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ON SOME EXIT CRITERIA FOR THE NEWTON METHOD

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Abstract

Newton's iterative method for a scalar equation $f(x) = 0$, is known to be convergent to the unique zero α of f, for f satisfying certain conditions involving f, f' and f''.

In thi case, if (x_n) is the sequence defined by Newton's method, a stopping inequality or an aposteriori estimation of the form $|x_{n+1} - \alpha| \leq |x_n - x_{n+1}|$, holds.

The aim of this paper is to show that Newton's method converges under weaker conditions, involving only f and f', when a generalized stopping inequality is valid.

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1. Introduction

In this paper we consider the equation

$$f(x) = 0 \tag{1}$$

on an interval $[a, b] \subseteq R$, for a real-valued function f.

The (one-dimensional) Newton method is given by the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 \in [a,b], \quad n \geq 0 \tag{2}$$

assuming $f' \neq 0$ on $[a,b]$.

The importance of the Newton iteration (2) rests on the fact that, under certain natural conditions on f, an estimate of the form

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^2, \tag{3}$$

holds.

Together with the simplicity and elegance of (2), the so-called "quadratic convergence" (3) makes Newton's method a focal point in the study of iterative methods for nonlinear equations.

However, the conditions as "f'' does not change its sign on $[a,b]$", in [4] or "$f'' > 0$ on $[a,b]$" in [5], are too strong. Even the Ostrowski theorem on the Newton method [7], assumes "f" there exist in the neighborhood of α". But, as shown by some practical examples, (see [2], [3]), the existence of the second derivative of f is not necessary for the convergence of the Newton method.

Theorem 1 in this paper gives conditions, stated in terms of f and f', for the convergence of the Newton method, which however provides a "linear convergence" instead of the classical quadratic convergence.

2. An extension of the Newton method

Let $f : [a,b] \to R$ be a real function which satisfies the following conditions

(f_1) $f(a) \cdot f(b) < 0$;

(f_2) $f \in C^1[a,b]$, $f'(x) \neq 0$, $x \in [a,b]$;

Then, from (f_1) we deduce that $f \in C[a,b]$ has a root $\alpha \in (a,b)$, and (f_2) implies that $f \in C[a,b]$ has only a root in $[a,b]$.

If, in addition, $f \in C^2[a,b]$ and $f''(x) \neq 0$ on $[a,b]$, the Newton iteration (2) converges to α, for $x_0 \in [a,b]$ arbitrary taken and (3) holds with $C = M_2/2m_1$, where $m_1 = \min\limits_{x\in[a,b]} |f'(x)|$ and $M_2 = \max\limits_{x\in[a,b]} |f''(x)|$, see [4].

If a stopping inequality is valid, i.e. an estimation of the form $|x_{n+1} - \alpha| \le |x_n - x_{n+1}|$, as shown in [4] or [5], then the numerical computation of the sequence (x_n) given by (2) is stopped when the distance between two successive approximations is less than a preassigned tolerance. Such an exit criterion is correct, because, from

$$|x_n - x_{n+1}| < \varepsilon,$$

and the stopping inequality we obtain

$$|x_n - \alpha| < \varepsilon.$$

In this paper we consider a stopping inequality of the form

$$|x_n - \alpha| \le c|x_n - x_{n+1}|, \tag{4}$$

where $c \ge 1$ is constant. Obviously, for $c = 1$, we obtain the stopping inequality from [5] and [4].

The main result of this paper is given by

Theorem 1. *Let $f : [a,b] \to R$, $a < b$, be a function which satisfies (f_1), (f_2) and*

(f_3): $2m > M$, where

$$m = \min_{x\in[a,b]} |f'(x)|, \quad M = \max_{x\in[a,b]} |f'(x)|.$$

Then the Newton iteration (2)converges to α, the unique solution of (1) and the following estimation

$$|x_n - \alpha| \le \frac{M}{m}|x_n - x_{n+1}|, \quad n \ge 0, \tag{5}$$

holds.

