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COMMUTATIVITY OF THE TOPOLOGICAL SEQUENCE ENTROPY ON FINITE GRAPHS

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Abstract: Let $f, g : G \to G$ be two continuous maps defined on a finite graph G. Denote by $h_A(f)$ the topological sequence entropy of f relative to the sequence of positive integers A. We prove for any sequence A the formula $h_A(f \circ g) = h_A(g \circ f)$.

1. Introduction

Let (X, d) be a compact metric space and consider maps $F : X \times \times X \to X \times X$ defined by $F(x, y) = (f(y), g(x))$, $(x, y) \in X \times X$, where $f, g : X \to X$ are continuous maps. These maps model economic phenomena called duopoly games (see [4], [12] or [11]). Notice that, for any $(x, y) \in X \times X$, it holds that

$$F^2(x, y) = F(F(x, y)) = (f \circ g(x), g \circ f(y)).$$

So, the dynamical behaviour of F must be connected in some sense with

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the dynamical behaviour of the maps $f \circ g$ and $g \circ f$. Following this idea, when $X = [0,1]$, some dynamical properties of F were studied in [10].

In this setting, in order to avoid unneccesary work, it is interesting to study which is the relationship between the dynamical properties of $f \circ g$ and $g \circ f$. For instance, in the case of the topological entropy, it is well known that $h(f \circ g) = h(g \circ f)$ (see [4] and [9]) and hence it is easy to see that $h(F) = h(f \circ g) = h(g \circ f)$. It is natural to think that a similar situation is held for others topological invariants. However, it was proved in [2] that the topological sequence entropy does not satisfy this property: for the sequence $A = (2^i)_{i=1}^{\infty}$ there are two continuous maps $f, g : X \to X$, with X a Cantor type set, such that $h_A(f \circ g) \neq \neq h_A(g \circ f)$. This situation is impossible when one considers the spaces $X = [0,1]$ or $X = S^1$; for any pair of continuous interval or circle maps f, g the formula $h_A(f \circ g) = h_A(g \circ f)$ holds for any increasing sequence of positive integers A (see [2]). In this paper we will extend this result for maps defined on finite graphs.

2. Preliminaries

Let (X,d) be a compact metric space. Let us denote by $C(X,X)$ and $\mathcal{I}$ the sets containing all the continuous maps $f : X \to X$ and all the increasing sequences of positive integers, respectively. For all $n \in \in \mathbb{N}$, f^n will denote the composition $f \circ \overset{n}{\ldots} \circ f$ (f^0 will be the identity). Given an $f \in C(X,X)$ and $A = (a_i)_{i=1}^{\infty} \in \mathcal{I}$, the topological sequence entropy (see [7]) is defined as follows. Let $Z \subset X$ and let $\varepsilon > 0$. A set $E \subset Z$ is said $(A, n, \varepsilon, Z, f)$-separated if for any $x, y \in E$, $x \neq y$ there is a $k \in \{1, 2, \ldots, n\}$ with $d(f^{a_k}(x), f^{a_k}(y)) > \varepsilon$. Denote by $s_n(A, \varepsilon, Z, f)$ the cardinality of any maximal $(A, n, \varepsilon, Z, f)$-separated contained in Z. It is easy to see that if $Z_1 \subset Z_2 \subseteq X$, then

$$s_n(A, \varepsilon, Z_1, f) \leq s_n(A, \varepsilon, Z_2, f). \tag{1}$$

It is also easy to check that for any $Z_1, Z_2 \subseteq X$ it holds that

$$s_n(A, \varepsilon, Z_1 \cup Z_2, f) \leq s_n(A, \varepsilon, Z_1, f) + s_n(A, \varepsilon, Z_2, f). \tag{2}$$

Let

$$s(A, \varepsilon, Z, f) := \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \varepsilon, Z, f).$$

The topological sequence entropy of f in Y is defined as the number

$$h_A(f, Y) := \lim_{\varepsilon \to 0} s(A, \varepsilon, Y, f)$$

and the topological sequence entropy of f is

$$h_A(f) := h_A(f, X).$$

Clearly, when $A = (i)_{i=0}^{\infty}$ this definition leads us to the classical topological entropy (see [3]). When one dimensional maps are consider, the topological sequence entropy is a useful tool to check if a continuous map is chaotic in the sense of Li–Yorke (see [6] and [8]).

