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HURWITZ AND TASOEV CONTINUED FRACTIONS WITH LONG PERIOD

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Abstract: Hurwitz and Tasoev continued fractions are regular continued fractions with a quasi-periodic pattern on the sequence of partial quotients. In previous works in this context the author obtained closed forms for the value of continued fractions of this type where the length of period was at most 4. Here a new method is developed yielding results with longer period.

1. Introduction

$\alpha = [a_0; a_1, a_2, \dots]$ denotes the regular (or simple) continued fraction expansion of a real number α, where

$$\alpha = a_0 + \theta_0, \quad a_0 = \lfloor \theta \rfloor ,$$
$$1/\theta_{n-1} = a_n + \theta_n, \quad a_0 = \lfloor \theta \rfloor \quad (n \ge 1).$$

Hurwitz continued fraction expansions, quasi-periodic simple continued fractions, have the form

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$$
\begin{aligned}
&[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty} = \\
&= [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots],
\end{aligned}
$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. Well-known examples are

$$
\begin{aligned}
e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots] = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}, \\
\tanh 1 &= \frac{e^2 - 1}{e^2 + 1} = [0; 1, 3, 5, 7, \dots] = [0; \overline{2k-1}]_{k=1}^{\infty}, \\
\tan 1 &= [1; 1, 1, 1, 3, 1, 1, 5, 1, \dots] = [1; \overline{2k-1, 1}]_{k=1}^{\infty}.
\end{aligned}
$$

It seems that every known example belongs to one of three types, e-type, tanh-type and tan-type. No concrete example where the degree of any polynomial exceeds 1 has been known.

Recently, the author [5] found more general forms of Hurwitz continued fractions belonging to tanh-type and tan-type. Namely,

$$
\begin{aligned}
&[0; ua, v(a+b), u(a+2b), v(a+3b), u(a+4b), v(a+5b), \dots] = \\
&= \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=0}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=0}^{n-1} (a+bi)^{-1}}
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
&[0; ua-1, 1, v(a+b)-2, 1, u(a+2b)-2, 1, v(a+3b)-2, 1, \dots] = \\
&= \frac{\sum_{n=0}^{\infty} (-1)^n (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=0}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (-1)^n (n!)^{-1} (uvb)^{-n} \prod_{i=0}^{n-1} (a+bi)^{-1}},
\end{aligned}
\tag{2}
$$

respectively. In [7], the author constituted more general forms of Hurwitz continued fractions of e-type, namely, the quasi-periodic continued fractions with period 3 whose partial quotients include at least one 1;

$$[0;\overline{u(a+bk)-1,1,v-1}]_{k=1}^{\infty}=$$

(3)
$$=\frac{\sum_{n=0}^{\infty}u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}b^{-n}(n!)^{-1}\left((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}$$

and

$$[0;\overline{v-1,1,u(a+bk)-1}]_{k=1}^{\infty}=$$

(4)
$$=\frac{\sum_{n=0}^{\infty}b^{-n}(n!)^{-1}\left(u^{-2n}v^{-2n-1}\prod_{i=1}^{n}(a+bi)^{-1}+u^{-2n-1}v^{-2n-2}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{\sum_{n=0}^{\infty}(uv)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}.$$

Tasoev continued fractions ([4], [5], [6], [7], [8], [12]) are also systematic but have hardly been known before. They are also quasi-periodic but $Q_j(k)$ includes exponentials in k instead of polynomials. In [5], the author found some more general Tasoev continued fractions. Namely,

(5)
$$[0;\overline{ua^k}]_{k=1}^{\infty}=\frac{\sum_{n=0}^{\infty}u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

(6)
$$[0;ua-1,\overline{1,ua^{k+1}-2}]_{k=1}^{\infty}=$$
$$=\frac{\sum_{n=0}^{\infty}(-1)^n u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

(7)
$$[0;\overline{ua^k,va^k}]_{k=1}^{\infty}=\frac{\sum_{n=0}^{\infty}u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}{\sum_{n=0}^{\infty}u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}$$

and

$$[0; ua-1, 1, va-2, \overline{1, ua^{k+1}-2, 1, va^{k+1}-2}]_{k=1}^{\infty} = \tag{8}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n}(a^i-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n}(a^i-1)^{-1}}.$$

We can safely say that Tasoev continued fractions are geometric and Hurwitz continued fractions are arithmetic ([6]). The Tasoev continued fractions corresponding to e-type Hurwitz continued fractions were also derived in [7];

$$[0; \overline{ua^k-1, 1, v-1}]_{k=1}^{\infty} = \tag{9}$$
$$= \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}\left((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}\right) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}$$

and

$$[0; \overline{v-1, 1, ua^k-1}]_{k=1}^{\infty} = \tag{10}$$
$$= \frac{\sum_{n=0}^{\infty}\left(u^{-2n} v^{-2n-1} a^{-n^2} + u^{-2n-1} v^{-2n-2} a^{-(n+1)^2}\right) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

The different types of Tasoev continued fractions with period 3 shown in [6] are

(11)
$$[0; \overline{ua^{2k-1}-1, 1, va^{2k}-1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}},$$

$$[0; ua, \overline{va^{2k}-1, 1, ua^{2k+1}-1}]_{k=1}^{\infty} = \tag{12}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}},$$

(13)

$$[0;\overline{ua^k-1,1,va^k-1}]_{k=1}^{\infty}=\frac{\sum_{n=0}^{\infty}u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}$$

and

$$[0;ua,\overline{va^k-1,1,ua^{k+1}-1}]_{k=1}^{\infty}=$$

(14)

$$=\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}.$$

Up to now, these are all Hurwitz and Tasoev continued fractions whose corresponding irrational numbers can be written explicitly. The length of the period is short and does not exceed 4. Of course, there are still more concrete Hurwitz and Tasoev continued fractions which belong to none of the above (See e.g. [12], [13]). However, each of them is not totally generalized. The Tasoev continued fractions in [8] can not be said to have explicit forms because they include recurrence relations.

In the case of $\sqrt{D}$, some technical methods have been studied yielding periodic simple continued fractions with long period (e.g. [9]). In this paper a new method is developed yielding quasi-periodic simple continued fractions with long period so that every partial quotient can be written explicitly.

