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A FIXED POINT THEOREM IN TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper we shall prove a fixed point theorem in topological vector space, using the theory of the topological semifield and a result of Kasahara.

1. First, we shall give some notations and definitions which will be used in the following text. By R we shall denote the set of all real numbers. Further, let X be a vector space over $\mathcal{K}$ (real or complex number field), R_Δ the set of all mappings from Δ into R with the Tychonoff product topology and the operations $+$ and scalar multiplication as usual. If $f, g \in R_\Delta$ we shall say that $f < g$ iff $f(t) \leqslant g(t)$, for every $t \in \Delta$ and $f \neq g$. By P_Δ we shall denote the cone of nonnegative elements in R.

DEFINITION 1. *The triplet* $(X, \|\cdot\|, \Phi)$ *is a* Φ*-paranormed space iff* $\|\cdot\|: X \to P_\Delta$, Φ *is a linear, continuous, positive mapping from* R_Δ *into* R_Δ *such that the following conditions are satisfied:*

1. $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|\lambda x\| = |\lambda| \, \|x\|$, *for every* $x \in X$ *and every* $\lambda \in \mathcal{K}$.
3. $\|x+y\| \leqslant \Phi(\|x\|) + \Phi(\|y\|)$, for every $x, y \in X$.

Let us denote by $\mathcal{U}$ the family of neighbourhoods of zero in R_Δ and for every $U \in \mathcal{U}$ we shall denote the set:

$$\{x \mid x \in X, \ \|x\| \in U\}$$

by V_U. Then X is a topological vector space in which $\{V_U\}_{U \in \mathcal{U}}$ is the family of neighbourhoods of zero in X.

In [2] it is proved that every Hausdorff topological vector space X is a paranormed space $(X, \|\cdot\|, \Phi)$ over a topological semifield R_Δ and we shall say that the triplet $(X, \|\cdot\|, \Phi)$ is *the associated paranormed space.*

DEFINITION 2. *Let* X *be a Hausdorff topological vector space and* $(X, \|\cdot\|, \Phi)$ *be the associated paranormed space. The set* $K \subset X$ *is of* Φ*-type iff for every* $n \in N$:

$$\Big\|\sum_{i=1}^{n} \lambda_i x_i\Big\| \leqslant \sum_{i=1}^{n} \lambda_i \, \Phi(\|x_i\|), \quad \textit{for every } x_i \in K-K \ (i=1, 2, \ldots, n)$$

and every $\lambda_i \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1.$

DEFINITION 3. *Let X be a topological vector space, $K\subset X$ and $\mathcal{V}$ be the fundamental system of neighbourhoods of zero in X. The set K is locally convex iff for every $W\in\mathcal{V}$ and every $x\in K$ there exists $U\in\mathcal{V}$ such that:* $\mathrm{co}\,((x+U)\cap K)\subset W+x$.

In [5] Bogdan Rzepecki proved the following fixed point theorem.

THEOREM A. *Let E be a Hausdorff topological vector space and let K be a nonempty, closed and convex subset of E. Suppose that T is a continuous mapping from K into a compact subset Z of K. Assume, moreover that the following condition is satisfied:*

(*) *For every x in Z and every neighbourhood W of x there exists a neighbourhood V of x such that:*

$$\mathrm{co}\,(V\cap Z)\subset W.$$

Then T has a fixed point in K.

Using Theorem A and the following proposition we shall obtain a fixed point theorem.

