Revista Colombiana de Matemáticas
Volumen 41 (2007), páginas 23–37

Curvature on reductive homogeneous spaces

Marlio Paredes*
Universidad del Turabo, Gurabo, Puerto Rico

Sofía Pinzón†
Universidad Industrial de Santander, Bucaramanga

Abstract. Here we consider the general flag manifold $\mathbb{F}_\Theta$ as a naturally reductive homogeneous space endowed with an U–invariant metric Λ^Θ and an invariant almost-complex structure J^Θ. The main objective of this work is to explore the *riemannian connection* associated with the metric Λ^Θ in order to calculate some classes of curvatures which should allow us to confirm, in a simple way, that flag manifolds are either not biholomorfically equivalent nor holomorphically isometric to any complex projective space.

Keywords. Homogeneous spaces, flag manifolds, riemannian connection, curvature.

2000 Mathematics Subject Classification. Primary: 54H25. Secondary: 47H10.

Resumen. Consideramos aquí la variedad bandera general $\mathbb{F}_\Theta$ como un espacio homogéneo naturalmente reductivo dotado con una métrica U–invariante Λ^Θ y una estructura cuasicompleja invariante J^Θ. El objetivo principal de este trabajo es explorar la *conexión riemanniana* asociada con la métrica Λ^Θ con el fin de calcular algunas clases de curvaturas las cuales nos permitan confirmar, de manera simple, que las variedades bandera no son bilomórficamente equivalentes ni holomórficamente isométricas a ningún espacio proyectivo complejo.

1. Introduction

The main purpose of this paper is to study the curvature on the generalized flag manifold associated with semi-simple complex Lie algebras and groups. Given a complex semi-simple Lie group G, its "fundamental homogeneous space" is the coset space $\mathbb{F}_\Theta = G/P_\Theta$ modulo a parabolic subgroup (Borel subgroup)

*Partially supported by COLCIENCIAS–COLOMBIA, Grant No. 240–2001.
†Partially supported by COLCIENCIAS–COLOMBIA, Grant No. 138-2004.

P_Θ of G, where Θ is a subset of simple roots of $\mathfrak{g}$, the Lie algebra of G. In the context of compact Lie groups, the spaces G/P_Θ are given by coset U/K_Θ where U is a compact real form of G and $K_\Theta = U \cap P_\Theta$ is the centralizer of a torus of U, when $\Theta = \emptyset$ the torus is maximal and we denote $\mathbb{F} = U/T$ as the maximal flag manifold. These spaces are also known generically as "generalized flag manifolds", since G/P_Θ can be identified with the concrete space of flags of subspaces of an n-dimensional complex vector space when G is the special linear group $Sl(n, C)$. We directly use the algebra (combinatorics) of root systems, which gives life to the theory of semi-simple Lie algebras, to find the form of the riemannian connection of $\mathbb{F}_\Theta$ associated to the invariant metric Λ^Θ and then we calculate some curvatures, in order to relate them with some topological and geometrical properties of $\mathbb{F}_\Theta$. In particular, the results reaffirm that a Kähler maximal flag manifold, different from $\mathbb{F}(2)$, can not be bi-holomorphic equivalent, or isometric holomorphic, to any projective space $\mathbb{C}P(n)$.

2. Preliminaries

Let G be a connected Lie group, H its closed subgroup, g an invariant riemannian metric on the homogenous space G/H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H, respectively. G/H is a reductive homogeneous space if the Lie algebra $\mathfrak{g}$ can be decomposed into a vector space direct sum of the $\mathfrak{h}$ and an $ad(H)$-invariant subspace $\mathfrak{m}$, that is, if

(1) $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \qquad \mathfrak{h} \cap \mathfrak{m} = 0;$
(2) $ad(H)\mathfrak{m} \subset \mathfrak{m}$.

Condition (2) implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We identify $\mathfrak{m}$ with the tangent space $T_{[H]}(G/H)$, the invariant metric g is completely defined by its value at the point $[H]$.

Recall that $(G/H, g)$ is naturally reductive [10] if

$$g\big([X,Y]_\mathfrak{m}, Z\big) = g\big(X, [Y,Z]_\mathfrak{m}\big),$$

for all X, Y, $Z \in \mathfrak{m}$. Here $[\cdot,\cdot]_\mathfrak{m}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{m}$ with respect to the reductive decomposition.

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra of $\mathfrak{g}$, that is, a nilpotent subalgebra such that its normalizer is itself or equivalently if $[X, \mathfrak{h}] \subset \mathfrak{h}$ then $X \in \mathfrak{h}$; α be a linear functional on the complex vectorial space $\mathfrak{h}$ and denote for $\mathfrak{g}_\alpha$ the linear space of $\mathfrak{g}$ given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \quad : \quad [H, X] = \alpha(H)\, X, \text{ for all } H \in \mathfrak{h}\}.$$

Note that for $\alpha = 0$, $\mathfrak{g}_\alpha = \mathfrak{h}$. The linear functional α is called a root (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$. In such case $\mathfrak{g}_\alpha$ is called a *root subspace*. Denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$ and by B the Cartan-Killing form in $\mathfrak{g} \times \mathfrak{g}$, that is,

$$B(X,Y) = \langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y),$$

for all $X, Y \in \mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple, B is not degenerated on $\mathfrak{g} \times \mathfrak{g}$, and its restriction to $\mathfrak{h} \times \mathfrak{h}$ is not degenerated either, for each $\alpha \in \Pi$ exists a unique $H_\alpha \in \mathfrak{h}$ such that $B(H, H_\alpha) = \langle H, H_\alpha \rangle = \alpha(H)$, for all $H \in \mathfrak{h}$. Let $(\alpha, \beta) = B\big(H_\alpha, H_\beta\big)$ then $(\cdot, \cdot)$ is a symmetric not degenerated bilinear form on $\mathfrak{h}^*$.

Theorem 2.1. [21] *If $\mathfrak{g}$ is a semi-simple complex Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ then*

(1) *$\mathfrak{g}$ admits a decomposition in root spaces $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$.*
(2) *The root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Pi$ have complex dimension one.*
(3) *If α and β are any two roots (including 0) and $\beta \neq -\alpha$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to B.*
(4) *If α is a not null root, then $\pi \cap \mathbb{Z}\{\alpha\} = \{\alpha, -\alpha\}$.*
(5) *For each $\alpha \in \Pi$ exists a vector $X_\alpha \in \mathfrak{g}_\alpha$ such that for all $\alpha, \beta \in \Pi$ we have:*
    (a) $[X_\alpha, X_{-\alpha}] = H_\alpha$, $[H, X_\alpha] = \alpha(H)\, X_\alpha$ *(for all $H \in \mathfrak{h}$);*
    (b) $[X_\alpha] = 0$ *if $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \Pi$;*
    (c) $\langle X_\alpha, X_\beta \rangle = 1$ *if $\alpha + \beta = 0$ and $\langle X_\alpha, X_\beta \rangle = 0$ in the other cases.* $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$, *if $\alpha + \beta \in \Pi$ with, $m_{\alpha\beta} \in \mathbb{R}$, and*

$$\begin{aligned} m_{-\alpha,-\beta} &= -m_{\alpha,\beta} \\ m_{-\alpha,\alpha+\beta} &= m_{\alpha+\beta,-\beta} \\ &= m_{-\beta,-\alpha}. \end{aligned} \tag{2.1}$$

The set $\{X_\alpha : \alpha \in \Pi\}$ in this theorem satisfying item 5 is called a *Weyl base or Cartan-Weyl base* of $\mathfrak{g}$ modulo $\mathfrak{h}$.

