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HAAR SPACES AND POLYNOMIAL SELECTIONS

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Abstract: A theorem on the existence of selections belonging to a Haar space is proved. As consequences some generalizations of the second author's earlier results concerning polynomial selections and separation by polynomials (cf. [7]) are obtained.

1. Introduction

Recall that if X, Y are two sets, then a map $\Phi : X \longrightarrow 2^Y$ is called a multifunction (or set–valued function). A function $h : X \longrightarrow Y$ is a *selection* of Φ if $h(x) \in \Phi(x)$ for every $x \in X$. For a topological vector space Y by $\mathrm{cc}(Y)$ we denote the family of all nonempty, compact and convex subsets of Y.

In [7] a necessary and sufficient condition for two real functions defined on a real interval I to be separated by a polynomial of degree at

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most n was given. It was obtained using a consequence of the classical Helly's theorem proved in [2] by E. Behrends and K. Nikodem. Their result was recently generalized by M. Balaj and K. Nikodem ([1, Th. 1]) in the following way:

Theorem A. *Let D be a nonempty subset of a set X, $(Y, \|\cdot\|)$ be a normed space and for each $i \in \{1, \ldots, l+1\}$ $\Phi_i : D \longrightarrow \mathrm{cc}(Y)$ be a multifunction. Assume that $\mathcal{W}$ is an l-dimensional subspace of the vector space of all functions from X to Y and D has enough points for $f\|_D = 0$ to imply $f = 0$, for each $f \in \mathcal{W}$. If for every $l+1$ points $x_1, \ldots, x_{l+1} \in D$ there exists an $h \in \mathcal{W}$ such that $h(x_i) \in \Phi_i(x_i)$, $(1 \le$ $\le i \le l+1)$, then for some $i_0 \in \{1, \ldots, l+1\}$ there exists an $\overline{h} \in \mathcal{W}$ such that $\overline{h}(x) \in \Phi_{i_0}(x)$ for all $x \in D$.*

The main goals of this note is to show that Th. 1 of [7] can be generalized using Th. A instead of Behrends and Nikodem's result and to give an extended version of Th. 2 of [7] on separation by polynomials.

2. Haar spaces

Let us adopt the following definition (cf. [3], [4], [6]):

Definition. Let D be a set containing at least n elements. A linear subspace $\mathcal{H}_n(D)$ of $\mathbb{R}^D$ will be called an *n–dimensional Haar space on D*, if for any n distinct elements $x_1, x_2, \ldots, x_n$ of D and any $y_1, y_2, \ldots, y_n \in$ $\in \mathbb{R}$ there is exactly one $h \in \mathcal{H}_n(D)$ for which $h(x_j) = y_j$ $(1 \le j \le n)$.

By the above definition it follows immediately that any non–zero function $h \in \mathcal{H}_n(D)$ has at most $n-1$ zeros.

Borwein [3] gave some examples of Haar spaces as follows:

(i) $\mathrm{span}\{1, e^{\alpha_1 x}, \ldots, e^{\alpha_{n-1} x}\}$, where $\alpha_1, \ldots, \alpha_{n-1}$ are distinct non–zero real numbers, is an n–dimensional Haar space on any real interval;

(ii) $\mathrm{span}\{1, x, x^2, \ldots, x^{n-2}, f(x)\}$, where $f^{(n-1)}(x) > 0$, is an n–dimensional Haar space on any real interval;

(iii) $\mathrm{span}\{1, x^2, x^4, \ldots, x^{2n-2}\}$ is an n–dimensional Haar space on each interval $[a, b]$, where $a > 0$.

If $x_1, x_2, \ldots, x_n$ are n distinct elements of D, then for each $j \in$ $\in \{1, 2, \ldots, n\}$ let $c_j(\,\cdot\,; x_1, x_2, \ldots, x_n)$ be the unique function in $\mathcal{H}_n(D)$ satisfying

$$c_j(x_i; x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \qquad (1 \le i \le n).$$

The proof of the following lemma is elementary.

