# Lessons from Toy-Models for the Dynamics of Loop Quantum Gravity*

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**Abstract.** We review some approaches to the Hamiltonian dynamics of (loop) quantum gravity, the main issues being the regularization of the Hamiltonian and the continuum limit. First, Thiemann's definition of the quantum Hamiltonian is presented, and then more recent approaches. They are based on toy models which provide new insights into the difficulties and ambiguities faced in Thiemann's construction. The models we use are parametrized field theories, the topological BF model of which a special case is three-dimensional gravity which describes quantum flat space, and Regge lattice gravity.

*Key words:* Hamiltonian constraint; loop quantum gravity; parametrized field theories; topological BF theory; discrete gravity

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## 1 Introduction

The key feature of general relativity is the equivalence between gravity and geometry, so that geometry itself becomes a dynamical entity. Loop Quantum Gravity (LQG) [8, 87, 93, 105] is a quantization framework which particularly emphasizes those geometric content. It makes sense of quantum geometry of space, with quantum numbers of areas and volumes. And it is also more than that. It has been put to a rigorous mathematical status, which makes clear that it is a generic way to quantize background independent theories. This means that no background metric is available, and that general covariance must be satisfied at all steps. A key theorem asserts the uniqueness of the quantization map when diffeomorphism invariance is required [82]. As a result, the basic excitations are supported by embedded graphs, known as spin networks.

After the completion of the kinematical framework, the task was to make sense of the Hamiltonian of general relativity. This implies two steps: *i*) regularization within the loop quantization, and *ii*) the continuum limit. In a remarkable series of papers, [106, 107, 108], this task was carried out by Thiemann, and as a major success the quantization was shown to be anomaly-free, which means no classical degrees of freedom of the regularization are left over through quantization.

This operator in addition to its technical precision has a number of interesting properties like diffeomorphism covariance. It has also led to detailed quantization of mini-superspace models like Loop Quantum Cosmology (LQC). Although its use in the current formulation of dynamics

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in LQG is rather limited (spin foam models [10, 98] and alternative approaches like the Master constraint in the canonical theory [111] have been introduced to circumvent the difficulties faced by Thiemann's construction), as far as canonical LQG is concerned it still retains a fundamental importance. There are already a large number of reviews on this subject. *Our aim in this review is not merely to repeat what has been said above, but instead to offer a frank account of the state of the art in the field, focusing on the most difficult aspects, and some of the different ways in which researchers have tried to go beyond Thiemann's construction.*

Quickly after the release of Thiemann's proposal, several criticisms were made, see for instance [64, 81, 104]. As usual in quantum theory, the construction is plagued by numerous ambiguities, but those papers also call into question the physical content of the continuum limit induced by the quantum Hamiltonian. In a nutshell, the difficulty with Thiemann's Hamiltonian is that although the quantum algebra is mathematically anomaly-free, it was not possible to check explicitly that it implements the classical (Dirac) constraint algebra at the quantum level. In [104], this issue was traced back to the fact that the quantum Hamiltonian acts in an ultra-local way, only at the vicinity of nodes of the spin network excitations, so that physical states lack long range correlations.

In spite of those criticisms, there has been very few modifications of the proposal (see [2] for a recent improved regularization). However, as usual in physics, toy models are useful to get insights which should survive in the full theory, because they enable to disentangle the main issues. Here we will focus on

- parametrized field theory in two dimensions,
- topological BF field theory,
- as a specific case of the latter, 2+1 gravity,
- a canonical approach to discrete gravity (Regge calculus).

They aim at going deeper into the structure of the loop quantization, and explore new alternatives which are, sometimes surprisingly, fully amenable. Let us say a few words on those models (to avoid overflowing of references at this stage, we put them in the corresponding sections).

Due to the infinite number of ambiguities of the standard quantum Hamiltonian, there is a vast room for improvement in the definition of the operator. In the context of parametrized field theory, we will argue that the off-shell closure of the constraint algebra is a key requirement which not only has the potential to reduce the number of ambiguities, but also that in simple models, this requirement leads to physically correct results. The main results appeared in [77, 78, 79].

Three-dimensional gravity is well-known as an exactly solvable topological field theory [113], where transition amplitudes have been evaluated [115]. However, one can try to forget as most as possible that it is topological, and treat it instead just as geometrodynamics with the loop quantization. The main question is then: What is the Wheeler–DeWitt equation, which would quantize the scalar Hamiltonian of the theory? Hence, this model addresses, in a successful way as we will show, the issue of the regularization in the loop quantization, and proves that it is possible to reproduce non-trivially the expected results of the topological formulation. The main references for the work we review are [27, 29, 115].