2. Hurwitz's theorem

It is known that if α is a Hurwitz number, and the continued fraction expansion of α is Hurwitz' one, then

$$\beta=\frac{a\alpha+b}{c\alpha+d}$$

is also a Hurwitz number ([10, Satz 4.4, p. 119]). Raney [11] gave a method to obtain the continued fraction of β from that of α. This method is well applicable to find the initial part of partial quotients in β, but it is very hard to write down the continued fraction of β completely and explicitly except in some special cases, e.g. $\beta=2\alpha$ ([2], [14]).

In fact, Hurwitz described a method to obtain the continued fraction expansion $(a\alpha+b)/d$ from the continued fraction expansion of α. In most practical cases it is enough to consider the rational linear forms of α. According to Satz 4.1 (p. 111) in [10], which is essentially from Hurwitz [3] and Châtelet [1], it says

Lemma 1. *Let $[a_0; a_1, a_2, \dots]$ be the regular continued fraction of an irrational number α and denote its n-th convergent by $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$. Moreover, let $\beta = (r_0\alpha + t_0)/s_0$, where r_0, s_0 and t_0 are integers with $r_0 > 0$, $s_0 > 0$ and $r_0 s_0 = N > 1$. For an arbitrary index $\nu \ge 1$ we have*

$$\frac{r_0[a_0; a_1, \dots, a_{\nu-1}] + t_0}{s_0} = \frac{r_0 p_{\nu-1} + t_0 q_{\nu-1}}{s_0 q_{\nu-1}} = [b_0; b_1, \dots, b_{\mu-1}]$$

where the index μ is adjusted such that $\mu \equiv \nu \pmod 2$. Denote its convergent by

$$\frac{p'_{\mu-1}}{q'_{\mu-1}} = [b_0; b_1, \dots, b_{\mu-1}].$$

Then three integers t_1, r_1 and s_1 are uniquely given satisfying the matrix formula

$$\begin{pmatrix} r_0 & t_0 \\ 0 & s_0 \end{pmatrix} \begin{pmatrix} p_{\nu-1} & p_{\nu-2} \\ q_{\nu-1} & q_{\nu-2} \end{pmatrix} = \begin{pmatrix} p'_{\mu-1} & p'_{\mu-2} \\ q'_{\mu-1} & q'_{\mu-2} \end{pmatrix} \begin{pmatrix} r_1 & t_1 \\ 0 & s_1 \end{pmatrix},$$

where $r_1 > 0$, $s_1 > 0$, $r_1 s_1 = N$, $-s_1 \le t_1 \le r_1$ and $\beta = [b_0; b_1, \dots, b_{\mu-1}, \beta_\mu]$ with $\beta_\mu = (r_1\alpha_\nu + t_1)/s_1$.

In theory, as described in [10, pp. 112–114] we consider the form:

$$\alpha = [a_0 + Ng_0; a_1, \dots, a_{\nu_1-1}, a_{\nu_1} + Ng_1, a_{\nu_1+1}, \dots, a_{\nu_2-1}, \\ a_{\nu_2} + Ng_2, a_{\nu_2+1}, \dots, a_{\nu_3-1}, a_{\nu_3} + Ng_3, \dots],$$

where g_0 is an integer, g_i $(i = 1, 2, \dots)$ and N are positive integers, from which we calculate

$$[b_0; b_1, \dots, b_{\mu_1-1}] = \frac{r_0[a_0; a_1, \dots, a_{\nu_1-1}] + t_0}{s_0},$$

$$\beta = \frac{r_0\alpha + t_0}{s_0} = [b_0; b_1, \dots, b_{\mu_1-1}, \beta_{\mu_1}], \quad \mu_1 \equiv \nu_1 \pmod 2;$$

$$[b_{\mu_1}; b_{\mu_1+1}, \dots, b_{\mu_2-1}] = \frac{r_1[a_{\nu_1}; a_{\nu_1+1}, \dots, a_{\nu_2-1}] + t_1}{s_1},$$

$$\beta_{\mu_1} = \frac{r_1\alpha_{\nu_1}+t_1}{s_1} = [b_{\mu_1}; b_{\mu_1+1}, \ldots, b_{\mu_2-1}, \beta_{\mu_2}], \quad \mu_2-\mu_1 \equiv \nu_2-\nu_1 \pmod 2;$$

$$[b_{\mu_2}; b_{\mu_2+1}, \ldots, b_{\mu_3-1}] = \frac{r_2[a_{\nu_2}; a_{\nu_2+1}, \ldots, a_{\nu_3-1}]+t_2}{s_2},$$

$$\beta_{\mu_2} = \frac{r_2\alpha_{\nu_2}+t_2}{s_2} = [b_{\mu_2}; b_{\mu_2+1}, \ldots, b_{\mu_3-1}, \beta_{\mu_3}], \quad \mu_3-\mu_2 \equiv \nu_3-\nu_2 \pmod 2;$$

and so on. Finally, we obtain

$$\beta = \frac{r_0\alpha+t_0}{s_0} = [b_0 + r_0^2 g_0; b_1, \ldots, b_{\mu_1-1}, b_{\mu_1} + r_1^2 g_1, b_{\mu_1+1}, \ldots, b_{\mu_2-1}, \\ b_{\mu_2} + r_2^2 g_2, b_{\mu_2+1}, \ldots, b_{\mu_3-1}, b_{\mu_3} + r_3^2 g_3, \ldots].$$

Three examples were shown in [10, pp. 114–118]. If $\alpha = [1+2g_0; 1+2g_1, \overline{2g_k}]_{k=2}^{\infty}$, then

$$2\alpha - 1 = [1+4g_0; g_1, \overline{1, 1, g_k - 1}]_{k=2}^{\infty}.$$

If $\alpha = [1+2g_0; 2g_1, \overline{1+2g_k}]_{k=2}^{\infty}$, then

$$2\alpha - 1 = [1+4g_0; \overline{g_{3k-2}, 2+4g_{3k-1}, g_{3k}, 1, 1}]_{k=1}^{\infty}.$$

If $\alpha = [1+6g_0; \overline{2, 1, 1+6g_k}]_{k=1}^{\infty}$, then

$$(2\alpha - 1)/3 = [4g_0; \overline{1, 1, 3, 1, -1+g_{2k-1}, 1, 1, 3, 1, 4g_{2k}}]_{k=1}^{\infty}.$$

3. Some quasi-periodic continued fractions with longer period

Our ideas lie in combining the known general Hurwitz (or Tasoev) continued fractions and Hurwitz's Theorem to obtain the new Hurwitz (or Tasoev) continued fractions with longer period. We shall first look at some new examples.

