Mathematica Pannonica
17/1 (2006), 131–138

ON NEARRINGS WITH DERIVATION

M. A. **Khan**

Department of Mathematics, Eritrea-Institute of Technology, Mainefhi, P.O. Box 10373, Asmara, Eritrea

M. S. **Khan**

Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, P C 123, Al-Khodh, Muscat, Sultanate of Oman

Received: June 2005

MSC 2000: 16 W 20, 16 N 60, 16 Y 30

Keywords: Nearring, near-field, prime nearring, derivation, commutativity.

Abstract: In the present paper, it is shown that the multiplicative or additive commutativity of nearring N if N admits a non-zero derivation F or G such that $[F(x), G(x)] = [x, y]$ for all $x, y \in B$, where N is a nearring and $B \subseteq N$. Further, we investigate under appropriate non-zero ideals of a nearring must be a commutative ring. Finally, we provide a counterexample in connection with the extension of semiprime nearring.

1. Introduction

Throughout the paper, N will denote a zero-symmetric left nearring with multiplicative center Z. For any $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. A nearring N is distributively generated $(d - g)$ if it contains a multiplicative subsemigroup of distributive elements which generates the additive group $(N, +)$ (for references see [8]).

E-mail addresses: moharram_a@yahoo.com, mohammad@squ.edu.om

An element x of N is said to be distributive if $(y+z)x = yx + zx$ for all $y, z \in N$; N is said to be distributive if all the elements of N are distributive.

An ideal of a nearring N is defined to be a normal subgroup I of $(N, +)$ such that
(i) $NI \subseteq I$,
(ii) $(x+a)y - xy \in I$ for all $x, y \in N$ and $a \in I$.

In a $(d-g)$-nearring (ii) may be replaced by (ii)$'$ $IN \subseteq I$.

A nearring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields $x0 = 0$). A nearring N is said to be prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$.

If N is zero-symmetric then $xI = \{0\}$ or $Ix = \{0\}$ and $IN \subseteq I$ implies that $x = 0$ for all $x \in N$. For preliminary definitions and results related to nearrings, we refer Pilz [9]. A natural example of prime near ring was presented in Bell [3].

By a multiplicative derivation D on N we mean a mapping $D : N \to N$ such that $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. If the multiplicative derivation D is also an additive endomorphism of N, then D is called a derivation.

If D is an additive endomorphism of N then, as noted in [10, Prop. 1], $D(xy) = D(x)y + xD(y)$ if and only if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$.

In the literature, some recent results on rings deal with commutativity of prime and semiprime rings admitting suitably-constrained derivations. It is natural to look for comparable results on nearrings, and this has been done in [1], [2], [3] and [4]. The strong commutativity preserving (SCP)-derivations are motivated by recent studies of mappings F in rings having the property that $[F(x), F(y)] = 0$ whenever $[x, y] = 0$ (for references see [5]). In [4], Bell and Mason established commutativity of nearrings admitting derivations which are SCP-derivations on its subsets. The aim of this paper is to study the commutativity of nearring with the following constraints: First, with suitably-restricted right cancellation property on N, we prove main Th. 2.1, which is a generalization of [6, Cor. 1]. Secondly, we deal with a type of derivation which is more general than SCP-derivations defined in [7]. Finally, we establish that a nearring N turn out to be a commutative ring if N satisfies $[F(x), D(y)] = [x, y]$ for all x and y in some well-behaved ideal of N.

2. Some results on nearrings

The following are the main results:

Theorem 2.1. *Let N be a nearring which has right cancellation property. If N admits a mapping F and a non-zero derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$, then $(N, +)$ is abelian.*

Theorem 2.2. *Let N be a nearring having no zero-divisors. If N admits a mapping F and a non-zero commuting derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$, then N is a commuting ring with no idempotent except 0 or 1.*

Theorem 2.3. *Let N be a non-zero nearring such that $xN = N$ for all non-zero $x \in N$. If N admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$, then N is a division ring.*

Remark 2.1. A strong commutativity preserving derivation (SCP-derivation) is a derivation D if $[x, y] = [D(x), D(y)]$ for all $x,\ y \in N$. Clearly, such derivations preserve commutativity, in the sense that, if $[D(x), D(y)] = 0$ then $[x, y] = 0$. Every derivation is an SCP-derivation when N is a commutative nearring. Th. 2.2 is an extension of [4, Th. 2], and Th. 2.3 is a generalization of [4, Th. 4].