Lemma 1. *Let $x_1, \dots, x_n$ be distinct elements of D and $y_1, \dots, y_n \in \mathbb{R}$. The unique function $h \in \mathcal{H}_n(D)$, for which $h(x_j) = y_j$ $(1 \le j \le n)$, is given by the formula*

$$h(x) = \sum_{j=1}^{n} c_j(x; x_1, x_2, \dots, x_n) y_j.$$

As a consequence of Lemma 1 we get

$$\mathcal{H}_n(D) = \operatorname{span}\{c_j(\,\cdot\,; x_1, x_2, \dots, x_n) \;:\; 1 \le j \le n\}$$

for every distinct points $x_1, \dots, x_n \in D$. In particular, $\mathcal{H}_n(D)$ is n–dimensional space of functions.

3. Main result

Our main result is the following

Theorem 1. *Let $\mathcal{H}_n(D)$ be an n–dimensional Haar space and $\Phi_i : D \longrightarrow \mathrm{cc}(\mathbb{R})$ $(1 \le i \le n+1)$ be set–valued functions. Assume that:*

(i) *for each $x \in D$, $\bigcap_{i=1}^{n+1} \Phi_i(x) \neq \emptyset$,*

(ii) *for any $n+1$ distinct elements $x_1, x_2, \dots, x_{n+1} \in D$ there exists an index $i \in \{1, 2, \dots, n+1\}$ such that*

$$\Phi_i(x_i) \cap \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \Phi_j(x_j) \neq \emptyset.$$

Then there exist $i_0 \in \{1, 2, \dots, n+1\}$ and $h \in \mathcal{H}_n(D)$ such that $h(x) \in \Phi_{i_0}(x)$ for all $x \in D$.

Proof. On account of Th. A it is enough to prove that for any $n+1$ points $x_1, x_2, \dots, x_{n+1} \in D$ there exists $h \in \mathcal{H}_n(D)$ such that $h(x_j) \in \Phi_j(x_j)$, $(1 \le j \le n+1)$.

Consider firstly $n+1$ distinct elements $x_1, x_2, \dots, x_{n+1} \in D$. By (ii) there exist $y_j \in \Phi_j(x_j)$ $(1 \le j \le n+1)$ such that

$$y_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) y_j.$$

for some $i \in \{1, \dots, n+1\}$. Since $\mathcal{H}_n(D)$ is a vector space, the function

$$h(x)=\sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})y_j$$

is an element of $\mathcal{H}_n(D)$ and satisfies

$$h(x_j)=y_j\in\Phi_j(x_j)\qquad(1\le j\le n+1).$$

Suppose now that the points $x_1,x_2,\dots,x_{n+1}\in D$ are not distinct, for instance $x_1=x_2=\dots=x_{k_1}=:\bar{x}_1$, $x_{k_1+1}=x_{k_1+2}=\dots=x_{k_2}=$ $=:\bar{x}_2$, $\dots$, $x_{k_l+1}=x_{k_l+2}=\dots=x_{n+1}=:\bar{x}_{l+1}$. By (i) we can choose $y_1\in\bigcap_{i=1}^{k_1}\Phi_i(\bar{x}_1)$, $y_2\in\bigcap_{i=k_1+1}^{k_2}\Phi_i(\bar{x}_2)$, $\dots$, $y_{l+1}\in\bigcap_{i=k_l+1}^{n+1}\Phi_i(\bar{x}_{l+1})$. Clearly there exists at least one function $h\in\mathcal{H}_n(D)$ such that $h(\bar{x}_i)=$ $=y_i$ $(1\le i\le l+1)$. Hence $h(x_i)\in\Phi_i(x_i)$ $(1\le i\le n+1)$ and the proof is complete. ◊

4. Applications

The following result extends under many aspects Th. 1 of [7].

Corollary 1. *Let $\mathcal{H}_n(D)$ be an n–dimensional Haar space and $F:$ $:D\longrightarrow \mathrm{cc}(\mathbb{R})$. The following statements are equivalent:*

(i) *there exists an $h\in\mathcal{H}_n(D)$ such that $h(x)\in F(x)$ for all $x\in D$;*

(ii) *for any $n+1$ distinct elements $x_1,x_2,\dots,x_{n+1}\in D$ there exists an index $i\in\{1,2,\dots,n+1\}$ such that*

$$F(x_i)\cap\sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})F(x_j)\neq\emptyset.$$

Proof. The implication (i)⇒(ii) follows immediately from Lemma. To prove the converse we take $\Phi_i=F$, $i=1,2,\dots,n+1$. Then the condition of Th. 1 are automatically fulfilled, hence the result follows from Th. 1. ◊

Next we present the following generalization of Th. 2 of [7].