Let $\alpha = [1+5g_0; \overline{2, 1, 1+5g_k}]_{k=1}^{\infty}$, and consider the continued fraction of $\beta = (\alpha - 1)/5$. Note that $[1; 2, 1] = 4/3$ and $[1; 2] = 3/2$. Since $([1; 2, 1] - 1)/5 = 1/15 = [0; 14, 1]$ with $[0; 14] = 1/14$, we have

$$\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 15 & 14 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 0 & 5 \end{pmatrix}.$$

Since $([1; 2, 1] - 4)/5 = -8/15 = [-1; 2, 7]$ with $[-1; 2] = -1/2$, we have

$$\begin{pmatrix} 1 & -4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -8 & -1 \\ 15 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Since $[1; 2, 1]/5 = 4/15 = [0; 3, 1, 2, 1]$ with $[0; 3, 1, 2] = 3/11$, we have

$$\begin{pmatrix}1&0\\0&5\end{pmatrix}\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}4&3\\15&11\end{pmatrix}\begin{pmatrix}1&-3\\0&5\end{pmatrix}.$$

Since $([1;2,1]-3)/5=-1/3=[-1;1,2]$ with $[-1;1]=0/1$, we have

$$\begin{pmatrix}1&-3\\0&5\end{pmatrix}\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}-1&0\\3&1\end{pmatrix}\begin{pmatrix}5&3\\0&1\end{pmatrix}.$$

Since $5[1;2,1]+3=29/3=[9;1,2]$ with $[9;1]=10/1$, we have

$$\begin{pmatrix}5&3\\0&1\end{pmatrix}\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}29&10\\3&1\end{pmatrix}\begin{pmatrix}1&-1\\0&5\end{pmatrix}.$$

We can then obtain the continued fraction with period 17:

$$\frac{\alpha-1}{5}=[g_0;\overline{14,1,-1+g_{5k-4},2,7,g_{5k-3},3,1,2,1,}\\ \overline{-1+g_{5k-2},1,2,9+25g_{5k-1},1,2,g_{5k}}]_{k=1}^{\infty}.$$

In fact, simultaneously we also have the continued fractions:

$$\frac{\alpha-4}{5}=[-1+g_0;\overline{2,7,g_{5k-4},3,1,2,1,-1+g_{5k-3},1,2,9+25g_{5k-2},}\\ \overline{1,2,g_{5k-1},14,1,-1+g_{5k}}]_{k=1}^{\infty},$$

$$\frac{\alpha}{5}=[g_0;\overline{3,1,2,1,-1+g_{5k-4},1,2,9+25g_{5k-3},1,2,g_{5k-2},}\\ \overline{14,1,-1+g_{5k-1},2,7,g_{5k}}]_{k=1}^{\infty},$$

$$\frac{\alpha-3}{5}=[-1+g_0;\overline{1,2,9+25g_{5k-4},1,2,g_{5k-3},14,1,-1+g_{5k-2},}\\ \overline{2,7,g_{5k-1},3,1,2,1,-1+g_{5k}}]_{k=1}^{\infty}.$$

and

$$5\alpha+3=[9+25g_0;\overline{1,2,g_{5k-4},14,1,-1+g_{5k-3},2,7,g_{5k-2},}\\ \overline{3,1,2,1,-1+g_{5k-1},1,2,9+25g_{5k}}]_{k=1}^{\infty}.$$

By the way, the continued fraction of $(\alpha-2)/5$ is not in this group, but

$$\frac{\alpha-2}{5}=[-1+g_0;\overline{1,6,1,1,-1+g_k}]_{k=1}^{\infty}.$$

Now, put $g_0=0$ and $g_k=(ua^k-2)/5$ $(k\ge 1)$ and set $\alpha-1=$ $=[0;\overline{2,1,ua^k-1}]_{k=1}^{\infty}$ in (10) with $v=3$. Then we have a new Tasoev continued fraction with period 17 as $(\alpha-1)/5$.

Theorem 1. *Let* $u \equiv 2$, $a \equiv 1 \pmod 5$ *with* $a > 1$. *Then*

$$\frac{\sum_{n=0}^{\infty}\left(3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}= \tag{15}$$
$$=\left[0;\overline{14,1,\frac{ua^{5k-4}-7}{5},2,7,\frac{ua^{5k-3}-2}{5},3,1,2,1,}\right.$$
$$\left.\overline{\frac{ua^{5k-2}-7}{5},1,2,5ua^{5k-1}-1,1,2,\frac{ua^{5k}-2}{5}}\right]_{k=1}^{\infty}.$$

Simultaneously, we have the following results as $(\alpha-4)/5$ or $(\alpha+1)/5$, $\alpha/5$, $(\alpha-3)/5$ or $(\alpha+2)/5$, and, $5\alpha+3$ or $5\alpha-6$, respectively.

Corollary 1.