Recall that a point $x \in X$ is said periodic if there exists a positive integer n such that $f^n(x) = x$. The smallest positive integer satisfying this condition is called the period of x. A point x is eventually periodic if there exists a positive integer k such that $f^k(x)$ is periodic. Denote by $\mathrm{Per}(f)$ and $\mathrm{EPer}(f)$ the sets of periodic and eventually periodic points of f, respectively.

A finite graph (or simply a graph) G is a connected Hausdorff space which has a finite subspace V (points of V are called vertices) such that $G \setminus V$ is a disjoint union of finite number of open subsets $e_1, e_2, \ldots, e_k$ (called edges), each of them homeomorphic to an open interval of the real line, and one or two vertices are attached at the boundary of each edge. A graph G can be embedded in a closed ball of radius one, and hence G is a compact metric space. As usual, denote by d the metric on G. For any edge e_i, denote by $|e_i|$ its diameter. Since the number of edges of G is finite, let

$$\lambda = \lambda(G) = \min\{|e_i| : e_i \text{ is an edge of } G\}. \tag{3}$$

For $x, y \in e_i$, let $[x, y] \subseteq e_i$ be the arc of G connecting x and y. For complementary information on graphs and dynamic properties of continuous maps defined on graphs see for instance [1].

3. Proof of the commutativity formula

The commutativity formula for the topological sequence entropy is deeply connected with the surjectivity of the maps f and g. More precisely, let X be a compact metric space and let $f \in C(X, X)$. Let $Y = \bigcap_{n \geq 0} f^n(X)$. Then, we have the following result (see [2]).

Theorem 1. *If $h_A(f|_Y) = h_A(f)$ holds for any $f \in C(X, X)$ and any $A \in \mathcal{I}$, then*

$$h_A(f \circ g) = h_A(g \circ f)$$

for any $f, g \in C(X, X)$ and any $A \in \mathcal{I}$.

So, given a finite graph G, in order to prove the commutativity formula for maps $f, g \in C(G,G)$ it suffices to prove that $h_A(f|_Y) = h_A(f)$ for any $f \in C(G,G)$ and any $A \in \mathcal{I}$. Previously, we need some useful definitions and several easy lemmas.

Notice that $f^n(G)$ is a finite graph for all $n \in \mathbb{N}$. Denote by V_n the set of vertices of $f^n(G)$ for all $n = 0, 1, 2, \ldots$ Since $f^{n+1}(G) \subseteq f^n(G)$ for all $n \in \mathbb{N}$, it is clear that if $v \in V_n$ and $v \notin V_{n+1}$, then $v \notin Y$. Denote by V_∞ the set of vertices of $Y = \bigcap_{n \geq 0} f^n(G)$. Here, we will also consider as vertices of Y those points obtained as limit points of sequences $(v_i)_{i=0}^\infty$ with $v_i \in V_i$. In order to illustrate this, consider the following example. Let $\mathbb{Y} = \{z \in \mathbb{C} : z^3 \in [0,1]\}$. Denote by B_1 the branch of $\mathbb{Y}$ with vertices 0 and 1, that is, $B_1 = [0,1]$. Denote by B_2 and B_3 the others two branches of $\mathbb{Y}$ with vertices v_2 and v_3. Define $f : \mathbb{Y} \to \mathbb{Y}$ as follows. If $x \in B_1$, then let $f(x) = x/2$, and define f on $B_2 \cup B_3$ satisfying that $f(B_2 \cup B_3) = B_2 \cup B_3$ and continuous ($f(0) = 0$). Notice that $Y = \bigcap_{n \geq 0} f^n(\mathbb{Y}) = B_2 \cup B_3$ and the vertices of Y are v_2, v_3 and 0.

Let λ be defined in (3). The following result is obvious.