Three-dimensional gravity is a special case of the topological BF theory [25, 72] which can be defined in arbitrary dimensions. Though gravity and BF theory are not equivalent in dimension four and higher, since they differ by the presence and the lack of local degrees of freedom, it turns out that they are still related [41, 58, 60, 72]. One of these relations consists in formulating gravity as a BF model supplemented with constraints, and this is the main road to the spin foam quantization attempts to quantum gravity. If one wants to understand how spin foams may be able to circumvent the difficulties faced in the Hamiltonian quantization, one should therefore

start with a good understanding of the topological model. However, until recently, its loop (and spin foam) quantization was not so much developed. It is mainly known in the three-dimensional case, thanks to [59, 88], and progress in higher dimensions have only appeared very recently [34, 37]. A pedagogical review is [10] but it is now ten years old. *We consider the section on the BF model of the present review as an update to this review, proving it is now much better understood.*

Since general relativity describes the dynamics of the geometry, it is interesting to see how the quantization of the 2+1 scalar constraint describes flat quantum geometry. This geometry is naturally discrete since it is supported by graphs, so that it makes contact with the canonical approach to discrete gravity, which looks for a Hamiltonian respecting the gauge symmetries, and hence the physical degrees of freedom of the full theory. Such a Hamiltonian can be reached through the so-called perfect discrete action, obtained after coarse-graining. The material of the review is borrowed from [13, 14].

The literature is quite rich and we have certainly left interesting works behind. For instance, we have not been able to cover most of the work done in certain mini-superspace models. Similarly, very little will be said on spin foam models, though parts of the work presented in the review were motivated by linking canonical LQG to spin foams, especially [29, 88] in 2+1 dimensions. In higher dimensions, the most recent works are [1, 3, 33].

As we will show, the technology used in loop quantum gravity, especially in the case of quantum flat space, relies heavily on SU(2) re-coupling methods. While the basic tools are very well-known for a long time [112, 119], research in loop quantum gravity has led over the last years to interesting side-products on advanced quantum angular momentum theory. They include exact results, such as new recursion formulae for Wigner $6j$-symbols and more complicated objects inspired by spin foam models [31, 33], and also new asymptotic expansion in the regime of large angular momenta. In particular, the expansion of the $6j$-symbol is now understood beyond the usual leading order [32, 54, 55], asymptotics of arbitrary Wigner coefficients with some large and small quantum numbers have been obtained in [28] completing the exciting results of [116, 117, 118] based on an improved WKB approximation. The coherent states technology introduced in the context of quantum gravity in [83] has been very useful in deriving new asymptotics for $15j$-symbols and other objects from spin foams [20, 21].

Such developments in quantum angular momentum theory have not appeared only in the quantum gravity community [5, 7], and is now considered as a key topic in modern physics [6], beyond the traditional use in spectroscopy and atomic/molecular physics, for instance in quantum computing [38], in the presence of topological order [80]. This is not so surprising since the reference model for topological order and fault-tolerant computing is the Kitaev model [75], which is actually a lattice BF model with the group $\mathbb{Z}_2$. We think those progress on quantum angular momenta in the study of quantum flat space is one of the major consequences of quantum gravity outside of the field and would deserve their own review.

Such models lying at the border between toy models for condensed matter and for quantum gravity have been recently proposed in order to investigate the second key issue, the continuum limit in background independent quantization schemes. Since the project has been started very recently, we will not comment further, but refer to [16, 45] instead.

A different road towards a generic framework for taking the continuum limit is under active development: tensor models and group field theories, which generalize matrix models to dimensions higher than two. A paragraph on those models is included but we refer to the reviews [57, 92] and more recently [69] for more details.

Since we review quite advanced works, on the dynamics itself, we include for completeness some basic material of the loop quantization and its kinematical aspects in Section 2. Then, we give an introduction to Thiemann's construction in Section 3. In the interest of pedagogy, we restrict to the Euclidean Hamiltonian constraint. This is because the full Lorentzian constraint

is the sum of this Euclidean term with an additional piece, so that it is in any cases necessary to understand the Euclidean part. A small paragraph on the master constraint program which was launched by Thiemann after criticisms of its construction is included. Throughout this section, we deliberately focus on the unsatisfactory aspects of the construction, for two reasons: *i*) its mathematical consistency is already emphasized in previous reviews, *ii*) to stress the issues which we address later using toy models.