$$\frac{\sum_{n=0}^{\infty}\left(-8\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$
$$=[-1;\overline{2,7,\frac{ua^{5k-4}-2}{5},3,1,2,1,\frac{ua^{5k-3}-7}{5},1,2,5ua^{5k-2}-1,}$$
$$\overline{1,2,\frac{ua^{5k-1}-2}{5},14,1,\frac{ua^{5k}-7}{5}}]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty}\left(7\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$
$$=[0;\overline{2,7,\frac{ua^{5k-4}-2}{5},3,1,2,1,\frac{ua^{5k-3}-7}{5},1,2,5ua^{5k-2}-1,}$$
$$\overline{1,2,\frac{ua^{5k-1}-2}{5},14,1,\frac{ua^{5k}-7}{5}}]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty}\left(4\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$
$$=[0;\overline{3,1,2,1,\frac{ua^{5k-4}-7}{5},1,2,5ua^{5k-3}-1,1,2,\frac{ua^{5k-2}-2}{5},}$$

$$\overline{14, 1, \frac{ua^{5k-1}-7}{5}, 2, 7, \frac{ua^{5k}-2}{5}}]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty}\left(-5\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$=[-1;\overline{1, 2, 5ua^{5k-4}-1, 1, 2, \frac{ua^{5k-3}-2}{5}, 14, 1, \frac{ua^{5k-2}-7}{5},}$$

$$\overline{2, 7, \frac{ua^{5k-1}-2}{5}, 3, 1, 2, 1, \frac{ua^{5k}-7}{5}}]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty}\left(10\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{5\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$=[0;\overline{1, 2, 5ua^{5k-4}-1, 1, 2, \frac{ua^{5k-3}-2}{5}, 14, 1, \frac{ua^{5k-2}-7}{5},}$$

$$\overline{2, 7, \frac{ua^{5k-1}-2}{5}, 3, 1, 2, 1, \frac{ua^{5k}-7}{5}}]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty}\left(29\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+5\cdot 3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$=[9;\overline{1, 2, \frac{ua^{5k-4}-2}{5}, 14, 1, \frac{ua^{5k-3}-7}{5}, 2, 7, \frac{ua^{5k-2}-2}{5},}$$

$$\overline{3, 1, 2, 1, \frac{ua^{5k-1}-7}{5}, 1, 2, 5ua^{5k}-1}]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty}\left(2\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+5\cdot 3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$=[0;\overline{1, 2, \frac{ua^{5k-4}-2}{5}, 14, 1, \frac{ua^{5k-3}-7}{5}, 2, 7, \frac{ua^{5k-2}-2}{5},}$$

$$\overline{3, 1, 2, 1, \frac{ua^{5k-1}-7}{5}, 1, 2, 5ua^{5k}-1}]_{k=1}^{\infty}.$$

Next, put $g_0 = 0$ and $g_k = (u(a+bk)-2)/5$ $(k \ge 1)$ and set $\alpha - 1 = [0; \overline{2, 1, u(a+bk)-1}]_{k=1}^{\infty}$ in (4) with $v = 3$. Then we have a new Hurwitz continued fraction with period 17 as $(\alpha-1)/5$.

Theorem 2. *Let* $b \equiv 0$ *and* $ua \equiv 2 \pmod 5$. *Then*

(16)

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} + 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}} =$$

$$= [0; \overline{14, 1, \frac{u(5bk+a-4b)-7}{5}, 2, 7, \frac{u(5bk+a-3b)-2}{5}, 3, 1, 2, 1,}$$
$$\overline{\frac{u(5bk+a-2b)-7}{5}, 1, 2, 5u(5bk+a-b)-1, 1, 2, \frac{u(5bk+a)-2}{5}}]_{k=1}^{\infty}.$$

Remark. The condition implies that $u \equiv 1$ and $a \equiv 2$, $u \equiv 2$ and $a \equiv 1$, $u \equiv 3$ and $a \equiv 4$, or $u \equiv 4$ and $a \equiv 3 \pmod 5$ with $b \ge 5$.

Simultaneously, we have the following results as $(\alpha-4)/5$ or $(\alpha+1)/5$, $\alpha/5$, $(\alpha-3)/5$ or $(\alpha+2)/5$, and, $5\alpha+3$ or $5\alpha-6$, respectively.

Corollary 2.

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(-8\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} + 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}} =$$

$$= \Big[-1; \overline{2, 7, \frac{u(5bk+a-4b)-2}{5}, 3, 1, 2, 1, \frac{u(5bk+a-3b)-7}{5}, 1, 2, 5u(5bk+a-2b)-1,}$$
$$\overline{1, 2, \frac{u(5bk+a-b)-2}{5}, 14, 1, \frac{u(5bk+a)-7}{5}}\Big]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(7\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} + 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}} =$$

$$= \Big[0; \overline{2, 7, \frac{u(5bk+a-4b)-2}{5}, 3, 1, 2, 1, \frac{u(5bk+a-3b)-7}{5}, 1, 2, 5u(5bk+a-2b)-1,}$$
$$\overline{1, 2, \frac{u(5bk+a-b)-2}{5}, 14, 1, \frac{u(5bk+a)-7}{5}}\Big]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(4\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}+3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}=$$

$$=\Big[0;\overline{3,1,2,1,\frac{u(5bk+a-4b)-7}{5},1,2,5u(5bk+a-3b)-1,1,2,\frac{u(5bk+a-2b)-2}{5},}$$
$$\overline{14,1,\frac{u(5bk+a-b)-7}{5},2,7,\frac{u(5bk+a)-2}{5}}\Big]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(-5\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}+3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}=$$

$$=\Big[-1;\overline{1,2,5u(5bk+a-4b)-1,1,2,\frac{u(5bk+a-3b)-2}{5},14,1,\frac{u(5bk+a-2b)-7}{5},}$$
$$\overline{2,7,\frac{u(5bk+a-b)-2}{5},3,1,2,1,\frac{u(5bk+a)-7}{5}}\Big]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(10\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}+3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{5\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}=$$

$$=\Big[0;\overline{1,2,5u(5bk+a-4b)-1,1,2,\frac{u(5bk+a-3b)-2}{5},14,1,\frac{u(5bk+a-2b)-7}{5},}$$
$$\overline{2,7,\frac{u(5bk+a-b)-2}{5},3,1,2,1,\frac{u(5bk+a)-7}{5}}\Big]_{k=1}^{\infty},$$

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(29\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}+5\cdot 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}=$$

$$=\Big[9;\overline{1,2,\frac{u(5bk+a-4b)-2}{5},14,1,\frac{u(5bk+a-3b)-7}{5},2,7,\frac{u(5bk+a-2b)-2}{5},}$$
$$\overline{3,1,2,1,\frac{u(5bk+a-b)-7}{5},1,2,5u(5bk+a)-1}\Big]_{k=1}^{\infty}$$

or

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(2\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}+5\cdot 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}=$$

$$= \Big[0; 1, 2, \overline{\frac{u(5bk+a-4b)-2}{5}, 14, 1, \frac{u(5bk+a-3b)-7}{5}, 2, 7, \frac{u(5bk+a-2b)-2}{5},}$$
$$\overline{3, 1, 2, 1, \frac{u(5bk+a-b)-7}{5}, 1, 2, 5u(5bk+a)-1}\Big]_{k=1}^{\infty}.$$

4. More possibilities

As seen in the previous section, if the pattern under the period of the initial irrational number α matches one of the patterns of (1) to (4) or (5) to (14), then many new quasi-periodic continued fractions with longer period can be created having the explicit forms. Furthermore, since the new continued fractions which are derived from (1) to (4) or (5) to (14) have the explicit forms too, more new continued fractions can be created from there again.