Theorem 2.2. [21] *Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Π the associated root system . We denote for $\mathfrak{h}_\mathbb{R}$ the subspace of $\mathfrak{g}$ generated on $\mathbb{R}$ for $H_\alpha, \alpha \in \Pi$.*

(1) *The restriction of the Cartan-Killing form B of $\mathfrak{g}$ to $\mathfrak{k}$ is real and strictly positive on $\mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R}$.*
(2) $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + \sqrt{-1}\mathfrak{h}_\mathbb{R}$.

Theorem 2.3. [21] *Let $\Pi^+ \subset \Pi$ be the set of positive roots of the pair $(\mathfrak{g}, \mathfrak{h})$. Suppose that l is the rank of $\mathfrak{g}$, then there exists a root subset $\Sigma = \{\alpha_1 ..., \alpha_l\}$ with the following properties:*

(i) *Each $\alpha_i \in \Sigma, 1 \leq i \leq l$, can not be written as a sum of other positive roots.*
(ii) *Each root $\alpha \in \Pi$ can be written as a linear combination of elements of σ, with coefficient integers, that is $\alpha = \sum_{i=1}^{l} n_i \alpha_i$ with n_i integer number for $i = 1, \dots, l$.*

A root subset Σ with the properties listed in the Theorem 2.3 will be called a simple system of roots.

Definition 2.4. [15] *A real Lie algebra is said to be compact if its Cartan-Killing form is negative definite on it.*

Theorem 2.5. [15] *All semi-simple complex Lie algebra* $\mathfrak{g}$ *admits compact real forms. If* $\mathfrak{u}_1$ *and* $\mathfrak{u}_2$ *are two compact real forms of* $\mathfrak{g}$*, then there is an automorphism* ϕ *of* $\mathfrak{g}$ *such that* $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$ *therefore, the two real forms are isomorphic.*

Definition 2.6. [8] *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{a}$ *a subalgebra of* $\mathfrak{g}$*. We said that* $\mathfrak{a}$ *is* a Borel subalgebra *if it is a soluble maximal subalgebra.*

Definition 2.7. [8] *Let* $\mathfrak{g}$ *be a Lie algebra. A subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$ is called a parabolic subalgebra*, if* $\mathfrak{p}$ *contains any Borel subalgebra.*

3. Flag manifolds as a naturally reductive homogeneous space

A flag manifold is a naturally reductive homogeneous space. In fact it is the homogeneous space $G/C(S)$ where G is a semi-simple Lie group and $C(S)$ is the centralizer of the torus S (not necessarily maximal in G.) When S is a maximal torus, the flag manifold is called maximal or total and we will denote it by $\mathbb{F}$.

For example, in the classical case G is the special unitary group and $C(S)$ must be conjugated to a subgroup of the form $S(U_{n_1} \times U_{n_2} \times \cdots \times U_{n_k})$, with $n_1, n_2, \ldots, n_k$ positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$. If $m_i = n_1 + \cdots + n_i$, the quotient $SU_n/S(U_{n_1} \times \cdots \times U_{n_k})$ can be identified with the set $\mathbb{F}(m_1, \ldots, m_k)$ of "partial flags" $\{0\} = E_0 \subset E_{m_1} \subset \cdots \subset E_{m_k-1} \subset E_{m_k} = \mathbb{C}^n$, where E_i is an i-dimensional subspace of $\mathbb{C}^n$. The case $n_r = 1$ for all $1 \leq r \leq k$ is denoted by $\mathbb{F}(n)$ and it can be identified with the set of the "total flags" $\{0\} = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n$.

Now, if we consider the general case, flag manifolds have a characterization in terms of root theory as follows: let $\mathfrak{g}$ be a semi-simple complex Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, we denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. In the sequel we fix a Weyl basis of $\mathfrak{g}$ as in item 5 of the Theorem 2.1. Let $\Pi^+ \subset \Pi$ a choice of positive roots. We denote with Σ the corresponding simple root system. Let Θ be a subset of Σ and $\langle\Theta\rangle$ the root set generated by Θ. The complementary set $\Pi \setminus \langle\Theta\rangle$ will be denoted as $\langle\Theta\rangle^\perp$ and any root in $\langle\Theta\rangle^\perp$ will be called a complementary root with respect to Θ. Put $\langle\Theta\rangle^+ = \langle\Theta\rangle \cap \Pi^+$, then, on $\mathfrak{g}$ we have the following decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\beta \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_{-\beta}, \tag{3.1}$$

where $\mathfrak{g}_\alpha$, $\alpha \in \Pi$, is the corresponding complex space to α. Now let $\mathfrak{p}_\Theta$ be the parabolic subalgebra of $\mathfrak{g}$ determined by Θ. Then,

$$\mathfrak{p}_\Theta = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\beta. \tag{3.2}$$

Thus, the equation (3.1) can be rewritten as

$$\mathfrak{g} = \mathfrak{p}_\Theta \oplus \sum_{\beta \in \Pi^+ \setminus \langle \Theta \rangle^+} \mathfrak{g}_{-\beta}. \tag{3.3}$$

The general flag manifold $\mathbb{F}_\Theta$ associated with the pair $\{\mathfrak{g}, \Theta\}$ corresponds to the homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$, where G is the complex Lie group whose Lie algebra is $\mathfrak{g}$ and P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G.

Consider the general flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. Let $\mathfrak{u}$ be a real compact form of $\mathfrak{g}$. Denote for U the connected Lie subgroup of G corresponding to $\mathfrak{u}$. Let $K_\Theta = P_\Theta \cap U$, by the construction K_Θ is the torus centralizer. Let $\mathfrak{t}_\Theta = \mathfrak{u} \cap \mathfrak{p}_\Theta$ be the real subalgebra and we will denote by $\mathfrak{t}_\Theta^{\mathbb{C}}$ its complexification. We can write,

$$\mathfrak{t}_\Theta^{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_{-\alpha}. \tag{3.4}$$

U acts transitively on $\mathbb{F}_\Theta$ and thus we can write $\mathbb{F}_\Theta = U/K_\Theta$. If $\Theta = \emptyset$, then $\mathbb{F}_\Theta = \mathbb{F}$ corresponds to the maximal flag manifold. Otherwise, $\mathbb{F}_\Theta$ corresponds to a partial flag manifold. $\mathfrak{u}$ is a real subspace generated by $i\mathfrak{h}_\mathbb{R}$, (see Theorem 2.2) and A_α, S_α, with $\alpha \in \Pi \setminus \Theta$, where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$. We have $\mathfrak{u}_\beta = \mathfrak{u} \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$, $\beta \in \Pi \setminus \langle \Theta \rangle$, and $\mathfrak{q}_\Theta = \sum_{\beta \in \Pi \setminus \langle \Theta \rangle} \mathfrak{u}_\beta$. Therefore,

(i) $\mathfrak{u} = \mathfrak{t}_\Theta \oplus \mathfrak{q}_\Theta, \quad \mathfrak{t}_\Theta \cap \mathfrak{q}_\Theta = \emptyset$;
(ii) $Ad(K_\Theta)\mathfrak{q}_\Theta \subset \mathfrak{q}_\Theta$ and this implies $[\mathfrak{t}_\Theta, \mathfrak{q}_\Theta] \subset \mathfrak{q}_\Theta$.