While our presentation of the standard quantum Hamiltonian may sound a bit pessimistic, this is to contrast with the present situation which seems better to us. In the following sections, we move on to present some exciting results which have been obtained in the very last years. In Section 4, we consider a simple two-dimensional generally covariant field theory to extract some interesting lessons. In Section 5, the diffeomorphism constraint itself is shown to admit a quantization rather satisfactorily if one is ready to examine all possible choices we are afflicted with in LQG. In Section 6, we present a different approach based on the topological BF theory. It contains some review material on the BF model, together with the most recent quantization of the scalar Hamiltonian. A discussion on how to get beyond the topological model is also included. Some conclusions are presented in Section 7.

## 2 Background material

This section presents some background material on loop quantum gravity. More details can be found in the reviews [8, 93, 105].

**Yang–Mills phase space.** The (unreduced, i.e. before imposing the constraints which generate the dynamics) phase space of 3+1 general relativity in Ashtekar–Barbero variables is quite similar to the Yang–Mills phase space. On the canonical surface $\Sigma$, we have a SU(2) connection 1-form $A_a^i$, where $i = 1, 2, 3$ is a $\mathfrak{su}(2)$ index and $a$ is the form index, and the canonical momentum $E_i^a$, which satisfy the Poisson brackets

$$\big\{A_a^i(x), E_j^b(y)\big\} = \gamma\ \delta_a^b\, \delta_j^i\, \delta^{(3)}(x-y).$$

The momentum is the non-Abelian version of the electric field. In 3+1 general relativity, it is the triad of the intrinsic geometry on $\Sigma$, related to the 3-metric $q_{ab}$ by $qq^{ab} = \delta^{ij} E_i^a E_j^b$. Here $\gamma$ is a free parameter of the classical theory known as the Immirzi parameter. The Yang–Mills phase space is the same with $\gamma = 1$. In three-dimensional Riemannian gravity, the phase space is also the same, with $\gamma = 1$ (but the form index just takes two values when $\Sigma$ is two-dimensional).

**Holonomies.** From a connection, locally seen as 1-form on $\Sigma$ with special rules for gauge transformations, one can build holonomies, whose geometric meaning is to enable parallel transport of vectors. The holonomy along a path $\ell : [0,1] \to \Sigma$ is $U_\ell(A) = \mathcal{P} \exp -\int_\ell A$, where $\mathcal{P}$ denotes the path ordering. The loop quantization also features *generalized connections*, which are objects which assign holonomies to any paths.

**Kinematical states of the loop quantization.** The loop quantization provides a natural kinematical Hilbert space to quantize the Yang–Mills phase space. The idea is to probe the connection only through a finite number of variables, similarly to a Wilson line which probes the connection along a given path. We introduce the holonomies $U_e(A)$ of $A$ along the path $e$. Given a graph $\Gamma$, with $E$ edges and $V$ vertices, and a function $f$ over $(\mathrm{SU}(2))^E$, we form the *cylindrical function* $\psi_{\Gamma,f} \in \mathrm{Cyl}_\Gamma$ supported on $\Gamma$ as

$$\psi_{\Gamma,f}(A) = f\big(U_{e_1}(A), \dots, U_{e_E}(A)\big).$$

We are moreover interested in SU(2) gauge invariant states. Gauge transformations act on holonomies only on their endpoints. If $h$ is a map from $\Sigma$ to SU(2), then the holonomy transforms as $U_e(A^h) = h(t(e)) U_e(A) h(s(e))^{-1}$, with $t(e)$, $s(e)$ being respectively the source and target

points of the path $e$. When focusing on a single graph $\Gamma$, this reduces gauge transformations to an action of $\mathrm{SU}(2)^V$ on the set of cylindrical functions over $\Gamma$. So from any function $f$ over $\mathrm{SU}(2)^E$, one gets an invariant function by averaging over the $\mathrm{SU}(2)^V$ action.

The algebra of such functions has a natural $\mathrm{SU}(2)^V$-invariant inner product which comes from the Haar measure on $(\mathrm{SU}(2))^E$, $d\mu_\Gamma = \prod_e dg_e$, giving

$$\langle \phi_{\Gamma,h} | \psi_{\Gamma,f} \rangle = \int \prod_{e=1}^{E} dg_e \, \bar{h}(g_1, \dots, g_E) f(g_1, \dots, g_E).$$

By completing this space for the corresponding norm, one obtains the Hilbert space $\mathcal{H}_\Gamma = L^2\big(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V, d\mu_\Gamma\big)$.