Proof. From (f_1) and (f_2) it results that (1) has a unique solution $\alpha \in (a,b)$. Let $x_0 \in [a,b]$, be the initial approximation. From (2) we obtain

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n) - f(\alpha)}{f'(x_n)}, \quad \text{for all } n = 0, 1, \ldots$$

and, using the mean value theorem, this yields

$$x_{n+1} - \alpha = \left[1 - \frac{f'(c_n)}{f'(x_n)}\right] \cdot (x_n - \alpha), \quad n \in N, \tag{6}$$

where $c_n = \alpha + \theta(x_n - \alpha)$, $0 < \theta < 1$.

In a similar manner we deduce, directly from (2),

$$x_{n+1} - x_n = \frac{f'(c_n)}{f'(x_n)}(x_n - \alpha), \quad \forall n \in N. \tag{7}$$

Now, from (f_2) it results

$$1 - \frac{f'(c_n)}{f'(x_n)} < 1, \quad n \in N.$$

On the other hand,

$$\frac{f'(c_n)}{f'(x_n)} = \left|\frac{f'(c_n)}{f'(x_n)}\right| = \frac{|f'(c_n)|}{|f'(x_n)|} \leq \frac{M}{m},$$

hence, in virtue of (f_3),

$$1 - \frac{f'(c_n)}{f'(x_n)} > -1 \quad \text{for each} \quad n \geq 0.$$

Since $[a, b]$ is compact and $f \in C^1[a, b]$, we obtain

$$k = \max_{x,y \in [a,b]} \left|1 - \frac{f'(y)}{f'(x)}\right| < 1,$$

which, together with (6), yields

$$|x_{n+1} - \alpha| \leq k|x_n - \alpha|, \quad n \in N,$$

and $0 < k < 1$.

By induction, we deduce

$$|x_n - \alpha| \leq k^n |x_0 - \alpha|, \quad n \in N,$$

so

$$x_n \to \alpha, \quad \text{as} \quad n \to \infty,$$

for each $x_0 \in [a, b]$.

However, $[a, b]$ is generally not an invariant set with respect to the iteration (2), i.e. it is possible to obtain at a certain step p, $x_p \notin [a, b]$.

In order to remove difficulties we define f over the whole real axis (and denote it by f too) as follows

$$f(x) = \begin{cases} f'(a)(x-a) + f(a), & \text{if } x < a \\ f(x), & \text{if } x \in [a, b] \\ f'(b)(x-b) + f(b), & \text{if } x > b. \end{cases} \tag{8}$$

If some iterate x_p does not lie in $[a, b]$, we have either $x_p < a$ or $x_p > b$. In the first case,

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)} = x_p - \frac{f'(a)(x_p - a) + f(a)}{f'(a)},$$

hence

$$x_{p+1} = a - \frac{f(a)}{f'(a)},$$

which shows $x_{p+1} > a$, because, from (f_1)-(f_2) it results $f(a)f'(a) < 0$.

In the second case,

$$x_{p+1} = x_p - \frac{f'(b)(x-b) + f(b)}{f'(b)} = b - \frac{f(b)}{f'(b)} < b,$$

because $f(b) \cdot f'(b) > 0$.

But (x_n) converges to α and $a < \alpha < b$. This means that, begining from a step $p_0 \geq 0$, we necessarly have

$$x_n \in [a, b].$$

The desired estimation (5) may be obtained from (7). The proof is now complete. □

Remarks.

1) The Newton iteration method given by (2) and (8) is called "the extended Newton method", see [2]. In fact, this algorithm consists in applying the Newton method on $[a, b]$ and the modified Newton method (see [4]) on $R \setminus [a, b]$.

2) If the conditions of Theorem 1 are satisfied, in order to obtain $|x_n - \alpha| < \varepsilon$ the iterative procedure must be stopped when $|x_{n+1} - x_n| < \frac{m}{M}\varepsilon$.

3) Relation (6) shows a linear rate of convergence for the extended Newton method. However, if f'' exists on $[a,b]\backslash\{\alpha\}$, the convergence is quadratic [4].