We begin with the following known results which will be used extensively. The proofs of results (a), (b) and (c) can be found in [3], whereas (d) is proved in [7].

Result (a). Let D be a derivation on a nearring N. Then N satisfies the following partial distributive law: $(xD(y) + D(x)y)z = xD(y)z + D(x)yz$, for all $x, y, z \in N$.

Result (b). If D is a derivation on a nearring N and suppose $u \in N$ is not a left zero divisor. Let $[u, D(u)] = 0$. Then $(N, +)$ is abelian.

Result (c). Let a nearring N has no non-zero divisors of zero. If N admits a non-trivial commuting derivation D, then $(N, +)$ is abelian.

Result (d). A $(d-g)$ nearring with identity 1 is a ring if N is distributive or $(N, +)$ is abelian. Then (x, u) is a constant for every $x \in N$.

In the sequel, we establish the following lemmas.

Lemma 2.1. *Let N be a nearring which admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$, then constants in N are multiplicatively central. In addition, if N has identity 1, then $(N, +)$ is abelian.*

Proof. Let c be a constant in N. Replacing y by c in the hypothesis, we get $[x, c] = [F(x), D(c)] = [F(x), 0] = 0$ for all $x \in N$. This implies that $c \in Z$. $\Diamond$

Next, if N has unity 1, then $1+1 \in Z$ and hence $[1+1, x+y] = 0$ for all $x, y \in N$. This implies that $x+y+x+y = x+x+y+y$, and hence, $y+x = x+y$ gives the required result.

Lemma 2.2. *Let N be a nearring which admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$. Then F is commuting on N if and only if D is commuting on N.*

Proof. If F is commuting on N, then $0 = [F(D(y), D(y)] = [D(y), y]$ for all $y \in N$, that is, D is commuting on N, then $0 = [F(x), D(F(x))] = [x, F(x)]$ for all $x \in N$. ◊

Lemma 2.3. *Let N be a nearring with identity 1 which admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x,\ y \in N$. Then $(zy+z)x = zyx + zx$ for all x, y and $z \in N$.*

Proof. Clearly, $D(1) = 0$ and $[x, y+1] = [F(x), D(y+1)] = [F(x), D(y)] = [x, y]$, we have $(y+1)x = yx + x$ for all $y \in N$. Left multiplying by z yields the required result. ◊

Proof of Theorem 2.1. By our hypothesis, we have

$$[F(x), D(xD(x))] = [x, xD(x)] \quad \text{for all } x \in N.$$

This gives that $[F(x), xD^2(x) + D(x)^2] = x[F(x), D^2(x)]$.

In view of Result (a), this yields

$$F(x)xD^2(x) + F(x)D(x)^2 - (xD^2(x)F(x) + D(x)^2F(x)) =$$
$$= xF(x)D^2(x) - xD^2(x)F(x).$$

This implies that $F(x)xD^2(x) + F(x)D(x)^2 - D(x)^2F(x) = xF(x)D^2(x)$ for all $x \in N$. Clearly, by our hypothesis, $[F(x), D(x)] = 0$, the last equation implies that $F(x)xD^2(x) = xF(x)D^2(x)$ for all $x \in N$.

Now two cases arise: (i)If $D^2(x) = 0$, then $D(x)$ is a constant and hence by Lemma 2.1, $D(x)$ is central, in particular, $[D(x), x] = 0$ for all $x \in N$.

(ii) If $D(x) \neq 0$, then $D^2(x)$ can be cancelled and we find $[F(x), x] = 0$ for all $x \in N$, i.e., F is commuting on N, which yields, by Lemma 2.2, D is commuting on N.

Combining of Result (c) and the obtained result, we get the required result. ◊

Proof of Theorem 2.2. For all $x \in N$, $[D(x), x] = 0$, in view of Lemma 2.2 yields that $[F(x), x] = 0$ for all $x \in N$. For any $x, y \in N$, we have

$$x[x, y] = [x, xy] = [F(x), D(xy)] = [F(x), xD(y) + D(x)y].$$

By an application of Result(a), it gives

$$x[x,y] = F(x)xD(y) + F(x)D(x)y - (xD(y)F(x) + D(x)yF(x)).$$

Further, in view of Result (c), $(N,+)$ is abelian and since $[F(x), D(x)] = 0$, the last equation reduces to

$$x[x,y] = x[F(x), D(y)] = x[F(x), D(y)] + D(x)[F(x), y].$$

This implies that

$$D(x)[F(x), y] = 0, \quad \text{for all } x, y \in N, \tag{2.1}$$

Hence

$$[F(x), y] = 0. \tag{2.2}$$

Replacing y by $D(y)$ in (2.2), we have

$$0 = [F(x), D(y)] = [x, y] \quad \text{for all } x, y \in N,$$

which yields N is a commutative ring.