The union of all functions which are cylindrical with respect to some graph is denoted $\mathrm{Cyl} = \cup_\Gamma \mathrm{Cyl}_\Gamma$. It comes equipped with a natural Hermitian inner product inherited from the measures $d\mu_\Gamma$. The completion of Cyl then leads to the kinematical Hilbert space of the loop quantization $\mathcal{H}_{\mathrm{kin}}$. The latter can be seen as the set of square-integrable functions over the set $\bar{\mathcal{A}}$ of generalized connections, $\mathcal{H}_{\mathrm{kin}} = L^2(\bar{\mathcal{A}}, d\mu_{\mathrm{AL}})$, where $\mu_{\mathrm{AL}}$ is known as the Ashtekar–Lewandowski measure. $\mathcal{H}_{\mathrm{kin}}$ can also be constructed from the set of Hilbert spaces $\mathcal{H}_\Gamma$ for all possible graphs $\Gamma$, using projective techniques. Importantly, the measure $\mu_{\mathrm{AL}}$ is *invariant* under diffeomorphisms of $\Sigma$.

**The spin network basis.** It is an orthogonal basis of the above $L^2\big(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V, d\mu_\Gamma\big)$. The first idea is to expand any element of $L^2(\mathrm{SU}(2)^E, d\mu_\Gamma)$ onto the matrix elements of the irreducible representations of $\mathrm{SU}(2)^E$, labeled by $E$ spins $j_e \in \mathbb{N}/2$. Then, rotation invariance at vertices imposes to contract the matrix elements of edges meeting at a vertex with intertwiners $\iota_v$. If $v$ is a node of $\Gamma$ with ingoing links $(e_{\mathrm{in}})$, and outgoing links $(e_{\mathrm{out}})$, then an intertwiner $\iota_v$ is a map: $\otimes_{e_{\mathrm{in}}} \mathcal{H}_{j_e} \to \otimes_{e_{\mathrm{out}}} \mathcal{H}_{j_e}$, (where $\mathcal{H}_j$ is the carrier space of the spin $j$ representation) which commutes with the group action. Thus, to form the spin network basis on $\mathcal{H}_\Gamma$, we need to assign spins to edges and a basis of intertwiners at each node. For a trivalent vertex with ingoing edges for example, there is a single intertwiner $\iota_{j_1 j_2 j_3} : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \mapsto \mathbb{C}$ up to normalization, its components in the standard magnetic number basis being the Wigner $3jm$-symbol $\langle j_1, m_1; j_2, m_2; j_3, m_3 | \iota_{j_1 j_2 j_3} | 0 \rangle = \left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$. Hence, spin network states are the following $\mathrm{SU}(2)^V$-invariant functions

$$s_\Gamma^{\{j_e, \iota_v\}}(g_1, \dots, g_E) = \sum_{\{m_e, n_e\}} \prod_{e=1}^{E} \langle j_e, m_e | g_e | j_e, n_e \rangle \prod_{v=1}^{V} \langle \otimes_{e \,\mathrm{in}} j_e, m_e | \iota_v | \otimes_{e \,\mathrm{out}} j_e, n_e \rangle,$$

which form an orthogonal set. This enables a transform between gauge invariant functions on $\Gamma$ and functions over the colorings $(j_e, \iota_v)$.

**Phase space on a graph.** There is a natural classical phase space on $\Gamma$, inherited from the full phase space, whose quantization leads to $L^2\big(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V, d\mu_\Gamma\big)$. Each link $e$ carries a group element $g_e \in \mathrm{SU}(2)$, which can be seen as the holonomy of the connection. In this picture, each node carries a local frame, and $g_e$ performs parallel transport from the frame of the source node of $e$ to its target node. The momenta $E^a_i$ give rise to fluxes when integrated along surfaces of codimension 1 on $\Sigma$. If it is done infinitesimally close to a node, source of a link $e$, we get the flux variable $E^i_e \in \mathfrak{su}(2)$, defined with respect to the frame over the source node. The Poisson bracket is the structure of the cotangent bundle over $\mathrm{SU}(2)^E$,

$$\big\{E^i_e, E^j_e\big\} = \epsilon^{ij}{}_k E^k_e, \qquad \big\{E^i_e, g_e\big\} = g_e \tau^i,$$

where the matrices $(\tau_i)_{i=1,2,3}$ are anti-hermitian matrices[1] satisfying $[\tau^i, \tau^j] = \epsilon^{ij}{}_k \tau^k$. All other brackets vanish. One can obviously define a flux variable in the frame of the target node of $e$, using the adjoint action of the holonomy on the initial flux, $\widetilde{E}_e = \mathrm{Ad}(g_e)\, E_e$.