Conditions (i) and (ii) above guarantee that $\mathbb{F}_\Theta$ is a reductive homogeneous space [10].

Now, we denote by b_0 the origin of $\mathbb{F}_\Theta$; here we are thinking $\mathbb{F}_\Theta$ like a homogeneous space of U. We identify $\mathfrak{q}_\Theta = T_{b_0}(\mathbb{F}_\Theta)$. This identification is given by $\{X \in \mathfrak{q}_\Theta\} \to \{X_{b_0} \in T_{b_0}(\mathbb{F}_\Theta)\}$, that is, by evaluation of $X \in \mathfrak{q}_\Theta$ in b_0 as a vectorial field on $T_{b_0}(\mathbb{F}_\Theta)$. The tangent space of $\mathbb{F}_\Theta$ in b_0 is identified with the subspace $\mathfrak{q}_\Theta = \mathfrak{u} \ominus \mathfrak{t} = \sum_{\beta \in \Pi \setminus \langle \Theta \rangle} \mathfrak{u}_\beta$, generated by $A_\alpha, S_\alpha, \alpha \in \Pi \setminus \langle \Theta \rangle$. Similarly, the complexificated tangent space of $\mathbb{F}_\Theta$ is identified with $\mathfrak{q}^{\mathbb{C}} = \mathfrak{g} \ominus \mathfrak{h} = \oplus_{\alpha \in \Pi \setminus \langle \Theta \rangle} \mathfrak{g}_\alpha$. By the item (ii) above, the action associated to K_Θ leaves $\mathfrak{q}_\Theta$ invariant and it splits in irreducible components, invariant by the adjoint action of K_Θ (see [20]). As $\mathfrak{q}_\Theta$ is generated by $A_\alpha, S_\alpha, \ \alpha \in \Pi \setminus \langle \Theta \rangle$, now we give some properties of these vectors (see [15], section 12.2) that we will use later.

$$\begin{array}{rclrcl}
[A_\alpha, S_{-\alpha}] & = & iH_\alpha, & \langle iH_\alpha, A_\beta \rangle & = & \langle iH_\alpha, S_\beta \rangle = \langle A_\alpha, S_\beta \rangle = 0 \\
[iH_\alpha, S_\beta] & = & -\beta(H_\alpha) A_\beta, & [S_\alpha, S_\beta] & = & -m_{\alpha,\beta} A_{\alpha+\beta} - m_{\alpha,-\beta} A_{\alpha-\beta} \\
[iH_\alpha, A_\beta] & = & \beta(H_\alpha) S_\beta, & [A_\alpha, A_\beta] & = & m_{\alpha,\beta} A_{\alpha+\beta} + m_{-\alpha,\beta} A_{\alpha-\beta} \\
\langle A_\alpha, A_\alpha \rangle & = & \langle S_\alpha, S_\alpha \rangle = -2, & [A_\alpha, S_\beta] & = & m_{\alpha,\beta} S_{\alpha+\beta} + m_{\alpha,-\beta} S_{\alpha-\beta}.
\end{array} \tag{3.5}$$

4. The almost complex manifold $\left(\mathbb{F}_\Theta, J^\Theta, \Lambda^\Theta\right)$

In this Section we will consider $\mathbb{F}_\Theta$ to join with an invariant almost complex structure J^Θ and an U-invariant riemannian metric $ds^2_{\Lambda^\Theta}$.

An invariant almost complex structure on $\mathbb{F}_\Theta$ is completely determined by its value $J^\Theta : \mathfrak{q}_\Theta \longrightarrow \mathfrak{q}_\Theta$. The map J^Θ satisfies $(J^\Theta)^2 = -1$ and commutes with the adjoint action of K_Θ. We denote with the same letter the real valued structure J^Θ and its complexification to $\mathfrak{q}_\Theta{}^{\mathbb{C}}$.

The invariance of J^Θ entails that $J^\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$ for all $\alpha \in \Pi \setminus \Theta$. The eigenvalues of J^Θ are $\pm i$ and the eigenvector in $\mathfrak{q}_\Theta{}^{\mathbb{C}}$ are X_α, $\alpha \in \Pi$. Hence $J^\Theta(X_\alpha) = i\varepsilon_\alpha X_\alpha$, with $\varepsilon_\alpha = \pm 1$ and satisfying $\varepsilon_{-\alpha} = -\varepsilon_\alpha$. As usual, eigenvectors associated to $+i$ are namely the type $(1,0)$, while $-i$-eigenvectors are namely the type $(0,1)$. An invariant almost complex structure on $\mathbb{F}_\Theta$ is completely prescribed by a set of signs $\{\varepsilon_\alpha\}_{\alpha\in\Pi\setminus\Theta}$, with $\varepsilon_{-\alpha} = -\varepsilon_\alpha$. In the sequel we abuse the notation to identify the invariant structure on $\mathbb{F}_\Theta$ with $J^\Theta = \{\varepsilon_\alpha\}_{\alpha\in\Pi}$.

An U-invariant riemannian metric $ds^2_{\Lambda^\Theta}$ on $\mathbb{F}_\Theta$ is completely determined by its values in the origin, that is, by an inner product $(\cdot,\cdot)$ in $\mathfrak{q}_\Theta$, invariant under the action associated to K_Θ ([3], [19], [20]). Such inner product has the form $(X,Y)_{\Lambda^\Theta} = -\langle \Lambda^\Theta \circ X, Y\rangle$, with $\Lambda^\Theta : \mathfrak{q}_\Theta \to \mathfrak{q}_\Theta$ positive definite with respect to the Cartan-Killing form and $\circ$ is the Hadamard product or product term by term. The inner product $(\cdot,\cdot)_{\Lambda^\Theta}$ admits a natural extension to a bilinear symmetric form on $\mathfrak{q}_\Theta^{\mathbb{C}}$ and we use the same notation $(\cdot,\cdot)_{\Lambda^\Theta}$ to this extension. Similarly, to the corresponding complexified form Λ^Θ we maintain the same notation too. K_Θ-invariance of $(\cdot,\cdot)_{\Lambda^\Theta}$ is equivalent to affirm that the Weyl base is a complex base of eigenvectors for the action of Λ^Θ, that is, in $\mathfrak{q}_\Theta^{\mathbb{C}}$ we have

$$\Lambda^\Theta X_\alpha = \lambda^\Theta_\alpha X_\alpha, \tag{4.1}$$

with $\lambda^\Theta_\alpha = \lambda^\Theta_{-\alpha} > 0$. for $\alpha \in \Pi \setminus \langle\Theta\rangle$.

For the real algebra $\mathfrak{q}_\Theta$, the elements of the canonical base A_α, S_α, with $\alpha \in \Pi \setminus \langle\Theta\rangle$, are eigenvectors to the same eigenvalue λ^Θ_α. In the sequel we will use Λ^Θ as synonymous of $ds^2_{\Lambda^\Theta}$ and in the case of the maximal flag manifold $\mathbb{F}$ we will use only Λ.