Taking $e \neq 0$, an idempotent element in N. Then, we have

$$D(e) = D(e^2) = eD(e) + D(e)e = 2eD(e).$$

This gives $eD(e) = 2eD(e)$, i.e., $eD(e) = 0$. Thus $D(e) = 0$, e is a constant, which is central by Lemma 2.1. Since $e(ex - x) = 0$ for all $x \in N$, e is a left identity element which is central, it follows that $e = 1$. ◊

Proof of Theorem 2.3. Taking any non-zero element $n \in N$. Then there exists an idempotent element e in N such that $ne = n$, $ne^2 = ne$ and $n(e^2 - e) = 0$. This shows that N has no zero divisors, the last equation implies that e is a non-zero idempotent which must be a left identity. Clearly, $D(e) = D(e^2) = eD(e) + D(e)e$ and hence $D(e) = D(e) + D(e)e$, i.e., $D(e)e = 0$. Thus $D(e)N = D(e)eN = 0$. This gives $D(e) = 0$, i.e., e is a constant, by Lemma 2.1, $e \in Z$. Thus, N has 1. Therefore, $xN = N$ for all $0 \neq x \in N$, by an application of Lemma 2.3 shows that N is distributive. In addition, using Lemma 2.1, $(N,+)$ is abelian and hence, by Lemma 2.3, N is a ring which must be a division ring. ◊

3. Commutativity results on ideals of nearrings

In this section, we prove the following results which show that nearring N turn out to be a commutative ring if N satisfies the property $[F(x), D(y)] = [x, y]$ for all $x, y \in I$, where I is an ideal of N. The following Th. 3.1 is a generalization of [6, Th. 3] or [4, Th. 3] and Th. 3.2 is an extension of [4, Th. 6].

Theorem 3.1. *Let N be a nearring and U be a non zero ideal of N which contains no zero divisors of N. If N admits a mapping F with the property that $F(U) \subseteq U$, and a non-zero derivation D such that D is commuting on U and $[F(x), D(y)] = [x, y]$ for all $x, y \in U$, then N is a commutative ring.*

Theorem 3.2. *Let N be a prime nearring and U a non zero ideal of N which is distributively generated $(d-g)$ nearring with identity. If N admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x, y \in U$, then N is a commutative ring.*

Remark 3.1. It is well known that in a prime ring N, the centralizer of any non-zero one sided ideal is equal to the center of N. In particular, if N has a non-zero central ideal then N must be commutative. Combining this facts together with Th. 1 of [5] gives the following result for prime rings.

Lemma 3.1. *Let N be a prime ring and U a non zero ideal of N. If N admits a mapping F and a derivation D such that $[F(x), D(y)] = [x, y]$ for all $x, y \in U$, then N is a commutative ring.*

Proof of Theorem 3.1. Without loss of generality, we first claim that:

If D is a non-zero derivation of N, then D is also a non-zero derivation of U. Taking $D(u) = 0$ for all $u \in U$. Then, $D(nu) = 0$ for all $n \in N$ and $u \in U$, and hence $D(n)u = 0$, gives that $D(n) = 0$ for all $n \in N$.

Secondly, we establish that: If u is a non-zero element of U, then $(N, +)$ is abelian. By application of Result (b), it follows that additive commutator (x, u) is constant for all $x \in N$ and $u \in U$. This implies that $n(x, u) = (nx, nu)$ is also constant for any $n \in N$. Thus, $D(n)(x, u) = 0$. But $(x, u) \in U$ and hence cannot be a non-zero divisors of zero. Thus $(x, u) = 0$ and $(U, +)$ is abelian. Further, if u is a non-zero element of U and $x, y \in N$ then $(nx, ny) = n(x, y) = 0$, yields that $(x, y) = 0$ for all $x, y \in N$. So we get $(N, +)$ is abelian.