Lemma 2. *Let $0 < \varepsilon < \lambda/2$. Then, there is a positive integer n_0 such that each connected component of $f^{n_0}(G) \setminus Y$ has length smaller than ε. Hence, each connected component of $f^{n_0}(G) \setminus Y$ is homeomorphic to an interval of the real line.*

Proof. It follows because f is uniformly continuous and the sequence $(f^i(G))_{i=0}^\infty$ decreases to Y. $\Diamond$

For $n \in \mathbb{N}$, $n \geq n_0$, denote by $C_1^n, C_2^n, \ldots, C_r^n$ the connected components of $f^n(G) \setminus Y$. Notice that, for $1 \leq i \leq r$, the closure of C_i^n, $\mathrm{Cl}(C_i^n) = [v, u]$ with $v \in V_\infty$ and $u \in V_n$. For any $v \in V_\infty$, let $i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, r\}$ be such that $v \in \mathrm{Cl}(C_{i_j}^n)$, $1 \leq j \leq s$. Define $C_v^n = \bigcup_{j=1}^s \mathrm{Cl}(C_i^n)$. Notice that it is possible that $C_v^n = \emptyset$ for some $v \in V_\infty$. It is also clear that $C_v^{n+1} \subseteq C_v^n$ for all $n \geq n_0$. We distinguish four types of vertices of Y in the following lemma.

Lemma 3. *Let $v \in V_\infty$. Under the conditions of Lemma 2, there is a $n_0 \in \mathbb{N}$ such that one and only one of the following possibilities holds:*

(a) $C_v^{n_0} = \emptyset$.

(b) $C_v^{n_0} = \{v\}$.

(c) $C_v^{n_0}$ *is infinite and* $f^{n_0}(C_v^{n_0}) \subseteq Y$.

(d) $C_v^{n_0}$ *is infinite and* $f^n(C_v^n) \nsubseteq Y$ *for all* $n \geq n_0$.

Proof. By Lemma 2, there is an $m_0 \in \mathbb{N}$ and some vertices $u \in V_{m_0}$ such that $C_v^{m_0} = \cup_u [v,u]$. If $C_v^{m_0} = \emptyset$ or $C_v^{m_0} = \{v\}$, then there is nothing to prove. Assume that $C_v^{m_0} \neq \emptyset$ and $C_v^{m_0} \neq \{v\}$. Clearly, $C_v^{m_0}$ must be infinite. Let $V_\infty^{m_0} = \{v \in V_\infty : C_v^{m_0} \text{ is infinite}\}$. For any $v \in V_\infty^{m_0}$ two possibilities hold: either $f^k(C_v^{m_0}) \not\subseteq Y$ for all $k \in \mathbb{N}$ or there is a $k_v \in \mathbb{N}$ such that $f^{k_v}(C_v^{m_0}) \subseteq Y$. Since $V_\infty^{m_0}$ is finite, let $k_{v_1}, \ldots, k_{v_j}$ be positive integers associated to $v_i \in V_\infty^{m_0}$, $1 \le i \le j$, such that $f^{k_{v_i}}(C_{v_i}^{m_0}) \subseteq Y$ for $1 \le i \le j$. Suppose also that if $v \in V_\infty^{m_0} \setminus \{v_1, \ldots, v_j\}$, then $f^k(C_v^{m_0}) \not\subseteq Y$ for all $k \in \mathbb{N}$. Let $n_0 = \max\{k_{v_1}, \ldots, k_{v_j}, m_0\}$. Notice that, since $C_v^{n_0} \subseteq C_v^{m_0}$ for all $v \in V_\infty$ and $f(Y) = Y$, it holds that $f^{n_0}(C_{v_i}^{n_0}) \subseteq Y$ for all $i = 1, \ldots, j$. This concludes the proof. ◊

Let $V_\infty^i = \{v \in V_\infty : v \text{ satisfies condition (d) in Lemma 3}\}$.

Lemma 4. $f(V_\infty^i) \subseteq V_\infty^i$. *Moreover, since V_∞^i is finite, each $v \in V_\infty^i$ is periodic or eventually periodic.*

Proof. Let $v \in V_\infty^i$. Then, there is a sequence of vertices $v_n \in V_n$, $v_n \neq v$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} v_n = v$ and holding that, if n is big enough, then $f^k[v, v_n] \not\subseteq Y$ for all $k \in \mathbb{N}$. Since f is continuous, $\lim_{n\to\infty} f(v_n) = f(v)$.