For example, we set

$$\alpha' = \big[4g_0+1; \overline{14, 1, 4g_{5k-4}, 2, 7, 4g_{5k-3}+1, 3, 1, 2, 1, 4g_{5k-2},}$$
$$\overline{1, 2, 4g_{5k-1}+2, 1, 2, 4g_{5k}+1}\big]_{k=1}^{\infty}$$

in order to use (15) or (16). Notice that every g_k must take a positive integral value. Otherwise, the regular pattern will collapse. Consider the continued fraction of $(2\alpha'-1)/2$. Then, after long calculations, we obtain the continued fraction with period 128;

$$\frac{2\alpha'-1}{2} = [4g_0; \overline{1,1,3,4,g_{30k-29},8,1,1,3,g_{30k-28},14,1,4+16g_{30k-27},1,2,g_{30k-26},}$$
$$\overline{2, 2, 1, 1, -1+g_{30k-25}, 1, 3, 3, 2, 4g_{30k-24}, 1, 29, g_{30k-23}, 3, 6, 2, 1,}$$
$$\overline{-1+g_{30k-22}, 1, 2, 11+16g_{30k-21}, 1, 2, g_{30k-20}, 59, 1, -1+g_{30k-19},}$$
$$\overline{2, 1, 2, 1, 1, 1, 4g_{30k-18}, 1, 3, 3, 2, -1+g_{30k-17}, 1, 11, g_{30k-16}, 1, 2,}$$
$$\overline{5+16g_{30k-15}, 3, 1, 2, 1, -1+g_{30k-14}, 1, 1, 1, 1, 1, 1, 3, 1, -1+g_{30k-13}, 1, 1, 3, 4,}$$
$$\overline{1+4g_{30k-12}, 5, 1, -1+g_{30k-11}, 1, 11, g_{30k-10}, 3, 1, 2, 1, 4+16g_{30k-9}, 1, 6, 1, 1,}$$
$$\overline{-1+g_{30k-8}, 1, 4, 2, 5, g_{30k-7}, 5, 1, 2+4g_{30k-6}, 5, 1, -1+g_{30k-5}, 1, 1, 14, 2,}$$
$$\overline{-1+g_{30k-4},1,6,1,1,5+16g_{30k-3},14,1,-1+g_{30k-2},2,2,1,1,g_{30k-1},5,1,4g_{30k}}]_{k=1}^{\infty}.$$

We set $g_0=0$, $g_k=(5ua^k-3)/4$ if $k \equiv 4 \pmod 5$; $g_k=(ua^k-7)/20$ otherwise. Apply the result $\alpha' = (\alpha-1)/5+1$ in the previous section. Then we obtain the Tasoev continued fraction with period 128 as $(2\alpha+3)/10$.

Theorem 3. *Let* $u \equiv 7$ *and* $a \equiv 1 \pmod{20}$. *Then*

$$\frac{\sum_{n=0}^{\infty}\left(17\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+2\cdot 3^{-2n-2}u^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{10\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$=\Big[0;1,1,3,4,\frac{ua^{30k-29}-7}{20},8,1,1,3,\frac{ua^{30k-28}-7}{20},14,1,\frac{4ua^{30k-27}-8}{5},1,2,$$
$$\frac{5ua^{30k-26}-3}{4},2,2,1,1,\frac{ua^{30k-25}-27}{20},1,3,3,2,\frac{ua^{30k-24}-7}{5},1,29,$$
$$\frac{ua^{30k-23}-7}{20},3,6,2,1,\frac{ua^{30k-22}-27}{20},1,2,20ua^{30k-21}-1,1,2,$$
$$\frac{ua^{30k-20}-7}{20},59,1,\frac{ua^{30k-19}-27}{20},2,1,2,1,1,1,\frac{ua^{30k-18}-7}{5},1,3,3,2,$$
$$\frac{ua^{30k-17}-27}{20},1,11,\frac{5ua^{30k-16}-3}{4},1,2,\frac{4ua^{30k-15}-3}{5},3,1,2,1,$$
$$\frac{ua^{30k-14}-27}{20},1,1,1,1,1,1,3,1,\frac{ua^{30k-13}-27}{20},1,1,3,4,$$
$$\frac{ua^{30k-12}-2}{5},5,1,\frac{5ua^{30k-11}-7}{4},1,11,\frac{ua^{30k-10}-7}{20},3,1,2,1,$$
$$\frac{4ua^{30k-9}-8}{5},1,6,1,1,\frac{ua^{30k-8}-27}{20},1,4,2,5,\frac{ua^{30k-7}-7}{20},5,1,$$
$$5ua^{30k-6}-1,5,1,\frac{ua^{30k-5}-27}{20},1,1,14,2,\frac{ua^{30k-4}-27}{20},1,6,1,1,$$
$$\frac{4ua^{30k-3}-3}{5},14,1,\frac{ua^{30k-2}-27}{20},2,2,1,1,\frac{5ua^{30k-1}-3}{4},5,1,\frac{ua^{30k}-7}{5}\Big]_{k=1}^{\infty}.$$