Definition 4.1. *Let J^Θ be an invariant almost complex structure on $\mathbb{F}_\Theta$. A triple of roots α, β, γ with $\alpha + \beta + \gamma = 0$ is said to be a $\{0,3\}$-triple if $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon_\gamma$ and a $\{1,2\}$-triple otherwise.*

Recall that an almost hermitian manifold is said to be Kähler if $d\,\Omega(X,Y,Z) = 0$, for all vectors X, Y, Z in its tangent space, and $(1,2)$-symplectic if $d\,\Omega(X,Y,Z) = 0$, when one of the vectors X, Y, Z is type $(1,0)$ and the other two are type $(0,1)$. Here Ω is the Kähler form which is given by

$$\Omega(X,Y) = ds^2_{\Lambda^\Theta}(X, JY) = -\langle \Lambda^\Theta \circ X, JY\rangle.$$

In the Weyl basis we have $\Omega(X_\alpha, X_\beta) = (X_\alpha, JX_\beta)_\Lambda = -\langle\, \Lambda X_\alpha, JX_\beta\,\rangle$, that is,

$$\Omega(X_\alpha, X_\beta) = \begin{cases} i\varepsilon_\alpha \lambda_\alpha, & \text{if } \beta = -\alpha, \\ 0, & \text{otherwise} , \end{cases}$$

for all $\alpha, \beta \in \Pi \setminus \langle\Theta\rangle$.

5. Riemannian connection on $\left(\mathbb{F}_\Theta, \Lambda^\Theta\right)$

Since $\mathbb{F}_\Theta$ is a naturally reductive homogeneous space, lets present a known result about this kind of spaces that will be very useful to calculate the riemannian connection in $\mathbb{F}_\Theta$.

Theorem 5.1. [10] *Let $M = G/H$ be a reductive homogeneous space with an $\mathrm{ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and an $\mathrm{ad}(H)$-invariant non-degenerate symmetric bilinear form B on $\mathfrak{m}$. Let g be the G-invariant metric corresponding to B. Then*

(1) *The riemannian connection for g is given by*

$$\nabla^{\mathfrak{m}}_X Y = \frac{1}{2}\,[X, Y]_{\mathfrak{m}} + U(X, Y),$$

where $U(X, Y)$ is the symmetric bilinear mapping on $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$, defined by

$$2B\big(U(X, Y), Z\big) = B\big(X, [Z, Y]_{\mathfrak{m}}\big) + B\big([Z, X]_{\mathfrak{m}}, Y\big),$$

for all $X, Y, Z \in \mathfrak{m}$.

(2) *The riemannian connection for g matches with the natural torsion-free connection if, and only if, B satisfies*

$$B\big(X, [Z, Y]_{\mathfrak{m}}\big) + B\big([Z, X]_{\mathfrak{m}}, Y\big) = 0, \qquad for X, Y, Z \in \mathfrak{m}.$$

Here we are interested in a symmetric bilinear application $U : \mathfrak{q}_\Theta \times \mathfrak{q}_\Theta \to \mathfrak{q}_\Theta$ satisfying $2\Lambda^\Theta\big(U(X, Y), Z\big) = \Lambda^\Theta\big(X, [Y, Z]_{\mathfrak{q}_\Theta}\big) + \Lambda^\Theta\big([Z, X]_{\mathfrak{q}_\Theta}, Y\big)$, for all $X, Y, Z \in \mathfrak{q}_\Theta$; or $2\big\langle\Lambda^\Theta \circ U(X, Y), Z\big\rangle = \big\langle\Lambda^\Theta \circ X, [Y, Z]_{\mathfrak{q}_\Theta}\big\rangle + \big\langle[Z, X]_{\mathfrak{q}_\Theta}, \Lambda^\Theta \circ Y\big\rangle$. Since

$$\big\langle[X, Y]_{\mathfrak{q}_\Theta}, Z\big\rangle = \big\langle X, [Y, Z]_{\mathfrak{q}_\Theta}\big\rangle, \tag{5.1}$$

we have,

$$2\big\langle\Lambda^\Theta \circ U(X, Y), Z\big\rangle = -\big\langle[\Lambda^\Theta \circ X, Y]_{\mathfrak{q}_\Theta}, Z\big\rangle + \big\langle[X, \Lambda^\Theta \circ Y]_{\mathfrak{q}_\Theta}, Z\big\rangle$$

and

$$2\Lambda^\Theta \circ U(X, Y) = \big[X, \Lambda^\Theta \circ Y\big]_{\mathfrak{q}_\Theta} - \big[\Lambda^\Theta \circ X, Y\big]_{\mathfrak{q}_\Theta}. \tag{5.2}$$

Using again Theorem 5.1 the *riemannian connection* ∇ in $\big(\mathbb{F}_\Theta, \Lambda^\Theta\big)$ is given by

$$2\nabla_X Y = [X, Y]_{\mathfrak{q}_\Theta} + 2U(X, Y), \tag{5.3}$$

then

$$2\nabla_X Y = [X, Y]_{\mathfrak{q}_\Theta} + {\Lambda^\Theta}^{-1} \circ \left(\big[X, \Lambda^\Theta Y\big]_{\mathfrak{q}_\Theta} - \big[\Lambda^\Theta X, Y\big]_{\mathfrak{q}_\Theta}\right), \tag{5.4}$$

with $X, Y \in \mathfrak{q}_\Theta$ and $(\Lambda^\Theta)^{-1}$ the inverse of Λ^Θ with respect to the Hadamard product. Note that $\left(\Lambda^\Theta\right)^{-1} = \left(\left(\lambda_\alpha^\Theta\right)^{-1}\right)_{\alpha\in\Pi\setminus\langle\Theta\rangle}$. Finally, in the Weyl basis we have

$$2U(X_\alpha, X_\beta) = \begin{cases} \frac{\lambda_\beta^\Theta - \lambda_\alpha^\Theta}{\lambda_{\alpha+\beta}^\Theta}[X_\alpha, X_\beta], & \text{if } \alpha+\beta \in \Pi\setminus\langle\Theta\rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore in the Weyl basis, the riemannian connection is characterized by the following proposition.

Proposition 5.2. *Consider* $(\mathbb{F}_\Theta, \Lambda^\Theta)$*,* α*,* β*,* $\alpha+\beta \in \Pi\setminus\langle\Theta\rangle$*, and* X_α*,* X_β*,* $X_{\alpha+\beta} \in \mathfrak{q}_\Theta$*, then*

$$\nabla_{X_\alpha} X_\beta = \frac{\lambda_{\alpha+\beta}^\Theta + \lambda_\beta^\Theta - \lambda_\alpha^\Theta}{2\lambda_{\alpha+\beta}^\Theta}[X_\alpha, X_\beta]. \tag{5.5}$$

Proof. Using equation (5.3), item 5 in Theorem 2.1, and equation (4.1) we obtain

$$\begin{aligned} 2\nabla_{X_\alpha} X_\beta &= [X_\alpha, X_\beta]_{\mathfrak{q}_\Theta} + 2U(X_\alpha, X_\beta), \\ &= m_{\alpha,\beta} X_{\alpha+\beta} + \frac{\lambda_\beta^\Theta - \lambda_\alpha^\Theta}{\lambda_{\alpha+\beta}^\Theta}[X_\alpha, X_\beta], \\ &= \frac{\lambda_{\alpha+\beta}^\Theta + \lambda_\beta^\Theta - \lambda_\alpha^\Theta}{\lambda_{\alpha+\beta}^\Theta}[X_\alpha, X_\beta]. \end{aligned}$$

☑

6. Generalized flag manifold and curvature

Since the beginning our main objective was to look for a handy way to calculate the riemannian connection on flag manifolds, Proposition 5.2 gives us (5.5) which is an easy expression to calculate the riemannian connection on $\mathbb{F}_\Theta = G/P_\Theta = U/K_\Theta$. Now we use it in order to understand, or at least to show the behavior of some type of curvatures on $\mathbb{F}_\Theta$.