Notice that $f(v) \notin Y \setminus V_\infty$. In the contrary case, by the continuity of f, it must exist a $k \in \mathbb{N}$ with $f(C_v^k) \subset Y \setminus V_\infty$, and this leads to a contradiction. Using a similar argument it can be proved that $f(v) \notin V_\infty \setminus V_\infty^i$. Finally, $f(v) \notin G \setminus Y$ because $f(v)$ has infinite preimages and any point in $G \setminus Y$ has a finite number of preimages. This concludes the proof. ◊

Now, a general lemma on topological sequence entropy previously proved in [2]. Let $\sigma : \mathcal{I} \to \mathcal{I}$ be the shift map defined by $\sigma(A) = \sigma((a_i)_{i=1}^\infty) = (a_{i+1})_{i=1}^\infty$ for all $A \in \mathcal{I}$.

Lemma 5. *Let (X, d) be a compact metric space and let $f \in C(X, X)$. Then, for any $A \in \mathcal{I}$ any $\varepsilon > 0$ and any $k \in \mathbb{N}$ it holds that*

$$s(A, 2\varepsilon, X, f) \le s(\sigma^k(A), \varepsilon, X, f) \le s(A, \varepsilon, X, f).$$

Now, we are ready to prove our main theorem.

Theorem 6. *Let $f : G \to G$ be continuous. Then, for any $A \in \mathcal{I}$ it holds that*

$$h_A(f) = h_A(f, Y).$$

Proof. Fix a positive real number $\varepsilon < \lambda/2$ (see (3)). Since V_∞ is finite, by Lemmas 2, 3 and 4, there is a positive integer n_0 satisfying the following conditions:

(C1) $\operatorname{diam}(\mathcal{C}_v^{n_0}) < \varepsilon$ for all $v \in V_\infty$.
(C2) If $v \in V_\infty \setminus V_\infty^i$, then $f^n(\mathcal{C}_v^{n_0}) \subset Y$ for all $n \geq n_0$.
(C3) $f(V_\infty^i) \subset V_\infty^i \subset \operatorname{Per}(f) \cup \operatorname{EPer}(f)$.
(C4) Let $v \in V_\infty$. If $f(v) = u \in V_\infty$, then there is a $\delta > 0$ such that if U is a neighborhood of diameter smaller than δ of some $w \in V_\infty$, $v \neq u$, then $f(\mathcal{C}_v^{n_0}) \cap U = \emptyset$. We can clearly assume that $\varepsilon \leq \delta$.

Let k be the first integer such that $a_{k+1} > n_0$. By Lemma 5, it holds that

$$s(A, 4\varepsilon, G, f) \leq s(\sigma^k(A), 2\varepsilon, G, f). \tag{4}$$

In what follows, we will work with $\sigma^k(A)$ instead of A.

Take a partition of $f^{n_0}(G) \setminus Y$ by connected sets with diameter smaller than ε homeomorphic to intervals. Let $\mathcal{P}_1 = \{P_1, P_2, \dots, P_r\}$ be the partition covering $f^{n_0}(G) \setminus Y$. Clearly, if $P_i \in \mathcal{P}_1$, then $f^j(P_i) \cap \cap P_i = \emptyset$ for any $j > n_0$. Let $\mathcal{P}_2 = \{\mathcal{C}_v^{n_0} : v \in V_\infty\}$. So, we can construct a partition of $G \setminus Y$ by

$$\mathcal{P} = \{P_1, P_2, \dots, P_r\} \cup \{\mathcal{C}_v^{n_0} : v \in V_\infty\}.$$

Fix $n \in \mathbb{N}$. Any $x \in G \setminus Y$ has associated a code $(C_1, C_2, \dots, C_l)$, $l \leq \leq n$, as follows; let l be the first integer such that $f^{a_{k+l+1}}(x) \in Y$. For $1 \leq i \leq l$, put $C_i = \mathcal{C}_v^{n_0}$ if $f^{a_{k+i}}(x) \in \mathcal{C}_v^{n_0}$. Notice that it is impossible that $f^{a_{k+i}}(x) \in P_j$ for some $1 \leq j \leq r$. Let

$$\mathcal{Z}(C_1, C_2, \dots, C_l) = \{x \in G \setminus Y \text{ with code } (C_1, C_2, \dots, C_l)\}.$$

Let E be an $(\sigma^k(A), n, \varepsilon, Y, f)$-separated set of maximal cardinality. We claim that

$$s_n(\sigma^k(A), 2\varepsilon, \mathcal{Z}(C_1, C_2, \dots, C_l), f) \leq \operatorname{Card}(E) = s_n(\sigma^k(A), \varepsilon, Y, f). \tag{5}$$