Corollary 2. *Let $I\subset\mathbb{R}$ be an interval and $\mathcal{H}_n(I)$ an n–dimensional Haar space. Let $f_1,\dots,f_{n+1},g_1,\dots,g_{n+1}$ be real functions defined on I with the following properties:*

(i) $\max\{f_1(x),\dots,f_{n+1}(x)\}\le\min\{g_1(x),\dots,g_{n+1}(x)\}$ *for each $x\in I$;*

(ii)*for any $x_1 < \cdots < x_{n+1}$ belonging to I there exists an index $i \in \{1, \ldots, n+1\}$ such that*

$$\big[f_i(x_i), g_i(x_i)\big] \cap$$
$$\cap \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})\big[f_j(x_j), g_j(x_j)\big] \neq \emptyset.$$

Then there exist $i_0 \in \{1, \ldots, n+1\}$ and $h \in \mathcal{H}_n(I)$ such that $f_{i_0}(x) \le h(x) \le g_{i_0}(x)$ for all $x \in I$.

Proof. By (i) we get $f_i \le g_i$ for all $i \in \{1, \ldots, n+1\}$. Let $\Phi_i(x) = \big[f_i(x), g_i(x)\big]$, $x \in I$, $i = 1, \ldots, n+1$. Applying (i) once again we obtain $\bigcap_{i=1}^{n+1} \Phi_i(x) \neq \emptyset$ for all $x \in I$. Then the multifunctions $\Phi_1, \ldots, \Phi_{n+1} :$ $: I \longrightarrow \mathrm{cc}(\mathbb{R})$ fulfil all the assumptions of Th. 1. Thus Th. 1 can be applied to obtain the desired conclusion. ◊

5. Separation by polynomials

Let $I \subset \mathbb{R}$ be an interval and $\mathcal{P}_{n-1}(I)$ be the set of all polynomials $p : I \longrightarrow \mathbb{R}$ of degree at most $n-1$. Clearly, $\mathcal{P}_{n-1}(I)$ is an n–dimensional Haar space on I. According to Lagrange interpolation theorem, in this particular case $(\mathcal{H}_n(I) = \mathcal{P}_{n-1}(I))$ for any distinct $x_1, x_2, \ldots, x_n \in I$ we have

$$c_j(x; x_1, x_2, \ldots, x_n) = \prod_{\substack{k=1\\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

With this significance for $c_j(x; x_1, x_2, \ldots, x_n)$ we have another generalization of Th. 2 of [7].

Corollary 3. *Let $f_1, \ldots, f_{n+1}, g_1, \ldots, g_{n+1}$ be real functions defined on I with the following properties:*

(i) $\max\{f_1(x), \ldots, f_{n+1}(x)\} \le \min\{g_1(x), \ldots, g_{n+1}(x)\}$ *for each $x \in I$;*

(ii) *for any $x_1 < \cdots < x_{n+1}$ belonging to I there exists an index $i \in \{1, \ldots, n+1\}$ such that*

$$f_i(x_i) \le \sum_{j \in S_1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{n+1}) g_j(x_j) +$$

$$+\sum_{j\in S_2} c_j(x_i;x_1,\dots x_{i-1},x_{i+1},\dots,x_{n+1})f_j(x_j)$$

and

$$g_i(x_i)\ge \sum_{j\in S_1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1}\dots,x_{n+1})f_j(x_j)+$$

$$+\sum_{j\in S_2} c_j(x_i;x_1,\dots x_{i-1},x_{i+1},\dots,x_{n+1})g_j(x_j)$$

where $S_1=\{j: 1\le j\le n+1,\ j-i \text{ is odd}\}$, $S_2=\{j: 1\le j\le n+ +1,\ j\ne i,\ j-i \text{ is even}\}$.

Then there exist $i_0\in\{1,\dots,n+1\}$ *and* $h\in\mathcal{P}_{n-1}(I)$ *such that* $f_{i_0}(x)\le h(x)\le g_{i_0}(x)$ *for all* $x\in I$.

Proof. If $x_1<\dots<x_{n+1}$ $(x_j\in I)$ it is not hard to see that $c_j(x_i;x_1,\dots,x_{i-1},x_{i+1}\dots,x_{n+1})$ is positive if $j\in S_1$ and negative if $j\in S_2$.