If we put $u=7$ and $a=21$, then we get

$$\frac{\sum_{n=0}^{\infty}\left(17\cdot 3^{-2n-1}7^{-2n}21^{-n^2}+2\cdot 3^{-2n-2}7^{-2n-1}21^{-(n+1)^2}\right)\prod_{i=1}^{n}(21^{2i}-1)^{-1}}{10\sum_{n=0}^{\infty}21^{-2n}21^{-n^2}\prod_{i=1}^{n}(21^{2i}-1)^{-1}}=$$

$$\begin{gathered}=[0;1,1,3,4,7,8,1,1,3,154,14,1,51860,1,2,1701708,2,2,1,1,1429434,1,3,3,2,\\ 120072568,1,29,630380989,3,6,2,1,13238000775,1,2,111199206521339,1,2,\\ 5837958342370,59,1,122597125189776,2,1,2,1,1,1,10298158515941296,1,3,3,2,\\ 54065332208691810,1,11,2838429940956320095 8,1,2,38148498406452942088 5,\\ 3,1,2,1,500699041584694864911,1,1,1,1,1,1,3,1,10514679873278592163158,\\ 1,1,3,4,8832331093554017417053 85,5,1,1159243456028964785988318 32,\\ 1,11,973764503064330420230187 40,3,1,2,1,\\ 327184873029615021197342967 56,1,6,1,1,42943014585136971532151264493,\\ 1,4,2,5,901803306287876402175176554381,5,1,\\ 18937869432045404445678707642008 34,5,1,3976952580729534933592528604821 74,\end{gathered}$$

$$1, 1, 14, 2, 835160041953202336054431007012568\,1, 1, 6, 1, 1,$$
$$280613774096275984914288818356222926\,9, 14, 1,$$
$$368305578501362230200004074092542591\,5, 2, 2, 1, 1,$$
$$193360428713215170855002138898584860608\,3, 5, 1,$$
$$649691040476402974072807186699245131644\,0, 1, 1, 3, 4,$$
$$341087796250111561388223773017103694113\,17, 8, 1, 1, 3, \dots].$$

We set $g_0 = 0$, $g_k = \frac{5u(a+bk)-3}{4}$ if $k \equiv 4 \pmod 5$; $g_k = \frac{u(a+bk)-7}{20}$ otherwise. Apply the result $\alpha' = (\alpha - 1)/5 + 1$ in the previous section. Then we obtain the Hurwitz continued fraction with period 128 as $(2\alpha + 3)/10$.

Theorem 4. *Let $b \equiv 0$ and $ua \equiv 7 \pmod{20}$. Then*

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(17\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} + 2\cdot 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{10\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}} =$$

$$= \Big[0; \overline{1, 1, 3, 4, \frac{u(30bk + a - 29b) - 7}{20}, 8, 1, 1, 3, \frac{u(30bk + a - 28b) - 7}{20}, 14, 1,}$$
$$\overline{\frac{4u(30bk + a - 27b) - 8}{5}, 1, 2, \frac{5u(30bk + a - 26b) - 3}{4}, 2, 2, 1, 1, \frac{u(30bk + a - 25b) - 27}{20},}$$
$$\overline{1, 3, 3, 2, \frac{u(30bk + a - 24b) - 7}{5}, 1, 29, \frac{u(30bk + a - 23b) - 7}{20}, 3, 6, 2, 1,}$$
$$\overline{\frac{u(30bk + a - 22b) - 27}{20}, 1, 2, 20u(30bk + a - 21b) - 1, 1, 2, \frac{u(30bk + a - 20b) - 7}{20},}$$
$$\overline{59, 1, \frac{u(30bk + a - 19b) - 27}{20}, 2, 1, 2, 1, 1, 1, \frac{u(30bk + a - 18b) - 7}{5}, 1, 3, 3, 2,}$$
$$\overline{\frac{u(30bk + a - 17b) - 27}{20}, 1, 11, \frac{5u(30bk + a - 16b) - 3}{4}, 1, 2,}$$
$$\overline{\frac{4u(30bk + a - 15b) - 3}{20}, 3, 1, 2, 1, \frac{u(30bk + a - 14b) - 27}{20}, 1, 1, 1, 1, 1, 1, 3, 1,}$$
$$\overline{\frac{u(30bk + a - 13b) - 27}{20}, 1, 1, 3, 4, \frac{u(30bk + a - 12b) - 2}{5}, 5, 1,}$$
$$\overline{\frac{5u(30bk + a - 11b) - 7}{4}, 1, 11, \frac{u(30bk + a - 10b) - 7}{20}, 3, 1, 2, 1, \frac{4u(30bk + a - 9b) - 8}{5},}$$
$$\overline{1, 6, 1, 1, \frac{u(30bk + a - 8b) - 27}{20}, 1, 4, 2, 5, \frac{u(30bk + a - 7b) - 7}{20},}$$
$$\overline{5, 1, 5u(30bk + a - 6b) - 1, 5, 1, \frac{u(30bk + a - 5b) - 27}{20}, 1, 1, 14, 2,}$$
$$\overline{\frac{u(30bk + a - 4b) - 27}{20}, 1, 6, 1, 1, \frac{4u(30bk + a - 3b) - 3}{5}, 14, 1, \frac{u(30bk + a - 2b) - 27}{20},}$$

$$\overline{2,2,1,1,\frac{5u(30bk+a-b)-3}{4},5,1,\frac{u(30bk+a)-7}{5}}\Big]_{k=1}^{\infty}.$$

Remark. The condition means that $u\equiv 1$ and $a\equiv 7$, $u\equiv 7$ and $a\equiv 1$, $u\equiv 3$ and $a\equiv 9$, $u\equiv 9$ and $a\equiv 3$, $u\equiv 11$ and $a\equiv 17$, $u\equiv 17$ and $a\equiv 11$, $u\equiv 13$ and $a\equiv 19$ or $u\equiv 19$ and $a\equiv 13$ (mod 20).

