A(\alpha_i).$$

Corollary 5.2. *For any $T\in \mathrm{Tol}\,L$ we have $(\overline{T})^*=T^*$.*
Proof. As the above (i) gives $A(\overline{T})=A(T)$, by applying Cor. 4.1 we deduce $(\overline{T})^*=T^*$. ◊
Proof of Main Theorem. First, assume that $\mathrm{Tol}\,L$ satisfies the identity (L_n) and consider $\theta_1,\theta_2,\ldots,\theta_n\in \mathrm{Con}\,L$. As $(\mathrm{Con}\,L,\wedge,^*)$ is a subalgebra of $(\mathrm{Tol}\,L,\wedge,^*)$, we can write

$$(\theta_1\wedge\theta_2\wedge\ldots\wedge\theta_n)^*\vee(\theta_1^*\wedge\theta_2\wedge\ldots\wedge\theta_n)^*\vee\ldots\vee(\theta_1\wedge\theta_2\wedge\ldots\wedge\theta_n^*)^*\geqq$$

$$(\theta_1\wedge\theta_2\wedge\ldots\wedge\theta_n)^*\sqcup(\theta_1^*\wedge\theta_2\wedge\ldots\wedge\theta_n)^*\sqcup\ldots\sqcup(\theta_1\wedge\theta_2\wedge\ldots\wedge\theta_n^*)^*=\nabla,$$

and this proves that $\mathrm{Con}L$ satisfies the identity (L_n).

Conversely, assume that $\mathrm{Con}L$ satisfies (L_n) and consider T_1, $T_2,\ldots,T_n\in \mathrm{Tol}\,L$. As $\overline{T}_i\in \mathrm{Con}\,L$, $1\le i\le n$, we have:

$$(\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n)^*\vee$$
$$\vee((\overline{T}_1)^*\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n)^*\vee\ldots\vee(\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge(\overline{T}_n)^*)^*=\nabla.$$

Let us introduce the notations: $\beta_0=\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n$, $\beta_1= =(\overline{T}_1)^*\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n$, $\beta_n=\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge(\overline{T}_n)^*$. Since by Cor. 5.2 we have $T_1^*=(\overline{T}_1)^*$, $T_2^*=(\overline{T}_2)^*,\ldots,T_n^*=(\overline{T}_n)^*$, in view of Remark 2.4 and applying Lemma 5.1 we obtain

$$A(T_1\wedge T_2\wedge\ldots\wedge T_n)=\bigcap_{i=1}^{n}A(T_i)=$$
$$=\bigcap_{i=1}^{n}A(\overline{T}_i)=A(\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n)=A(\beta_0),$$

$$A(T_1^*\wedge T_2\wedge\ldots\wedge T_n)=A(T_1^*)\cap A(T_2)\cap\ldots\cap A(T_n)=$$
$$=A((\overline{T}_1)^*)\cap A(\overline{T}_2)\cap\ldots\cap A(\overline{T}_n)=A((\overline{T}_1)^*\wedge\overline{T}_2\wedge\ldots\wedge\overline{T}_n)=A(\beta_1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A(T_1\wedge T_2\wedge\ldots\wedge T_n^*)=A(\overline{T}_1\wedge\overline{T}_2\wedge\ldots\wedge(\overline{T}_n)^*)=A(\beta_n).$$

Now, Cor. 4.1 gives that $(T_1\wedge T_2\wedge\ldots\wedge T_n)^*=\beta_0^*$, $(T_1^*\wedge T_2\wedge\ldots\wedge T_n)^*=\beta_1$, $\ldots$,$(T_1\wedge T_2\wedge\ldots\wedge T_n^*)^*=\beta_n^*$. Since in view of Prop. B there exist $s_0,s_1,..s_n\in S(L)$ such that $\beta_0^*=\theta(s_0),\beta_1^*=\theta(s_1),\ldots,\beta_n^*= =\theta(s_n)$, by Remark 2.3 we have $\beta_0^*\sqcup\beta_1^*\sqcup\ldots\sqcup\beta_n^*=\beta_0^*\vee\beta_1^*\vee\ldots\vee\beta_n^*$. Summarizing the above results we can write:

$$(T_1 \wedge T_2 \wedge \ldots \wedge T_n)^* \sqcup$$
$$\sqcup (T_1^* \wedge T_2 \wedge \ldots \wedge T_n)^* \sqcup \ldots \sqcup (T_1 \wedge T_2 \wedge \ldots \wedge T_n^*)^* =$$
$$= \beta_0^* \sqcup \beta_1^* \sqcup \ldots \sqcup \beta_n^* = \beta_0^* \vee \beta_1^* \vee \ldots \vee \beta_n^* =$$
$$= (\overline{T}_1 \wedge \overline{T}_2 \wedge \ldots \wedge \overline{T}_n)^* \vee$$
$$\vee ((\overline{T}_1)^* \wedge \overline{T}_2 \wedge \ldots \wedge \overline{T}_n)^* \vee \ldots \vee (\overline{T}_1 \wedge \overline{T}_2 \wedge \ldots \wedge (\overline{T}_n)^*)^* = \nabla,$$

thus $\mathrm{Tol}\, L$ satisfies the identity (L_n). ◊

Corollary 5.3. *Let L be an atomistic algebraic lattice. Then* $\mathrm{Tol}\, L$ *is a* (not necessarily distributive) *Stone lattice if and only if* $\mathrm{Con}\, L$ *is a Stone lattice.*

Proof. Substituting $n = 1$ in (L_n) we obtain the Stonean identity $x^* \vee x^{**} = 1$. ◊

Corollary 5.4. *If L is a weakly modular atomistic algebraic lattice, then* $\mathrm{Tol}\, L$ *is a* (not necessarily distributive) *Stone lattice.*

Proof. Since the congruence lattice of a weakly modular atomistic algebraic lattice is a Stone lattice (as noted by [9]), our assertion is an immediate consequence of Cor. 5.3. ◊

Problems. Let L denote an atomistic algebraic lattice.

1) Under what conditions Tol$L$ is distributive ?
2) Under what conditions we have Con(Tol$L$) $\cong$Con(Con$L$) ?

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