For reductive homogenous spaces, again [10] provides an expression for the curvature tensor, using it jointly with the equation (5.4), in b_0 we have

$$R(X,Y)_{b_0} = \left[\nabla^{\mathfrak{q}_\Theta} X, \nabla^{\mathfrak{q}_\Theta} Y\right] - \nabla[X,Y]_{\mathfrak{q}_\Theta} - ad\big([X,Y]_{\mathfrak{k}_\Theta}\big), \tag{6.1}$$

for all $X, Y \in \mathfrak{q}_\Theta$, with $\mathfrak{q}_\Theta$ and $\mathfrak{k}_\Theta$ as in (3.4). Here $\nabla^{\mathfrak{q}_\Theta}$ represents the riemannian connection on $\mathfrak{q}_\Theta$ and $[\ ]_{\mathfrak{q}_\Theta}$, $[\ ]_{\mathfrak{k}_\Theta}$ represent the bracket projection on the respective spaces.

We know that (see [9]) for each plane generated by the vectors X, Y in the tangent space, the sectional curvature of the plane is defined by

$$K(X,Y) = \Lambda^\Theta\big(R(X,Y)X, Y\big). \tag{6.2}$$

Thus, applying the equations (6.2) and (6.1) we have

$$K(X,Y) = \Lambda^{\Theta}\Big(\nabla_X\nabla_Y X - \nabla_Y\nabla_X X - \nabla_{[X,Y]_{\mathfrak{q}_\Theta}} X - \big[[X,Y]_{\mathfrak{t}_\Theta}, X\big], Y\Big). \quad (6.3)$$

Now suppose that $[X,Y]_{\mathfrak{t}_\Theta} = 0$. Using (6.3), (5.4) and the invariance of $\langle\cdot,\cdot\rangle$ (5.1) we have

$$\begin{aligned}
K(X,Y) &= \Lambda^{\Theta}(\nabla_X\nabla_Y X, Y) - \Lambda^{\Theta}(\nabla_Y\nabla_X X, Y) - \Lambda^{\Theta}\left(\nabla_{[X,Y]}X, Y\right), \\
&= \Lambda^{\Theta}\left(\tfrac{1}{2}[X,\nabla_Y X], Y\right) + \Lambda^{\Theta}\Big(\tfrac{1}{2}\left(\Lambda^{\Theta}\right)^{-1}[X,\Lambda^{\Theta}\nabla_Y X], Y\Big) \\
&\quad -\Lambda^{\Theta}\Big(\tfrac{1}{2}\left(\Lambda^{\Theta}\right)^{-1}[\Lambda^{\Theta}X,\nabla_Y X], Y\Big) - \Lambda^{\Theta}\left(\tfrac{1}{2}[[X,Y],X], Y\right) + \\
&\quad -\tfrac{1}{2}\left(\Lambda^{\Theta}\right)^{-1}[[X,Y],\Lambda^{\Theta}X], y + \Lambda^{\Theta}\Big(\tfrac{1}{2}\left(\Lambda^{\Theta}\right)^{-1}[\Lambda^{\Theta}[X,Y],X], Y\Big) \\
&= -\{\tfrac{1}{2}\langle[X,\nabla_Y X],\Lambda^{\Theta}Y\rangle + \tfrac{1}{2}\langle[X,\Lambda^{\Theta}\nabla_Y X],Y\rangle - \tfrac{1}{2}\langle[\Lambda^{\Theta}X,\nabla_Y X],Y\rangle \\
&\quad -\tfrac{1}{2}\langle[[X,Y],X],\Lambda^{\Theta}Y\rangle - \tfrac{1}{2}\langle[[X,Y],\Lambda^{\Theta}X],Y\rangle\tfrac{1}{2}\langle[\Lambda^{\Theta}[X,Y],X],Y\rangle\} \\
&= -\{-\tfrac{1}{2}\langle\nabla_Y X,[X,\Lambda^{\Theta}Y]\rangle - \tfrac{1}{2}\langle\Lambda^{\Theta}\nabla_Y X,[X,Y]\rangle + \tfrac{1}{2}\langle\nabla_Y X,[\Lambda^{\Theta}X,Y]\rangle \\
&\quad +\tfrac{1}{2}\langle\nabla_X X,[Y,\Lambda^{\Theta}Y]\rangle - \tfrac{1}{2}\langle\nabla_X X,[Y,Y]\rangle - \tfrac{1}{2}\langle\nabla_X X,[\Lambda^{\Theta}Y,Y]\rangle \\
&\quad -\tfrac{1}{2}\langle[X,Y],[X,\Lambda^{\Theta}Y]\rangle - \tfrac{1}{2}\langle[X,Y],[\Lambda^{\Theta}X,Y]\rangle + \tfrac{1}{2}\langle\Lambda^{\Theta}[X,Y],[X,Y]\rangle\} \\
&= \tfrac{1}{2}\langle[X,Y],[X,\Lambda^{\Theta}Y]\rangle - \tfrac{1}{4}\langle\left(\Lambda^{\Theta}\right)^{-1}[\Lambda^{\Theta}X,Y],[X,\Lambda^{\Theta}Y]\rangle \\
&\quad +\tfrac{1}{4}\langle\left(\Lambda^{\Theta}\right)^{-1}[X,\Lambda^{\Theta}Y],[X,\Lambda^{\Theta}Y]\rangle - \tfrac{3}{4}\langle\Lambda^{\Theta}[X,Y],[X,Y]\rangle \\
&\quad +\tfrac{1}{2}\langle[\Lambda^{\Theta}X,Y],[X,Y]\rangle + \tfrac{1}{4}\langle\left(\Lambda^{\Theta}\right)^{-1}[\Lambda^{\Theta}X,Y],[\Lambda^{\Theta}X,Y]\rangle \\
&\quad -\tfrac{1}{4}\langle\left(\Lambda^{\Theta}\right)^{-1}[X,\Lambda^{\Theta}Y],[\Lambda^{\Theta}X,Y]\rangle.
\end{aligned} \quad (6.4)$$

Proposition 6.1. *Consider the maximal flag manifold $\mathbb{F}$, and the basic vectors A_α, S_α, $\alpha\in\Pi$. Then*

(i) $K\big(A_\alpha, S_\beta\big) = K\big(S_\alpha, S_\beta\big) = K\big(A_\alpha, A_\beta\big) = -\xi_{\alpha,\beta}m^2_{\alpha,\beta} + \xi_{-\alpha,\beta}m^2_{-\alpha,\beta}$, *where*

$$\xi_{\alpha,\beta} = \lambda_\alpha + \lambda_\beta + \frac{\lambda_\alpha^2 + \lambda_\beta^2 - 2\lambda_\alpha\lambda_\beta}{2(\lambda_{\alpha+\beta})} - \frac{3\lambda_{\alpha+\beta}}{2}. \quad (6.5)$$

(ii) $K(A_\alpha, S_{-\alpha}) = -4\lambda_\alpha\alpha(H_\alpha)$.

Proof.

(i) It is immediately obtained using (6.4) and (3.5) and the property $m^2_{\alpha,-\beta} = m^2_{-\alpha,\beta}$.