[1] They read: $\tau^i = -\frac{i}{2}\sigma^i$, in terms of the Pauli matrices $(\sigma^i)$.

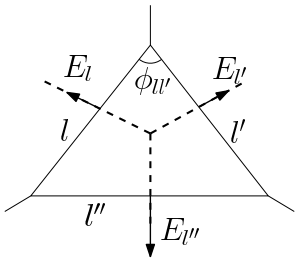


**Figure 1.** The dual graph, in dashed lines, which supports the phase space, and the triangulation. The fluxes are the normals to the edges of the triangle, they encode the local embedding into flat 3-space. Their dot products are proportional to the cosines of the angles $\phi_{ee'}$.

**Geometric content.** For simplicity, we consider $\Gamma$ to be dual to a triangulation of $\Sigma$. Fluxes, which come from the triad, encode the *intrinsic geometry of the triangulation*. The main reason is that the Gauß law, which generates SU(2) transformations at a node $v$, reads

$$\sum_{e \text{ outgoing}} E_e^i - \sum_{e \text{ ingoing}} \widetilde{E}_e^i = 0. \tag{2.1}$$

Thinking of the node as dual to a tetrahedron, and the links dual its triangles, each of them carries a flux and this allows to interpret the above relation as the closure of the tetrahedron, where fluxes are the (oriented) normals to the triangles (in the frame of the tetrahedron). Areas are given by their norms, and dihedral angles by their dot products. This way, the geometry of each tetrahedron is completely specified. However, different tetrahedra will generically assign different lengths to their common edges. Such geometries are described in [47, 48, 61, 100] and are known as *twisted* geometries.

A similar interpretation holds in 2+1 gravity. Since fluxes are still three-dimensional vectors, one should rather think of nodes as dual to triangles locally embedded in flat 3-space. The Gauß law on a node is the closure of the dual triangle, so that fluxes based on a node represent the normals to the dual edges in the frame of the dual triangle, as shown in Fig. 1. Norms and dot products of fluxes give access to the lengths and $2d$ angles within the triangles. The closure relation in particular asserts that

$$\cos\phi_{e_1e_2} = \frac{E_{e_1}^2 + E_{e_2}^2 - E_{e_3}^2}{2|E_{e_1}|\,|E_{e_2}|}, \tag{2.2}$$

which is the standard formula to compute the angles of flat triangles from their lengths, namely the law of cosines.

In contrast with the 3+1 case, the geometries encoded into fluxes in 2+1 dimensions assign lengths unambiguously to the whole triangulation and are thus Regge geometries.

It is also possible to introduce dihedral angles between the top-simplices of the triangulation, which give a notion of *extrinsic geometry*, using fluxes and holonomies. The Hamiltonian of general relativity is precisely a relation between extrinsic and intrinsic geometries, and will be described later in the flat case.

## 3 Quantum Hamiltonian constraint: Thiemann's definition

In this section we review the quantization of Hamiltonian constraint, as proposed by Thiemann in [106, 107]. Instead of focusing on the details, we focus on the basic underlying ideas and what in our opinion are its major shortcomings.

In a remarkable series of papers, Thiemann proposed to quantize the Lorentzian Hamiltonian constraint of canonical gravity by using in a crucial way certain relations involving Poisson brackets at the classical level, together with suitable choices of intermediate regularization schemes. The final Hamiltonian constraint turned out to be well defined on $\mathcal{H}_{\rm kin}$. It showed glimpses of generating an anomaly free quantum Dirac algebra, as the commutator of two Hamiltonian constraints, when acting on any spin network state $T_s$ in $\mathcal{H}_{\rm kin}$, gave a null state when averaged over spatial diffeomorphisms.

However as we will see below, the definition given by Thiemann is unsatisfactory for several reasons. This is not a criticism of the construction itself as it was the first ever rigorous definition of the Hamiltonian constraint, but as it has been over fifteen years since Thiemann's seminal work, it is perhaps good to take stock of the current state of the art in the field.

As we argue below, the most worrisome aspect of Hamiltonian constraint is that there are infinitely many ambiguities which feed into its definition. Each such choice could give rise to (in the continuum limit) an inequivalent quantization of the Hamiltonian constraint. The preferred choices currently used in the literature are based on subjective criteria rather then appealing to certain physical requirements like local Lorentz invariance.