Thirdly, we prove that N is a commutative ring: Note that arguments used in the proof of Th. 2.2 of relation (2.1) are still valid in the present situation. Hence $D(x)[F(x), y] = 0$ for all $x, y \in U$. Clearly, $[F(x), y] \in U$ and hence the last equation implies that if $D(x) = 0$ then $0 = [D(x), F(y)] = [x, y]$. In particular, $[F(x), y] = 0$ for all $x, y \in$ $\in U$. But since D is non-zero on U and hence $[F(x), y] = 0$ for all $x, y \in U$. Replacing y by $yD(y)$ in the last obtained result, we have $0 = [F(x), yD(y)] = y[F(x), D(y)] = y[x, y]$ for all $x, y \in U$. We conclude that $[x, y] = 0$ for all $x, y \in U$. Now, If u is a non-zero element of

U and $n, m \in N$ then $u^2[n,m] = u^2nm - u^2mn = u(un)m - u(um)n = = unum - umun = 0$ thus, $[n,m] = 0$ for all $n, m \in N$ and hence, N is a commutative ring. ◊

Proof of Theorem 3.2. Let e be an identity element of U. Then $eu = u$ for all $u \in U$ and hence, we have $D(u) = eD(u) + D(e)u$. This give $eD(e)u = 0$ for all $u \in U$, so $eD(e) = 0$. Thus for each $u \in U$, $uD(e) = ueD(e) = 0$, i.e., $UD(e) = \{0\}$. This implies that $D(e) = = 0$ and hence $D(e+e) = 0$. Since Lemma 2.1 is valid in the present situation, we obtain that both e and $e+e$ commute with elements of U, and $(U,+)$ is abelian. Trivially, one can see that $U(n,m) = \{0\}$ for all $n, m \in N$ thus, $(n,m) = 0$ for all $n, m \in N$, yields $(N,+)$ is abelian.

Since U is a $(d-g)$ nearring with identity and $(U,+)$ is abelian, application of Result(d) gives that U is distributive. Let $u, v \in U$ and $m, n \in N$. Then

$$u\{(m+n)v - (mv+nv)\} = (um+un)v - (umv+unv) = 0.$$

This implies that $(m+n)v = mv+nv$. Putting of v by zv for any $z \in N$, gives that $(m+n)zv = mzv+nzv$. We obtain $\{(m+n)z-(mz+nz)\}U = = \{0\}$ and hence $(m+n)z = mz+nz$ for all $n, m, z \in N$, i.e., N is distributive. This indicates that N is a ring which is commutative by Lemma 3.1. ◊

Corollary 3.1. *Suppose N is a prime nearring admitting a derivation D and U is a non-zero ideal of N which is $(d-g)$ nearring with identity. If for each $x \in U$, there exists an integer $i = i(x) \geq 1$ such that $[D^i(x), D(y)] = [x,y]$ for all $y \in U$, then N is a commutative ring.*

Proof. By Th. 3.1, $(N,+)$ is abelian, which in the setting of $(d-g)$ nearrings forces N to be a ring. Further, it is clear that D is commuting on U, hence if $D \neq 0$, we can invoke [8, Th. 1 (2)] to the effect that a prime ring admitting a nontrivial commuting derivation must be commutative. Finally, if $D = 0$, Cor. 3.1 is obvious. ◊

Corollary 3.2. *Let N be a ring admitting a derivation D and U a non-zero ideal of N with identity. Then $U = N$.*

Remark 3.2. In view of Lemma 3.1, in the hypothesis of Th. 2.3 N can be extended to a field.

4. Counterexample

In ring theory, it is known that a mapping $F : N \to N$, where N is a ring, is called commuting if $[F(x), x] = 0$ holds for all $x \in N$. This theory has been initiated by a result of Posner [10, Posner's Second

Th.], which states that existence of a non-zero commuting derivation $D : N \to N$, where N is a prime ring forces the ring to be commutative. In general, Posner's Second theorem cannot be generalized on semiprime ring as shows the following example. Let N_1 be a noncommutative prime ring and let N_2 a commutative prime ring that admits a non-zero derivation $D : N_2 \to N_2$. Then $N = N_1 \bigoplus N_2$ is a noncommutative semiprime ring.

In this context, we construct an example in nearrings which shows that our Th. 3.2 cannot be extended to semiprime nearring.

Example 4.1. Let N_1 be a noncommutative prime ring and N_2 a noncommutative prime nearring admitting a non-zero commuting derivation δ. Then $N = N_1 \bigoplus N_2$ is a noncommutative semiprime nearring. Define $D : N \to N$ by $D(x, y) = (0, \delta(y_1))$. Then D is a non-zero commuting derivation on N. Now, we define $F : N \to N$ by $D(x, y) = = (x_1, 0)$. Then $[D(x), F(y)] = [x, y]$ for all $x, y \in N$.

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