PROPOSITION 1. *Let X be a topological vector space and K be a subset of X of Φ-type. Then K is a locally convex subset of X.*

Proof: We shall prove that for every $W\in\mathcal{V}$ and every $x\in K$ there exists $U\in\mathcal{V}$ such that:

$$\mathrm{co}\,(\,(x+U)\cap K)\subset x+W$$

which means that K is a locally convex subset of X. Let $W\in\mathcal{V}$. Let $(X, \|\cdot\|, \Phi)$ be the associated Φ paranormed space over the topological semifield R_Δ. Then there exists $\mu=\{t_1, t_2, \ldots, t_n\}\subset\Delta$ and $\varepsilon>0$ such that the following implication holds:

$$\|u\|\in U_{\mu,\varepsilon}\Rightarrow u\in W$$

where: $\quad U_{\mu,\varepsilon}=\{x|\quad \|x\|\,(t)<\varepsilon, \quad \text{for every } t\in\mu\}.$

Since the mapping Φ is linear and continuous there exists a neighbourhood U_1 of zero in R_Δ such that:

$$\|u\|\in U_1\Rightarrow \Phi\,(\|u\|)\in U_{\mu,\varepsilon}.$$

Suppose, further that U_2 is a symmetric neighbourhood of zero in X such that $U_2\subset V_{U_1}$. Let us prove that:

$$\mathrm{co}\,(\,(x+U_2)\cap K)\subset x+W. \tag{1}$$

Suppose that $u\in\mathrm{co}\,((x+U_2)\cap K)$. Then $u=\sum_{i=1}^{n}\lambda_i x_i$, where $x_i\in(x+U_2)\cap K$ $(i=1, 2, \ldots, n)$ and $\lambda_i\in[0, 1]$, $\sum_{i=1}^{n}\lambda_i=1$. Then we have:

$$\|u-x\|(t)=\|\sum_{i=1}^{n}\lambda_i\,(x_i-x)\|(t)\leqslant\sum_{i=1}^{n}\lambda_i\;\Phi\,(\|x_i-x\|)\,(t)<\varepsilon$$

for every $t\in\mu$. So, it follows that $\|u-x\|\in U_{\mu,\varepsilon}$ and consequently $u-x\in W$ which completes the proof, since (1) holds.

COROLLARY. *Let X be a topological vector space, K be a nonempty, closed and convex subset of X and T be a continuous mapping from K into a compact subset $Z \subset K$ of Φ type. Then T has a fixed point in K.*

In [4] the following problem is proposed:

PROBLEM. *If $K \subset X$, X is a topological vector space and K is a locally convex subset under which conditions K has the following property:*

$$A \subset K \text{ and } A \text{ is precompact} \Rightarrow \text{co } A \text{ is precompact.}$$

PROPOSITION 2. *Let X be a topological vector space, K be a subset of X of Φ type, $0 \in K$ and A is a precompact subset of K. Then* co A *is precompact.*

Proof: It is known that co A is precompact iff for every $W \in \mathcal{V}$ there exists a finite set $\{y_1, y_2, \ldots, y_n\} \subset \text{co } A$ such that:

$$\text{co } A \subseteq \bigcup_{i=1}^{n} \{y_i + W\}.$$

Let $W \in \mathcal{V}$ and $\varepsilon > 0$, $\mu = \{t_1, t_2, \ldots, t_m\} \subset \Delta$ such that:

$$\|u\| \in U_{\mu,\varepsilon} \Rightarrow u \in W.$$

Since Φ is a linear and continuous mapping there exists a neighbourhood V_1 of zero in R_Δ such that:

$$\|u\| \in V_1 \Rightarrow \Phi(\|u\|) \in U_{\mu, \frac{\varepsilon}{2}}$$

and an open, symmetric neighbourhood of zero in X such that:

$$u \in V_2 \Rightarrow \|u\| \in V_1.$$

Since the set A is precompact there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset A$ such that:

$$A \subseteq \bigcup_{i=1}^{n} \{x_i + V_2\}. \tag{1}$$

Let S be the subset of R^n consisting of all $s = (s_1, s_2, \ldots, s_n)$ such that:

$$s_i \geq 0, \quad i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} s_i = 1.$$