(ii) On the maximal flag manifold, $\mathfrak{t} = \mathfrak{h}$ and the only case where $[X,Y]_{\mathfrak{h}} \neq 0$ is when $X = A_\alpha$ and $Y = S_{-\alpha}$. Then, $[A_\alpha, S_{-\alpha}] = 2iH_\alpha$ and we obtain

$$\begin{aligned}
K\left(A_\alpha, S_{-\alpha}\right) &= \Lambda\left(\nabla_{A_\alpha}\nabla_{S_{-\alpha}}A_\alpha, S_{-\alpha}\right) - \Lambda\left(\nabla_{S_{-\alpha}}\nabla_{A_\alpha}A_\alpha, S_{-\alpha}\right) + \\
&\quad -\Lambda\left(\nabla_{[A_\alpha, S_{-\alpha}]}A_\alpha, S_{-\alpha}\right) - \Lambda\left(\Lambda\left([A_\alpha, S_{-\alpha}]_{\mathfrak{h}}\right)A_\alpha, S_{-\alpha}\right), \\
&= \Lambda\left(\tfrac{1}{2}[A_\alpha, \nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\right) + \Lambda\left(\tfrac{1}{2}\Lambda^{-1}[A_\alpha, \Lambda\nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\right) + \\
&\quad -\Lambda\left(\tfrac{1}{2}\Lambda^{-1}[\Lambda A_\alpha, \nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\right) - \Lambda\left(\tfrac{1}{2}[S_{-\alpha}, \nabla_{A_\alpha}A_\alpha], S_{-\alpha}\right) + \\
&\quad -\Lambda\left(\tfrac{1}{2}\Lambda^{-1}[S_{-\alpha}, \Lambda\nabla_{A_\alpha}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\right) + \\
&\quad +\Lambda\left(\tfrac{1}{2}\Lambda^{-1}[\Lambda A_\alpha, \nabla_{A_\alpha}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\right) + \\
&\quad -ds^2_\Lambda\left([[A_\alpha, S_{-\alpha}]_{\mathfrak{h}}, A_\alpha], S_{-\alpha}\right), \\
&= -\{\tfrac{\lambda_\alpha}{2}\langle[A_\alpha, \nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle + \tfrac{1}{2}\langle[A_\alpha, \Lambda\nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle + \\
&\quad -\tfrac{1}{2}\langle[\Lambda A_\alpha, \nabla_{S_{-\alpha}}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle - \tfrac{\lambda_\alpha}{2}\langle[S_{-\alpha}, \nabla_{A_\alpha}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle + \\
&\quad -\tfrac{1}{2}\langle[S_{-\alpha}, \Lambda\nabla_{A_\alpha}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle + \tfrac{1}{2}\langle[\Lambda A_\alpha, \nabla_{A_\alpha}A_\alpha]_{\mathfrak{q}}, S_{-\alpha}\rangle\} + \\
&\quad \lambda_\alpha\langle[2iH_\alpha, A_\alpha], S_{-\alpha}\rangle, \\
&= -\{-\tfrac{\lambda_\alpha}{2}\langle\nabla_{S_{-\alpha}}A_\alpha, [A_\alpha, S_{-\alpha}]_{\mathfrak{q}}\rangle - \tfrac{1}{2}\langle\Lambda\nabla_{S_{-\alpha}}A_\alpha, [A_\alpha, S_{-\alpha}]_{\mathfrak{q}}\rangle + \\
&\quad +\tfrac{1}{2}\langle\nabla_{S_{-\alpha}}A_\alpha, [\Lambda A_\alpha, S_{-\alpha}]_{\mathfrak{q}}\rangle + \tfrac{\lambda_\alpha}{2}\langle\nabla_{A_\alpha}A_\alpha, [S_{-\alpha}, S_{-\alpha}]_{\mathfrak{q}}\rangle + \\
&\quad +\tfrac{1}{2}\langle\Lambda\nabla_{A_\alpha}A_\alpha, [S_{-\alpha}, S_{-\alpha}]_{\mathfrak{q}}\rangle - \tfrac{1}{2}\langle\nabla_{A_\alpha}A_\alpha, [\Lambda A_\alpha, S_{-\alpha}]_{\mathfrak{q}}\rangle\} + \\
&\quad +2\lambda_\alpha\langle\alpha\left(H_\alpha\right)S_{-\alpha}, S_{-\alpha}\rangle, \\
&= -4\lambda_\alpha\alpha\left(H_\alpha\right).
\end{aligned} \tag{6.6}$$

☑

Note that in the last case in the proposition above $K(A_\alpha, S_{-\alpha}) < 0$, since $\alpha(H_\alpha)$ is a positive rational.

Now, lets consider $(\mathbb{F}, J, \Lambda)$ to be an almost Hermitian maximal flag manifold, and assume that $\alpha, \beta \in \Sigma$, then $\pm(\alpha - \beta)$ is not in Π and (6.5) is reduced to

$$K\big(A_\alpha, S_\beta\big) = K\big(S_\alpha, S_\beta\big) = K\big(A_\alpha, A_\beta\big) = -\xi_{\alpha,\beta} m^2_{\alpha,\beta}.$$

Remark 6.1. *Now assume that J is integrable, $(\mathbb{F}, J, \Lambda)$ is Kähler* [17] *and all zero-sum triple $\{\alpha, \beta, -(\alpha + \beta)\}$ must be of the type $\{1, 2\}$. Here we have the following cases*

(1) *If $\lambda_\alpha = \lambda_\beta + \lambda_{\alpha+\beta}$, we have*

$$K\big(A_\alpha, S_\beta\big) = K\big(S_\alpha, S_\beta\big) = K\big(A_\alpha, A_\beta\big) = -2\lambda_\beta\big(m_{\alpha,\beta}\big)^2 < 0.$$

(2) *If $\lambda_\beta = \lambda_\alpha + \lambda_{\alpha+\beta}$, we have*

$$K\big(A_\alpha, S_\beta\big) = K\big(S_\alpha, S_\beta\big) = K\big(A_\alpha, A_\beta\big) = -2\lambda_\alpha\big(m_{\alpha,\beta}\big)^2 < 0.$$

(3) *If* $\lambda_\alpha + \lambda_\beta = \lambda_{\alpha+\beta}$, *we have*

$$K(A_\alpha, S_\beta) = K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \frac{2\lambda_\alpha\lambda_\beta}{\lambda_\alpha + \lambda_\beta}(m_{\alpha,\beta})^2 > 0.$$

(4) *If* $\lambda_{\alpha+\beta} = 2\lambda_\alpha$, *we have*

$$K(A_\alpha, S_\beta) = K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \lambda_\alpha(m_{\alpha,\beta})^2 > 0.$$

Now, when $\alpha-\beta$ *is also a root, we have that* $\{\alpha, \beta, -(\alpha+\beta)\}, \{\beta, -\alpha, \alpha-\beta\}$ *are* $\{1,2\}$*-triples, then we have the following cases:*

(1) *If* $\lambda_\alpha = \lambda_\beta + \lambda_{\alpha+\beta}$, *then* $\lambda_{\alpha-\beta} = \lambda_\alpha + \lambda_\beta$ *and*

$$\begin{aligned} K(A_\alpha, S_\beta) &= K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta), \\ &= -2\lambda_\beta\left\{(m_{\alpha,\beta})^2 - \frac{\lambda_\alpha}{\lambda_\alpha+\lambda_\beta}(m_{\alpha,-\beta})^2\right\}. \end{aligned}$$