Let u and v be the right hand sides of the inequalities (1) and (2), respectively. Then $v\le u$ and $[f_i(x_i),g_i(x_i)]\cap[v,u]\ne\emptyset$ (otherwise $g_i(x_i)<v$ contradicting (2), or $u<f_i(x_i)$ contradicting (1)).

It is easy to verify that

$$[v,u]=\sum_{\substack{j=1\\ j\ne i}}^{n+1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1}\dots,x_{n+1})\big[f_j(x_j),g_j(x_j)\big].$$

Hence the condition (ii) of Cor. 2 is satisfied. Thus the result follows from Cor. 2. ◊

Theorems on selections of multifunctions as well as theorems on separation by functions belonging to some classes are often useful for investigating the problems of the stability of Hyers-Ulam type (see [1], [5], [7]). Our last result is in this spirit.

Corollary 4. *Let* $\varphi\in\mathcal{P}_{n-1}(I)$ *and* $f_1,f_2,\dots,f_{n+1}: I\longrightarrow\mathbb{R}$ *satisfy the following conditions:*

(i) $\big|f_i(x)-f_j(x)\big|\le\varphi(x)$, *for each* $i,j\in\{1,2,\dots,n+1\}$ *and all* $x\in I$;

(ii) *for any* $x_1<\dots<x_{n+1}$ *belonging to* I *there exists an index* $i\in\{1,\dots,n+1\}$ *such that*

$$(1)\quad \Big|f_i(x_i)-\sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})f_j(x_j)\Big|\leq \varphi(x_i).$$

Then there exists an $h\in\mathcal{P}_{n-1}$ such that

$$\big|f_i(x)-h(x)\big|\leq \frac{3}{2}\varphi(x)$$

for all $x\in I$, $i\in\{1,2,\dots,n+1\}$.

Proof. Let $g_i=f_i+\varphi$ $(1\leq i\leq n+1)$. Then, by (i) we immediately get

$$\max\{f_1(x),\dots,f_{n+1}(x)\}\leq \min\{g_1(x),\dots,g_{n+1}(x)\},\ \text{for each } x\in I.$$

Fix $x_1,\dots,x_n\in I$ such that $x_1<\cdots<x_{n+1}$. Since $\varphi\in\mathcal{P}_{n-1}(I)$, by Lemma 1 it admits the representation

$$\varphi(x)=\sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})\varphi(x_j).$$

Keeping in mind the fact that $c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})$ is positive if $j\in S_1$ and negative if $j\in S_2$, from (4) we succesively infer:

$$\begin{aligned} f_i(x_i)&\leq \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})g_j(x_j)\leq\\ &\leq \sum_{j\in S_1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})g_j(x_j)+\\ &+\sum_{j\in S_2} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})f_j(x_j)\end{aligned}$$

and

$$\begin{aligned} g_i(x_i)&\geq \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})f_j(x_j)\geq\\ &\geq \sum_{j\in S_1} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})f_j(x_j)+\\ &+\sum_{j\in S_2} c_j(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})g_j(x_j).\end{aligned}$$

Therefore the functions f_i, g_i $(1\leq i\leq n+1)$ satisfy the conditions of Cor. 3. Thus, for some $i_0\in\{1,2,\dots,n+1\}$ there exist $h'\in\mathcal{P}_{n-1}(I)$

such that $f_{i_0}(x) \le h'(x) \le g_{i_0}(x) = f_{i_0}(x) + \varphi(x)$. Let $h(x) = h'(x) - \frac{1}{2}\varphi(x)$. Hence $h \in \mathcal{P}_{n-1}(I)$ and $f_{i_0}(x) - \frac{1}{2}\varphi(x) \le h(x) \le f_{i_0}(x) + \frac{1}{2}\varphi(x)$, from which we obtain $\left|f_{i_0}(x) - h(x)\right| \le \frac{1}{2}\varphi(x)$, $x \in I$. For $i \in \{1, 2, \dots, n+1\}$ and $x \in I$ by (*i*) we get

$$\begin{aligned} \left|f_i(x) - h(x)\right| \le \left|f_i(x) - f_{i_0}(x)\right| + \left|f_{i_0}(x) - h(x)\right| \le \\ \le \varphi(x) + \frac{1}{2}\varphi(x) = \frac{3}{2}\varphi(x), \end{aligned}$$

which finishes the proof. $\Diamond$

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