Since S is a compact subset of R^n, for every $\delta > 0$ there exists a finite set of points $\{\beta^j\}_{j \in J}$ (J is a finite set), $\beta^j = (\beta_1^j, \ldots, \beta_n^j) \in R^n$ such that for every $s \in S$ there exists $\beta^j \in \{\beta^j\}_{j \in J}$ such that:

$$\sum_{i=1}^{n} |s_i - \beta_i^j| \leq \delta.$$

Let $\delta > 0$ be such that for every $i = 1, 2, \ldots, n$ and every $r = 1, 2, \ldots, m$:

$$\Phi(\|x_i\|)(t_r) < \frac{\varepsilon}{2\delta}.$$

We shall show that:

$$\operatorname{co} A \subseteq \bigcup_{j\in J} \left(\sum_{i=1}^{n} \beta_i^j x_i + W\right).$$

Suppose that $y \in \operatorname{co} A$. Then:

$$y=\sum_{k=1}^{N} \gamma_k y_k, \quad \text{where} \quad y_k \in A\ (k=1,2,\ldots,N),\quad \gamma_k \geqslant 0\ (k=1,2,\ldots,N),\ \sum_{k=1}^{N} \gamma_k = 1.$$

From (1) it follows that:

$$y_k = x_{i_k} + z_k, \quad z_k \in V_2 \quad (k=1,2,\ldots,N)$$

and so:

$$y=\sum_{k=1}^{N} \gamma_k x_{i_k} + \sum_{k=1}^{N} \gamma_k z_k = \sum_{i=1}^{n} \gamma_i' x_i + \sum_{k=1}^{N} \gamma_k z_k.$$

Suppose that β^j is such that:

$$\sum_{i=1}^{n} |\gamma_i' - \beta_i^j| \leqslant \delta$$

and let us show that:

$$(2) \qquad y \in \sum_{i=1}^{n} \beta_i^j x_i + W.$$

For every $r \in \{1, 2, \ldots, m\}$ we have that:

$$\|y - \sum_{i=1}^{n} \beta_i^j x_i\|(t_r) = \|\sum_{i=1}^{n} (\gamma_i' - \beta_i^j)\, x_i + \sum_{k=1}^{N} \gamma_k z_k\|(t_r) \leqslant$$

$$\leqslant \sum_{i=1}^{n} |\gamma_i' - \beta_i^j|\ \Phi(\|x_i\|)(t_r) + \sum_{k=1}^{N} \gamma_k\, \Phi(\|z_k\|)(t_r) < \frac{\varepsilon}{2\delta}\cdot\delta + \frac{\varepsilon}{2} = \varepsilon$$

and so:

$$\|y - \sum_{i=1}^{n} \beta_i^j x_i\| \in U_{\mu,\varepsilon}$$

which implies that:

$$y - \sum_{i=1}^{n} \beta_i^j x_i \in W$$

and so relation (2) is proved.

Now, let E be a Hausdorff topological vector space. In [5] a set function ψ on E (similar to a certain sense to the measure of non-compactness of Kuratowski) is introduced in the following way:

Let L be a linear space, let S be a cone in L generating the partial order $\leqslant_S$, and let S_∞ be a set containing S. If $S_\infty \setminus S \neq \emptyset$ it is assumed that in S_∞ linear operations are defined which are extensions of those in S and in S_∞:

$$x \leqslant y \quad \text{means} \quad \begin{cases} x, y \in S & \text{and} \quad x \leqslant_S y \\ x, y \in S_\infty \setminus S & \text{and} \quad x = y \\ x \in S & \text{and} \quad y \in S_\infty \setminus S \end{cases}$$

Further, let the function $\psi : 2^E \to S_\infty$ has the following properties:

1. $\psi(X \cup \{x\}) = \psi(X)$, for every $X \subset E$ and every $x \in E$.
2. If $X \subset Y$ then $\psi(X) \leq \psi(Y)$, for every $X, Y \subset E$.
3. If $\psi(X) = 0$, then X is a relatively compact subset of E.

In [5] the following theorem is proved.