(2) *If* $\lambda_\beta = \lambda_\alpha + \lambda_{\alpha+\beta}$, *then* $\lambda_{\alpha-\beta} = \lambda_\alpha + \lambda_\beta$ *and*

$$\begin{aligned} K(A_\alpha, S_\beta) &= K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \\ &= -2\lambda_\alpha\left\{(m_{\alpha,\beta})^2 - \frac{\lambda_\beta}{\lambda_\alpha+\lambda_\beta}(m_{\alpha,-\beta})^2\right\}. \end{aligned}$$

(3) *If* $\lambda_{\alpha+\beta} = \lambda_\alpha + \lambda_\beta$, *then we have two cases:*

- *If* $\lambda_\alpha = \lambda_\beta + \lambda_{\alpha-\beta}$, *we have*

$$\begin{aligned} K(A_\alpha, S_\beta) &= K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \\ &= -2\lambda_\beta\left\{-\frac{\lambda_\alpha}{\lambda_\alpha+\lambda_\beta}(m_{\alpha,\beta})^2 + (m_{\alpha,-\beta})^2\right\}. \end{aligned}$$

- *If* $\lambda_\beta = \lambda_\alpha + \lambda_{\alpha-\beta}$, *we have*

$$\begin{aligned} K(A_\alpha, S_\beta) &= K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \\ &= -2\lambda_\alpha\left\{-\frac{\lambda_\beta}{\lambda_\alpha+\lambda_\beta}(m_{\alpha,\beta})^2 + (m_{\alpha,-\beta})^2\right\}. \end{aligned}$$

Example 6.1. *Let us consider the invariant case* $(\mathbb{F}(3), J, \Lambda)$ *to be Kähler, in this case*

$$\Lambda = \begin{pmatrix} 0 & \lambda_\alpha & 2\lambda_\alpha \\ \lambda_\alpha & 0 & \lambda_\alpha \\ 2\lambda_\alpha & \lambda_\alpha & 0 \end{pmatrix}.$$

As $\alpha + 2\beta, 2\alpha + \beta$ *are not roots we have*

$$\begin{aligned} K(A_\alpha, S_\beta) &= K(S_\alpha, S_\beta) = K(A_\alpha, A_\beta) = \lambda_\alpha(m_{\alpha,\beta})^2, \\ K(A_\alpha, A_{\alpha+\beta}) &= K(A_\alpha, S_{\alpha+\beta}) = K(S_\alpha, A_{\alpha+\beta}) = \\ &= K(S_\alpha, A_{\alpha+\beta}) = K(S_\beta, A_{\alpha+\beta}) = K(S_\beta, A_{\alpha+\beta}) = 0, \\ K(A_\alpha, S_\alpha) &= K(A_\beta, S_\beta) = 4\lambda_\alpha\alpha(H_\alpha) > 0. \end{aligned}$$

Thus the scalar curvature *of* $\mathbb{F}(3)$ *is* $3\lambda_\alpha(m_{\alpha,\beta})^2 + 8\lambda_\alpha\alpha(H_\alpha) > 0$. *So we have that the* Ricci curvature $Ric(A_{\alpha+\beta}) = Ric(S_{\alpha+\beta}) = 0$ *and* $Ric(A_\alpha) = Ric(A_\beta) = Ric(S_\alpha) = Ric(S_\beta) = 2\lambda_\alpha(m_{\alpha,\beta})^2 + 4\lambda_\alpha\alpha(H_\alpha) > 0$. *In* $\mathbb{F}(n)$ *is the only case where* $Ric > 0$.

In the next sections we will study some type of curvatures, such as: *holomorphic bisectional curvature* and *sectional Kälherian curvature* on $(\mathbb{F}, J, \Lambda)$ in order to understand, through the possible values of these curvatures, some aspects of its geometry and its topology, (see for example [10], [18], [6], [16]).

7. Holomorphic bisectional curvature

Let (N, J, g) be a Hermitian riemannian manifold. $HBRiem^N(X, Y)$ denotes the holomorphic bisectional curvature of N, given by the following equation (see [10])

$$HBRiem^N(X, Y) = g(R^N(X, JX)Y, JY),$$

where R^N is the curvature tensor in N. In our case, $(\mathbb{F}, J, \Lambda)$, since J is an endomorphism it is easy to show that on basic vectors A_α, S_β we have

$$J(A_\alpha) = \varepsilon_\alpha S_\alpha, \qquad J(S_\alpha) = -\varepsilon_\alpha A_\alpha.$$

Then,

$$\begin{aligned} HBRiem\big(A_\alpha, S_\beta\big) &= \Lambda\big(R\big(A_\alpha, J\big(A_\alpha\big)S_\beta, J\big(S_\beta\big)\big), \\ &= -\Lambda\big(R\big(A_\alpha, \varepsilon_\alpha S_\alpha\big)S_\beta, \varepsilon_\beta A_\beta\big), \\ &= -\varepsilon_\alpha\varepsilon_\beta\Lambda\big(R\big(A_\alpha, S_\alpha\big)S_\beta, A_\beta\big). \end{aligned}$$

$$\begin{aligned} HBRiem\big(A_\alpha, A_\beta\big) &= \Lambda\big(R\big(A_\alpha, J\big(A_\alpha\big)\big)A_\beta, J\big(A_\beta\big)\big), \\ &= \Lambda\big(R\big(A_\alpha, \varepsilon_\alpha S_\alpha\big)A_\beta, \varepsilon_\beta S_\beta\big), \\ &= \varepsilon_\alpha\varepsilon_\beta\Lambda\big(R\big(A_\alpha, S_\alpha\big)A_\beta, S_\beta\big), \\ &= -\varepsilon_\alpha\varepsilon_\beta\Lambda\big(R\big(A_\alpha, S_\alpha\big)S_\beta, A_\beta\big). \end{aligned}$$

$$\begin{aligned} HBRiem\big(S_\alpha, S_\beta\big) &= \Lambda\big(R\big(S_\alpha, J\big(S_\alpha\big)\big)S_\beta, J\big(S_\beta\big)\big), \\ &= \Lambda\big(R\big(S_\alpha, -\varepsilon_\alpha A_\alpha\big)A_\beta, -\varepsilon_\beta A_\beta\big), \\ &= \varepsilon_\alpha\varepsilon_\beta\Lambda\big(R\big(S_\alpha, A_\alpha\big)S_\beta, A_\beta\big), \\ &= -\varepsilon_\alpha\varepsilon_\beta\Lambda\big(R\big(A_\alpha, S_\alpha\big)S_\beta, A_\beta\big). \end{aligned}$$

Therefore,

$$\begin{aligned} HBRiem\big(A_\alpha, S_\beta\big) &= HBRiem\big(A_\alpha, A_\beta\big) = HBRiem\big(S_\alpha, S_\beta\big) = \\ &= -\varepsilon_\alpha\varepsilon_\beta\left(m^2_{\alpha,\beta}\big(2\lambda_\beta - 2\lambda_\alpha + \lambda_{\alpha+\beta}\big) - \frac{\big(\lambda_\alpha - \lambda_\beta\big)^2}{\lambda_{\alpha-\beta}}m^2_{\alpha,-\beta}\right) \end{aligned}$$

while,

$$\begin{aligned} HBRiem\big(A_\alpha, S_{-\alpha}\big) &= \Lambda\big(R\big(A_\alpha, J\big(A_\alpha\big)\big)S_{-\alpha}, J\big(S_{-\alpha}\big)\big), \\ &= \varepsilon_\alpha^2 \Lambda\big(R\big(A_\alpha, S_{-\alpha}\big)S_{-\alpha}, A_\alpha\big), \\ &= -\Lambda\big(R\big(A_\alpha, S_{-\alpha}\big)A_\alpha, S_{-\alpha}\big), \\ &= -K\big(A_\alpha, S_{-\alpha}\big), \\ &= 4\alpha\big(H_\alpha\big)\lambda_\alpha > 0. \end{aligned}$$