4) A fixed point proof of Theorem 1 is given in [3], and the corresponding n-dimensioanl case is treated in [1].

5) A classical fixed point argument, based on Edelstein's fixed point theorem shows that condition (f_3) in Theorem 1 may be weakened. We also obtain

Theorem 2. *If f satisfies (f_1), (f_2) and the following condition*

(f'_3) $2m \geq M$,

then the conclusion of Theorem 1 remains true.

Remarks. The condition (f'_3) is similar to condition (F_4) in [5], but there $f \in C^2[a,b]$, and the studied method is a general one given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} g(x_n),$$

where g is a certain function.

The proof of Theorem 1 may be easily extended to this procedure, assuming $x_n \in [a,b]$, for each $n \in N$, see [1].

3. Examles

The best result until now about the convergence of Newton's method in the scalar case seems to be the well-known Ostrowski's theorem (see [7], Theorem 7.2) which assumes that f'' exists in a neighborhood of α. Our Theorem 2 is better than Ostrowski's theorem, as shown by Example 2.

For the convergence of the extended Newton method, the numerical tests was performed on an IBM PC, under MATHCAD.

Example 1. For $f(x) = \mathrm{tg}x$, $x \in [a,b] \subset (\frac{\pi}{2}, \frac{3\pi}{2})$, we have $f'(x) > 0$, but

$f''(\pi) = 0$ and $\alpha = \pi$ is the unique solution of the equation $f(x) = 0$ in $[a, b]$. The Demidovich's theorem does not apply. The convergence of the Newton method is a consequence of Theorem 1 or 2 in this paper, or Ostrowski's theorem.

For $a = \frac{7\pi}{12}$, $b = \frac{17\pi}{12}$ and $x_0 = a$, the constant c in the stopping inequality is $c = \dfrac{M}{m} = 14.76$. After 7 iterations we obtain π with exact digits: $x_0 = 1.83$; $x_1 = 2.08$; $x_2 = 2.51$; $x_3 = 2.99$; $x_4 = 3.139$; $x_5 = 3.141592644$; $x_6 = 3.141592653589794$ and $x_7 = 3.141592653589793$.

Taking $x_0 = \frac{2\pi}{3}$ we obtain the solution after 5 iterations : $x_0 = 2.09$; $x_1 = 2.527$; $x_2 = 2.998$; $x_3 = 3.139$; $x_4 = 3.141592648$; and $x_5 = 3.141592653589793$.

Example 2. [3] Let $f : [-1, 1] \to R$ be given by $f(x) = -x^2 + 2x$, if $x \in [-1, 0)$ and $f(x) = x^2 + 2x$, if $x \in [0, 1]$. The equation $f(x) = 0$ has only the solution $\alpha = 0$ on $[-1, 1]$. Since f'' does not exist in 0, neither Theorem 1 nor Ostrowski's theorem does apply. We have $m = 2$, $M = 4$, the condition (f'$_3$) in Theorem 2 is satisfied, hence the Newton method is convergent.

Indeed, if we start with $x_0 = 0.5$, we obtain $x_1 = 0.833333$; $x_2 = 0.0032051$; $x_3 = 0.0000129$; $x_4 = 0.0000001$ and $x_5 = 0$.

Remarks. The basic idea in proving the convergence of the Newton method in the classical form, i.e. $f \in C^2[a, b]$ and $f'' \neq 0$ on $[a, b]$, is to show that the sequence (x_n) is monotonous. As shown by the following example, in the conditions of Theorem 1 or 2, (x_n) is not generally monotonous.

Example 3. For $f : [1.1, 4] \to R$, $f(x) = xe^{-x} - 2e^{-2}$, we have $f' < 0$, $f''(2) = 0$ and $\alpha = 2$ is the unique zero of f.

Starting from $x_0 = 1.2$, we obtain $x_1 = 2.7067103$; $x_2 = 1.9169384$; $x_3 = 2.0002041$; $x_4 = 2.00...$ and $x_5 = 2$, hence (x_n) is not monotonous.

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