If we put $u=3$, $a=9$ and $b=20$, then we get

$$\frac{\sum_{n=0}^{\infty}20^{-n}(n!)^{-1}(17\cdot 3^{-2n-1}3^{-2n}\prod_{i=1}^{n}(9+20i)^{-1}+2\cdot 3^{-2n-2}3^{-2n-1}\prod_{i=1}^{n+1}(9+20i)^{-1})}{10\sum_{n=0}^{\infty}9^{-2n}20^{-n}(n!)^{-1}\prod_{i=1}^{n}(9+20i)^{-1}}=$$

$$\begin{aligned}=[&0,1,1,3,4,4,8,1,1,3,7,14,1,164,1,2,333,2,2,1,1,15,1,3,3,2,76,1,29,22,3,\\ &6,2,1,24,1,2,11339,1,2,31,59,1,33,2,1,2,1,1,1,148,1,3,3,2,39,1,11,1083,\\ &1,2,741,3,1,2,1,48,1,1,1,1,1,1,3,1,51,1,1,3,4,221,5,1,1457,1,11,61,3,\\ &1,2,1,1028,1,6,1,1,66,1,4,2,5,70,5,1,7334,5,1,75,1,1,14,2,78,1,6,1,1,1317,\\ &14,1,84,2,2,1,1,2208,5,1,364,1,1,3,4,94,8,1,1,3,97,14,1,1604,1,2,2583,2,\\ &2,1,1,105,1,3,3,2,436,1,29,112,3,6,2,1,114,1,2,47339,1,2,121,59,1,123,2,1,\\ &2,1,1,1,508,1,3,3,2,129,1,11,3333,1,2,2181,3,1,2,1,138,1,1,1,1,1,\ldots].\end{aligned}$$

Note that we obtain a different result if we try to get the continued fraction of $(2\alpha+3)/10$ directly from the continued fraction α. In this case, we must put $\alpha=[1+20g_0;\overline{2,1,1+20g_k}]_{k=1}^{\infty}$ instead of $\alpha=$ $=[1+5g_0;\overline{2,1,1+5g_k}]_{k=1}^{\infty}$. Then for this new α we get the continued fraction expansion with period 80 as $(2\alpha+3)/10$.

$$\begin{aligned}&[4g_0;\overline{1,1,3,4,-1+g_{20k-19},1,6,1,1,1+16g_{20k-18},14,1,-1+g_{20k-17},5,1,}\\ &\overline{9+100g_{20k-16},5,1,-1+g_{20k-15},3,1,2,1,16g_{20k-14},1,6,1,1,-1+g_{20k-13},1,1,3,4,}\\ &\overline{4g_{20k-12},5,1,1+25g_{20k-11},1,2,1+16g_{20k-10},3,1,2,1,-1+g_{20k-9},2,1,2,1,1,1,}\\ &\overline{-1+4g_{20k-8},1,3,3,2,-1+g_{20k-7},1,2,39+400g_{20k-6},1,2,-1+g_{20k-5},1,3,3,2,}\\ &\overline{-1+4g_{20k-4},1,29,g_{20k-3},14,1,16g_{20k-2},1,2,2+25g_{20k-1},5,1,-1+4g_{20k}}]_{k=1}^{\infty}.\end{aligned}$$

Put $g_0=0$ and $g_k=(ua^k-2)/20$ $(k\ge 1)$ and set $\alpha-1=$ $=[0;\overline{2,1,ua^k-1}]_{k=1}^{\infty}$ in (10) with $v=3$. Then we obtain the Tasoev continued fraction with period 80 as $(2\alpha+3)/10$.

Theorem 5. *Let* $u\equiv 2$ *and* $a\equiv 1$ (mod 20). *Then*

$$\frac{\sum_{n=0}^{\infty}(17\cdot 3^{-2n-1}u^{-2n}a^{-n^2}+2\cdot 3^{-2n-2}u^{-2n-1}a^{-(n+1)^2})\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{10\sum_{n=0}^{\infty}(3u)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}=$$

$$
\begin{aligned}
= \Big[0; &\overline{1,1,3,4,\frac{ua^{20k-19}-22}{20},1,6,1,1,\frac{4ua^{20k-18}-3}{5},14,1,\frac{ua^{20k-17}-22}{20},5,1,}\\
&\overline{5ua^{20k-16}-1,5,1,\frac{ua^{20k-15}-22}{20},3,1,2,1,\frac{4ua^{20k-14}-8}{5},1,6,1,1,}\\
&\overline{\frac{ua^{20k-13}-22}{20},1,1,3,4,\frac{ua^{20k-12}-2}{5},5,1,\frac{5ua^{20k-11}-6}{4},1,2,}\\
&\overline{\frac{4ua^{20k-10}-3}{5},3,1,2,1,\frac{ua^{20k-9}-22}{20},2,1,2,1,1,1,\frac{ua^{20k-8}-7}{5},1,3,}\\
&\overline{3,2,\frac{ua^{20k-7}-22}{20},1,2,20ua^{20k-6}-1,1,2,\frac{ua^{20k-5}-22}{20},1,3,3,2,}\\
&\overline{\frac{ua^{20k-4}-7}{5},1,29,\frac{ua^{20k-3}-2}{20},14,1,\frac{4ua^{20k-2}-8}{5},1,2,}\\
&\overline{\frac{5ua^{20k-1}-2}{4},5,1,\frac{ua^{20k}-7}{5}}\Big]_{k=1}^{\infty}.
\end{aligned}
$$

When $u=2$ and $a=21$, we have

$$
\frac{\sum_{n=0}^{\infty}\left(17\cdot 3^{-2n-1}2^{-2n}21^{-n^2}+2\cdot 3^{-2n-2}2^{-2n-1}21^{-(n+1)^2}\right)\prod_{i=1}^{n}(21^{2i}-1)^{-1}}{10\sum_{n=0}^{\infty}6^{-2n}21^{-n^2}\prod_{i=1}^{n}(21^{2i}-1)^{-1}} =
$$

$$
\begin{aligned}
= [0; &1,1,3,4,1,1,6,1,1,705,14,1,925,5,1,1944809,5,1,408409,3,1,\\
&2,1,137225792,1,6,1,1,180108853,1,1,3,4,15129143744,5,1,\\
&1985700116451,1,2,26687809565121,3,1,2,1,35027750054221,2,1,\\
&2,1,1,1,2942331004554655,1,3,3,2,15447237773911945,1,2,\\
&129756797300860347239,1,2,681223185829516822 9,1,3,3,2,\\
&57222747609679413 1327,1,29,30041942495081691894 74,14,1,\\
&1009409267834744847663296,1,2,331212416008275653 13951952,5,1,\\
&1112873717787806194548785 59,1,1,3,4,58425870183859825213811 2441,\\
&1,6,1,1,196310923817769012718405780545,14,1,\\
&257658087510821829192907586965,5,1,5410819837727258413051059 32628809,\\
&5,1,11362721659227242667407224585 2049,3,1,2,1,\\
&381787447750035353624882746062 88832,1,6,1,1,\\
&501096025171921401632658604207 54093,1,1,3,4,\\
&42092066114441397737143322753433 43904,5,1,\\
&552458367752043345300006111138813887451,1,2,\\
&742504046258746256083208213370565864 7361,3,1,2,1,\\
&974536560714604461109210780048867697 4661,2,1,2,1,1,1,\\
&818610711000267747331737055241048865871615,1,3,3,\ldots].
\end{aligned}
$$

Put $\underline{g_0 = 0 \text{ and } g_k = (u(a+bk)-2)/20}$ $(k \geq 1)$ and set $\alpha - 1 = [0; \overline{2, 1, u(a+bk)-1}]_{k=1}^{\infty}$ in (4) with $v = 3$. Then we obtain the Hurwitz continued fraction with period 80 as $(2\alpha+3)/10$.