THEOREM B. *Let E be a Hausdorff topological vector space, K be a closed and convex subset of E such that for every compact subset Z of K is the condition (*) satisfied. Suppose, moreover, that the mapping ψ satisfies 1., 2. and 3. and $\psi(K) \in S$, $\psi(\overline{\text{co}}\, X) = \psi(X)$, for each $X \subset K$. If $T : K \to K$ is a continuous mapping such that $\Theta < \psi(X)\ (X \subset K)$ implies $\psi(T(X)) < \psi(X)$ then there exists a fixed point of the mapping T.*

From Theorem B we have the following Corollary.

COROLLARY. *Let E be a Hausdorff topological vector space, K be a closed and convex subset of E of Φ-type, $T : K \to K$ be a continuous mapping and the mapping ψ satisfies all the conditions of Theorem B. If $\Theta < \psi(X)\ (X \subset K)$ implies $\psi(T(X)) < \psi(X)$ then there exists a fixed point of the mapping T.*

Remark: If E is a Hausdorff topological vector space and ψ is defined by $\psi(X) = 0$ if $\overline{X}$ is a compact subset of E and $\psi(X) = 1$ if $\overline{X}$ is not a compact subset of E $(X \subset E)$ then the function ψ has in E the properties 1., 2, and 3., where $S = S_\infty = [0, \infty)$. From the Proposition it follows that $\psi(\overline{\text{co}}\, X) = \psi(X)$, for each $X \subset K$, if E is complete. It is easy to see that in the Proposition we can suppose that $0 \notin K$.

DEFINITION 1. *Let E be a Hausdorff topological vector space, $\emptyset \neq K \subset E$ and $T : K \to E$. The mapping T is a generalized condensing iff:*

(a) *T is continuous.*

(b) *If $\emptyset \neq A \subset K$ and $T(A) \subset A$ such that $A \setminus \overline{\text{co}}\, T(A)$ is compact then A is relatively compact.*

From Satz 1.17 [4] and the Proposition we obtain the following Corollary.

COROLLARY. *Let E be a complete Hausdorff topological vector space, K be a nonempty, closed and convex subset of Φ-type of E. If T is generalized condensing mapping from K into K then there exists a fixed point of the mapping T.*

2. In [3] the following theorem is proved:

Let X be a Hausdorff locally convex topological vector space and G be a nonempty complete convex subset of X and let $T : G \to G$ be continuous. If:

(i) *$\{T^n\}_{n \in N}$ is an equicontinuous family of functions.*

(ii) *There exists $M \subseteq G$ which is an attractor for compact sets under T,*

then T has a fixed point.

We shall prove a generalization of this theorem, where X is a Hausdorff topological vector space and G is a nonempty complete, convex subset of Φ-type of X.

First, we shall give some definitions and theorems which we shall need later[3].

DEFINITION 5. *Let L be a Hausdorff topological vector space and $K \subset L$ be nonempty. Then a family F of mappings from K into itself is said to be equicontinuous (on K) if for each $x \in K$ and each neighbourhood U of the origin 0, there exists a neighbourhood V of 0 such that $y \in K$ and $y - x \in V$ imply $Ty - Tx \in U$, for all $T \in F$.*

DEFINITION 6. *Let X be a Hausdorff topological semigroup. S is said to act on X if there is a continuous map $\pi : S \times X \to X$ such that $\pi(s_1, \pi(s_2, x)) = \pi(s_1 s_2, x)$, for any $s_1, s_2 \in S$ and $x \in X$.*

If $s \in S$ then:

$$\Gamma_n(s) = \overline{\{s^m \mid m \geqslant n\}}, \quad \Gamma(s) = \Gamma_1(s), K(s) = \cap \{\Gamma_n(s) \mid n \geqslant 1\}.$$

DEFINITION 7. *Let X be a topological space and $T : X \to X$. A subset M of X is said to be an attractor for compact sets under T iff:*

(i) *M is a nonempty compact and $T(M) \subseteq M$.*

(ii) *For ary compact subset C of X and any open neighbourhood U of M, there exists an integer N such that $T^n(C) \subseteq U$, for all $n \geqslant N$.*

In [6] Wallace proved the following theorem.