In order to see this, let F be an $(\sigma^k(A), n, 2\varepsilon, \mathcal{Z}(C_1, C_2, \dots, C_l), f)$-separated set of maximal cardinality. Since $f|_Y$ is surjective and E is maximal, any $x \in F$ has associated a point $y \in E$ such that $d(f^{a_{k+i}}(x), f^{a_{k+i}}(y)) < \varepsilon$ for $l < i \leq n$. Notice that different $x_1, x_2 \in F$ have associated different points $y_1, y_2 \in E$. This is due to the following fact: if x_1 and x_2 have associated the same y_1, then

$$\begin{aligned} &d(f^{a_{k+i}}(x_1), f^{a_{k+i}}(x_2)) \leq \\ &\leq d(f^{a_{k+i}}(x_1), f^{a_{k+i}}(y_1)) + d(f^{a_{k+i}}(y_1), f^{a_{k+i}}(x_2)) < \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

for all $l < i \leq n$. Since x and y have the same code $(C_1, \dots, C_l)$, $d(f^{a_{k+i}}(x), f^{a_{k+i}}(y)) < 2\varepsilon$ for $1 \leq i \leq l$. Then, x_1, x_2 would not be

$(\sigma^k(A), n, 2\varepsilon, \mathcal{Z}(C_1, C_2, \ldots, C_l), f)$-separated points. This proves our claim.

Let $G_k = \{x \in G : f^{a_{k+1}}(x) \in Y\}$. Notice that $G = G_k \cup (\cup_{l=1}^n \cup \cup_{(C_1,\ldots,C_l)} \mathcal{Z}(C_1, \ldots, C_l))$. Notice also that

$$s(\sigma^k(A), 2\varepsilon, Y, f) = s(\sigma^k(A), 2\varepsilon, G_k, f).$$

By (2) and (5),

$$s_n(\sigma^k(A), 2\varepsilon, G, f) \leq$$

$$\leq s_n(\sigma^k(A), 2\varepsilon, Y, f) + \sum_{l=1}^{n} \sum_{(C_1,\ldots,C_l)} s_n(\sigma^k(A), 2\varepsilon, \mathcal{Z}(C_1, C_2, \ldots, C_l), f)$$

$$\leq s_n(\sigma^k(A), 2\varepsilon, Y, f) \left(1 + \sum_{l=1}^{n} \sum_{(C_1,\ldots,C_l)} \mathrm{Card}\{(C_1, C_2, ..., C_l) : C_i \in \mathcal{P}\}\right).$$

So, we must compute the cardinality of $\{(C_1, C_2, \ldots, C_l) : C_i \in \in \mathcal{P}\}$ for some $1 \leq l \leq n$. First of all, notice that $C_i \in \mathcal{P}_2$ for all $1 \leq i \leq l$. Then

$$\mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}\} = \mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}_2\}.$$

So, we will estimate $\mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}_2\}$. Notice that if $1 < l$, then, by (C2) and (C4), we must consider only codes $\mathcal{C}_v^{n_0}$ with $v \in V_\infty^i$. Notice also that, by (C2) and (C4), if $C_1 = \mathcal{C}_v^{n_0}$, then $C_i = \mathcal{C}_u^{n_0}$ with $f^{a_{k+i}}(v) = u \in V_\infty^i$. Then

$$\mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}_2\} = \mathrm{Card}(V_\infty^i).$$

If $1 = l$, then obviusly $\mathrm{Card}\{(C_l)\} \leq \mathrm{Card}(V_\infty)$. In any case

$$\mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}\} \leq \mathrm{Card}(V_\infty).$$

Hence

$$\sum_{l=1}^{n} \mathrm{Card}\{(C_1, \ldots, C_l) : C_i \in \mathcal{P}\} \leq \sum_{l=1}^{n} \mathrm{Card}(V_\infty) \leq n\mathrm{Card}(V_\infty).$$

So

$$s_n(\sigma^k(A), 2\varepsilon, G, f) \leq (1 + n\mathrm{Card}(V_\infty)) s_n(\sigma^k(A), \varepsilon, Y, f). \tag{6}$$