Theorem 6. *Let* $ua \equiv 2$ *and* $b \equiv 0 \pmod{20}$. *Then*

$$\frac{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\left(17\cdot 3^{-2n-1}u^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} + 2\cdot 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\right)}{10\sum_{n=0}^{\infty}(3u)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}} =$$

$$= \Big[0; \overline{1, 1, 3, 4, \frac{u(20bk+a-19b)-22}{20}, 1, 6, 1, 1, \frac{4u(20bk+a-18b)-3}{5}, 14, 1,}$$
$$\overline{\frac{u(20bk+a-17b)-22}{20}, 5, 1, 5u(20bk+a-16b)-1, 5, 1,}$$
$$\overline{\frac{u(20bk+a-15b)-22}{20}, 3, 1, 2, 1, \frac{4u(20bk+a-14b)-8}{5}, 1, 6, 1, 1,}$$
$$\overline{\frac{u(20bk+a-13b)-22}{20}, 1, 1, 3, 4, \frac{u(20bk+a-12b)-2}{5}, 5, 1,}$$
$$\overline{\frac{5u(20bk+a-11b)-6}{4}, 1, 2, \frac{4u(20bk+a-10b)-3}{5}, 3, 1, 2, 1,}$$
$$\overline{\frac{u(20bk+a-9b)-22}{20}, 2, 1, 2, 1, 1, 1, \frac{u(20bk+a-8b)-7}{5}, 1, 3, 3, 2,}$$
$$\overline{\frac{u(20bk+a-7b)-22}{20}, 1, 2, 20u(20bk+a-6b)-1, 1, 2, \frac{u(20bk+a-5b)-22}{20},}$$
$$\overline{1, 3, 3, 2, \frac{u(20bk+a-4b)-7}{5}, 1, 29, \frac{u(20bk+a-3b)-2}{20}, 14, 1,}$$
$$\overline{\frac{4u(20bk+a-2b)-8}{5}, 1, 2, \frac{5u(20bk+a-b)-2}{4}, 5, 1, \frac{u(20bk+a)-7}{5}}\Big]_{k=1}^{\infty}.$$

Remark. The condition means that $u \equiv 1$ and $a \equiv 2$, $u \equiv 2$ and $a \equiv 1$, $u \equiv 11$ and $a \equiv 2$, $u \equiv 2$ and $a \equiv 11$, $u \equiv 6$ and $a \equiv 7$, $u \equiv 7$ and $a \equiv 6$, $u \equiv 6$ and $a \equiv 17$, $u \equiv 17$ and $a \equiv 6$, $u \equiv 9$ and $a \equiv 18$, $u \equiv 18$ and $a \equiv 9$, $u \equiv 13$ and $a \equiv 14$, $u \equiv 14$ and $a \equiv 13$, $u \equiv 18$ and $a \equiv 19$ or $u \equiv 19$ and $a \equiv 18 \pmod{20}$.

When $u = 7$, $a = 6$ and $b = 20$, we have

$$\frac{\sum_{n=0}^{\infty} 20^{-n}(n!)^{-1}\left(17\cdot 3^{-2n-1}7^{-2n}\prod_{i=1}^{n}(6+20i)^{-1} + 2\cdot 3^{-2n-2}u^{-2n-1}\prod_{i=1}^{n+1}(6+20i)^{-1}\right)}{10\sum_{n=0}^{\infty}21^{-2n}20^{-n}(n!)^{-1}\prod_{i=1}^{n}(6+20i)^{-1}} =$$

$$= [0; 1, 1, 3, 4, 8, 1, 6, 1, 1, 257, 14, 1, 22, 5, 1, 3009, 5, 1, 36, 3, 1, 2, 1, 704, 1, 6, 1, 1,$$
$$50, 1, 1, 3, 4, 232, 5, 1, 1626, 1, 2, 1153, 3, 1, 2, 1, 78, 2, 1, 2, 1, 1, 1, 343, 1, 3, 3, 2,$$

$92, 1, 2, 40039, 1, 2, 106, 1, 3, 3, 2, 455, 1, 29, 121, 14, 1, 2048, 1, 2, 3377, 5, 1, 567,$
$1, 1, 3, 4, 148, 1, 6, 1, 1, 2497, 14, 1, 162, 5, 1, 17009, 5, 1, 176, 3, 1, 2, 1, 2944, 1, 6,$
$1, 1, 190, 1, 1, 3, 4, 792, 5, 1, 5126, 1, 2, 3393, 3, 1, 2, 1, 218, 2, 1, 2, 1, 1, 1, 903, 1, 3,$
$3, 2, 232, 1, 2, 96039, 1, 2, 246, 1, 3, 3, 2, 1015, 1, 29, 261, 14, 1, 4288, 1, 2, 6877,$
$5, 1, 1127, 1, 1, 3, 4, 288, 1, 6, 1, 1, 4737, 14, 1, 302, 5, 1, 31009, 5, 1, 316, 3, 1, \dots].$

If we would like to find cases with different congruent conditions, then we change c in the initial continued fraction $[1+20g_0; \overline{2, 1, c+20g_k}]_{k=1}^{\infty}$. In order to fit this to (10) we set $ua^k \equiv c+1 \pmod{20}$ so that every g_k is a positive integer. Concerning (4), we set $u(a+bk) \equiv c+1 \pmod{20}$.

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