Now suposse that $(\mathbb{F}, J, \Lambda)$ is Kähler and take $\alpha, \beta \in \Sigma$, then $\alpha - \beta$ is not root; therefore,

$$HBRiem\big(A_\alpha, S_\beta\big) = -\varepsilon_\alpha \varepsilon_\beta m_{\alpha,\beta}^2 \big(2\lambda_\beta - 2\lambda_\alpha + \lambda_{\alpha+\beta}\big).$$

If $\{\alpha, \beta, -(\alpha+\beta)\}$ is a $\{1,2\}$-triple the only interesting case is when $\lambda_{\alpha+\beta} = 2\lambda_\alpha$, then,

$$HBRiem\big(A_\alpha, S_\beta\big) = -2m_{\alpha,\beta}^2 \lambda_\alpha < 0.$$

The previous calculations jointly with a result due to Siu and Yau [18] implies that if $(\mathbb{F}, \Lambda, J)$ is Kähler, then it can not be biholomorphically equivalent to any projective space $\mathbb{C}P(n)$.

8. Kählerian sectional curvature

Let M be a Kähler manifold of complex dimension n, $x \in M$ and let P be a plane in T_xM, that is, a real 2-dimensional subspace of T_xM. Let X, Y be an orthonormal base of P. Define $\rho(P)$, the angle between P and $J(P)$, by

$$\cos\rho(P) = |g(X, JY)|,$$

where g is the metric on M. Denote by $K(P)$ the sectional curvature of P. Then the Kählerian sectional curvature of P is denoted $K^*(P)$ and given by

$$K^*(P) = \frac{4K(P)}{1 + 3cos^2\rho(P)}.$$

In our case to the maximal flag manifold $\mathbb{F}$, normalizing A_α and S_β, $\alpha, \beta \in \Pi$, then they are an orthonormal base for $\mathfrak{q}$. If $P = span\{A_\alpha, S_\beta\} \subset \mathfrak{q}$ we have

$$\begin{aligned} \cos\rho(P) &= |\Lambda(S_\alpha, J(S_\beta))|, \\ &= |\Lambda(A_\alpha, -\varepsilon_\beta A_\beta)|, \\ &= |\lambda_\alpha \langle A_\alpha, A_\beta \rangle|. \end{aligned}$$

Thus $\cos\rho(P)$ is different from zero only when $\beta = \pm\alpha$ and in this case $\cos\rho(P) = 1$, because of the normalization of the base. Thus,

$$K^*(P) = K(P) = -4\lambda_\alpha \alpha(H_\alpha) < 0.$$

So if $(\mathbb{F}, J, \Lambda)$ is Kähler then it can not be holomorphically isometric to any projective space $\mathbb{C}P(n)$ (see [10] p. 369).

Given the results about curvatures in $\mathbb{F}$, one question appears in order to continue this work: Is it possible to characterize, with this behavior, flag manifolds in the same way that projective spaces are characterized?

Acknowledgments: The first author is grateful to the *Universidad Industrial de Santander* for the financial support during the development of the present work. The authors would like to thank Department of Mathematics of Universidade Estadual de Campinas (IMECC-UNICAMP) for their hospitality during the tenure of our projects.

References

[1] J. F. Adams, Lectures on Lie Groups, University of Chicago Press, Chicago 1983.

[2] M. Black, *Harmonic Maps into Homogeneous Spaces*, Pitman Research Notes, Math series 255. Longman, Harlow, 1991.

[3] A. Borel, Kählerian coset spaces of semi-simple Lie groups, *Proc. Nat. Acad. of Sci.* **40** (1954), 1147–1151.

[4] A. Borel & F. Hirzebruch, Characteristic classes and homogeneous spaces I, *Amer. J. Math.* **80** (1958), 458–538.

[5] N. Cohen, C. J. Negreiros & L. A. San Martin, Characterization of (1,2)–symplectic metrics on flag manifolds, *Contemporary Math.* **288** (2002), 300–304.

[6] J. Eells & L. Lemaire, Another report on harmonic maps, *Bull. London Math. Soc.* **20** (1988), 385–524.

[7] S. Helgason, Differential geometry, Lie groups and symetric spaces, Acad. Press, New York, 1978.

[8] J. E. Humphreys, *Introduction to Lie algebras and representation theory,* Springer-Verlag, New York, 1973.

[9] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. 1, Interscience Publishers, Mew York, 1963.

[10] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. 2, Interscience Publishers, New York, 1969.

[11] X. Mo & C. J. Negreiros, $(1,2)$-Symplectic structure on flag manifolds, *Tôhoku Math. J.* **52** (2000), 271–282.

[12] M. Paredes, *Aspectos da geometria complexa das variedades bandeira*, Ph. D. Thesis, Universidade Estadual de Campinas, Brasil, 2000.

[13] M. Paredes, Families of $(1,2)$–symplectic metrics on flag manifolds, *Internat. J. Math. Math. Sci.* **29** (2002), 651–664.

[14] S. Pinzón, *Variedades bandeira, f–Estruturas e métricas* $(1,2)$*–simpléticas*, Ph. D. Thesis, Universidade Estadual de Campinas, Brasil, 2003.

[15] L. A. San Martin, *Algebras de Lie*, Editora da Unicamp, Campinas-S.P., 1999.

[16] J. Sacks & K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. of Math.* **113** (1981), 1–24.

[17] L. A. B. San Martin & C. J. Negreiros, Invariant almost Hermitian structures on flag manifolds, *Adv. in Math.*, **178** (2003), 277–310.

[18] Y. T. Siu & S. T. Yau, Compact Kähler manifolds of positive bisectional curvature, *Invent. Math.* **59** (1980), 189–204.

[19] E. C. Licurgo, *Estruturas Quase-Hermitianas Invariantes e Ideais Abelianos*, Dissertação de Mestrado, Universidade Estadual de Campinas, Campinas-S.P., 2003.
[20] R. C. Silva, *Estruturas Quase-Hermitianas Invariantes em Espaços Homogêneos de Grupos Semi-simples*, Ph. D. Thesis, Universidade Estadual de Campinas, Brasil, 2003.
[21] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups, I.*, Springer-Verlag, Berlin, 1972.
[22] J. A. Wolf & A. Gray, Homogeneous spaces defined by Lie group automorphisms, I., *J. Diff. Geom.* **2** (1968), 77–114.
[23] J. A. Wolf & A. Gray, Homogeneous spaces defined by Lie group automorphisms, II., *J. Diff. Geom.* **2** (1968), 115–159.

(Recibido en octubre de 2006. Aceptado en marzo de 2007)

School of Science and Technology
Universidad del Turabo
PO Box 3030. PR 00778-3030
Gurabo, Puerto Rico
e-mail: maparedes@suagm.edu

Escuela de Matemáticas
Universidad Industrial de Santander
Apartado Aéreo 678
Bucaramanga, Colombia
e-mail: spinzon@uis.edu.co