THEOREM C. *Suppose that S acts on X. Let $s \in S$ be such that $\Gamma(s)$ is compact and let $A \subseteq X$ be nonempty compact such that $sA \supset A$. Then for each $s_1 \in \Gamma(s)$, $s_1 A = A$ and s_1 acts as a homomorphism on A. In particular, the unit of $K(s)$ acts as identity map on A.*

Let X be a compact Hausdorff and $S = C(X, X)$ be the family of all continuous maps on X into itself equipped with a compact open topology. For $f, g \in S$ define $f \cdot g = f \circ g$, the composition of g followed by f. Then S is a Hausdorff topological semigroup. If $\pi : S \times X \to X$ is defined by $\pi(f, x) = f(x)$, for all $f \in S$, $x \in X$ then π is (jointly) continuous and S acts on X. If $S = C(X, X)$ and $T \in S$ such that the family $\{T^n\}_{n \in N}$ is equicontinuous then $\Gamma(T)$ is compact [3]. Further for $A = \bigcap_{n=1}^{\infty} T^n(X)$ is $T(A) = A$ and so from Theorem C it follows that the unit r of $K(T)$ acts as an identity map on A. In [3] it is proved that r is a retraction of X onto A.

As in [3] we shall prove the following theorem.

THEOREM. *Let G be a nonempty, complete, convex subset of Φ type of Hausdorff topological vector space X and $T : G \to G$. If:*

(i) *$\{T^n\}_{n \in N}$ is an equicontinuous family of functions,*

(ii) *There exists $M \subseteq G$ which is an attractor for compact sets under T, then T has a fixed point.*

Proof: Let $Y = \overline{\text{co}}\, M$. From the Proposition 2, it follows that the set Y is compact. Let:

$$X = \overline{\bigcup_{n=0}^{\infty} T^n(Y)}, \quad \text{where} \quad T^{\circ}(Y) = Y.$$

In [3] it is proved that X is compact and let $A = \bigcap_{n=1}^{\infty} T^n(X)$. Then there exists a retraction $r : X \to A$ and $g = T \cdot r$ is a continuous mapping Y into Y. From the Proposition 1, it follows that there exists $y_0 \in Y$ such that $g(y_0) = y_0$. As in [3] it follows that $y_0 \in A$ and $Ty_0 = y_0$.

REFERENCES

[1] O. Hadžić, *On the admissibility of topological vector space*, Acta Scientiarum Mathematicarum, T 42, Fasc. 1—2, (1980), 81—85.

[2] S. Kasahara, *On formulations of topological linear spaces by topological semifields*, Math. Jap. 19 (1974), 121—134.

[3] H. M. Ko, K. K. Tan, *Attractors and a fixed point theorem in locally convex space*, CMUC, 21, 1 (1980), 71—79.

[4] C. Krauthausen, *Der Fixpunktsatz von Schauder in nicht notwendig konvexen Räumen sowie Anwendungen auf Hammersteinsche Gleichungen*, Dok. Diss., Aachen, 1976.

[5] B. Rzepecki, *Remarks on Schauder's Fixed Point Theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phy., 24 (1976), 589—603.

TEOREMA O NEPOKRETNOJ TAČKI U VEKTORSKO TOPOLOŠKIM PROSTORIMA

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REZIME

Korišćenjem nekih rezultata S. Kasahare [2] u ovom radu je, pored ostalog dokazana sledeća teorema o nepokretnoj tački, koja je uopštenje teoreme iz rada [3].

TEOREMA. *Neka je G neprazan, kompletan, konveksan podskup Φ tipa Hausdorffovog vektorsko topološkog prostora X i $T:G\to G$ tako da su zadovoljeni sledeći uslovi:*

(i) $\{T^n\}_{n\in N}$ *je podjednako neprekidna familija.*

(ii) *Postoji $M\subseteq G$ tako da je M atraktor za kompaktne skupove u odnosu na T.*
Tada postoji nepokretna tačka preslikavanja T.