Then, by Lemma 5 and (6), we have that

$$s(A, 4\varepsilon, G, f) \leq s(\sigma^k(A), 2\varepsilon, G, f) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \log s_n(\sigma^k(A), 2\varepsilon, G, f) \leq$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \log \left((1 + n\mathrm{Card}(V_\infty)) s_n(\sigma^k(A), \varepsilon, Y, f)\right) =$$

$$= s(\sigma^k(A), \varepsilon, Y, f) \leq$$
$$\leq s(A, \varepsilon, Y, f).$$

Since obviously $s(A, \varepsilon, Y, f) \leq s(A, \varepsilon, G, f)$, we obtain taking limits when ε tends to zero that

$$h_A(f, Y) = h_A(f),$$

which ends the proof. ◊

Theorem 7. *Let $f, g : G \to G$ be two continuous maps. Then for all $A \in \mathcal{I}$ it follows*

$$h_A(f \circ g) = h_A(g \circ f).$$

Proof. It follows by Ths. 1 and 6. ◊

Final remarks

It seems that the commutativity formula for the topological sequence entropy is a one dimensional property. As we have mentioned above, in [2] an example showing that Th. 7 does not hold for arbitrary compact metric spaces has been constructed. We also conjecture that Th. 7 does not hold in general in the case of two dimensional maps, for example triangular maps ($F : [0,1]^2 \to [0,1]^2$ is said triangular if it has the form $F(x, y) = (f(x), g(x, y))$, $(x, y) \in [0,1]^2$). Our conjecture is supported by the following result.

Theorem 8. *There is a triangular map F and an increasing sequence of positive integers A such that $h_A(F, Y) = 0$ and $h_A(F) > 0$.*

Proof. By [5], there is a triangular map $F_\alpha(x, y) = (\alpha x, g(x, y))$, $\alpha \in \in (0, 1)$ satisfying that:

(a) F is non-chaotic in the sense of Li–Yorke (see [5] for the definition).

(b) There is an increasing sequence of positive integers such that $h_A(F) > 0$.

It is easy to see that $\bigcap_{n \geq 0} F^n([0,1]^2) \subset \{0\} \times [0,1]$. On the other hand, the map $g_0 : [0,1] \to [0,1]$ given by $g_0(y) = F(0, y)$ is non-chaotic (if g_0 was chaotic, then F would be also chaotic). By Franzová–Smítal Theorem (see [6]), $h_A(g_0) = 0$. Then, we conclude that

$$h_A(F, Y) \leq h_A(g_0) = 0 < h_A(F),$$

which ends the proof. ◊

References

[1] ALSEDA, Ll., LLIBRE, J. and MISIUREWICZ, M.: Low-dimensional combinatorial dynamics, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), 1687–1704.

[2] BALIBREA, F., CÁNOVAS, J. and JIMENEZ LOPEZ, V.: Commutativity and non-commutativity of the topological sequence entropy, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1693–1709.

[3] BOWEN, R.: Entropy for group endomorphism and homogeneous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401–414.

[4] DANA, R. A. and MONTRUCCHIO, L.: Dynamic complexity in duopoly games, *Journal of Economic Theory* **44** (1986), 40–56.

[5] FORTI, G. L., PAGANONI, L. and SMITAL, J.: Strange traingular maps of the square, *Bull. Austral. Math. Soc.* **51** (1995), 395–415.

[6] FRANZOVA, N. and SMITAL, J.: Positive sequence topological entropy characterizes chaotic maps, *Proc. Amer. Math. Soc.* **112** (1991), 1083–1086.

[7] GOODMAN, T. N. T.: Topological sequence entropy, *Proc. London Math. Soc.* **29** (1974), 331–350.

[8] HRIC, R.: Topological sequence entropy for maps of the circle, *Comment. Math. Univ. Carolinae* **41** (2000), 53–59.

[9] KOLYADA, S. and SNOHA, L.: Topological entropy of nonautononous dynamical systems, *Random and Comp. Dynamics* **4** (1996), 205–233.

[10] LINERO, A.: Cuestiones sobre dinámica topológica de algunos sistemas bidimensionales y medidas invariantes de sistemas unidimensionales asociados, PhD Thesis, Universidad de Murcia, 1998.

[11] PUU, T.: Chaos in duopoly pricing, *Chaos, Solitons & Fractals* **1** (1991), 573–581.

[12] RAND, D.: Exotic phenomena in games and doupoly models, *J. Math. Econ.* **5** (1978), 173–184.