The classical, density one, Euclidean Hamiltonian constraint is given by

$$H_E[N] = \int_\Sigma d^3x\, N(x) \epsilon_i^{\ jk} \frac{F^i_{ab} E^a_j E^b_k}{\sqrt{q}}(x),$$

(we have set $G = 1$). As we will see shortly, it is only for density weight one, that the continuum constraint operator can be well defined on $\mathcal{H}_{\rm kin}$. However it is important to note that, it is a priori not required that the domain for the quantum Hamiltonian should be in $\mathcal{H}_{\rm kin}$, and could very well be a subspace of Cyl$^*$ in which case other density weighted constraints could also admit a continuum limit.

One of the ingenious features of Thiemann's construction is the following. The density one constraint, although non-polynomial in the basic fields could be written as

$$H_E[N] = \int_\Sigma N(x)\, {\rm Tr}(F(x) \wedge \{A(x), V(R)\}),$$

where $R$ is any open region containing the point $x$. $V(R)$ is the volume of this region, as a function of the field $E$.

As the elementary variables in the theory are holonomies along (piecewise analytic) edges and fluxes along (semi-analytic) surfaces, we need to first approximate $H_E[N]$ by a sequence of functionals of holonomies and fluxes such that the limit of the sequence (in the suitable sense) is $H_E[N]$. This is accomplished as follows. Choose a cell decomposition of the underlying manifold $\Sigma$ (it can be simplicial or cubical for convenience). For each 3-cell $\triangle$ in $T$, choose a vertex $v(\triangle)$, a family of edges $\mathcal{S}_e = \{s_1(\triangle), \ldots, s_{M_s}(\triangle)\}$ and a family of loops $\mathcal{S}_L = \{\alpha_1(\triangle), \ldots, \alpha_{M_L}(\triangle)\}$ and consider

$$H_{T(\epsilon)E}[N] = \sum_{\triangle \in T(\epsilon)} \sum_{i=1}^{M_s} \sum_{J=1}^{M_L} \mathcal{C}^{iJ}\ {\rm Tr}(h_{\alpha_J(\triangle)} h_{s_i(\triangle)} \{h^{-1}_{s_i(\triangle)}, V(R_{v(\triangle)})\},$$

where $C^{iJ}$ are constants independent of $\epsilon$. Each choice of $(T(\epsilon), \{C^{iJ}\}, \mathcal{S}_e, \mathcal{S}_L\ )$ such that,

$$\lim_{\epsilon \to 0} H_{T(\epsilon)E}[N] = H_E[N]$$

is an acceptable net of regulators. Classically one choice is as good as the next (in the limit of infinite refinement) but quantum mechanically it is not clear which of the choices are indistinguishable in the continuum limit.

As $H_{T(\epsilon)}[N]$ is a function of holonomies and volume, it can be quantized on $\mathcal{H}_{\rm kin}$. Given a spin network $\mathbf{s} = (\gamma, \vec{j}, \vec{c})$ let $T(\epsilon, \gamma)$ be a one parameter family of triangulations adapted to the graph $\gamma$. By adapted all we mean is that each vertex $v'$ in $V(\gamma)$ should be either in the interior of precisely one cell $\triangle \in T(\epsilon, \gamma)\ \forall\, \epsilon$, or $v'$ be a vertex of $T(\epsilon, \gamma)$ such that for all $\epsilon > 0$, the number of 3-cells saturating $v'$ remain constant. One can now prescribe the action of $\hat{H}_{T(\epsilon,\gamma)}[N]$ on $|\mathbf{s}\rangle$ as,

$$\hat{H}_{T(\epsilon,\gamma)}[N]|\mathbf{s}\rangle = \sum_{\triangle\in T(\epsilon,\gamma)} \sum_{i=1}^{M_s} \sum_{J=1}^{M_L} \mathcal{C}^{iJ}\ \mathrm{Tr}\left(\hat{h}_{\alpha_J(\triangle)} \hat{h}_{s_i(\triangle)} \left[\hat{h}^{-1}_{s_i(\triangle)}, \hat{V}(R_{v(\triangle)})\right]\right)|\mathbf{s}\rangle. \tag{3.1}$$

Let $\mathcal{H}_\gamma$ be the subspace of $\mathcal{H}_{\rm kin}$ spanned by spin networks based on graph $\gamma$. There are two definitions of volume operators available in the literature. One is known as AL (Ashtekar–Lewandowski) volume operator and the other is known as RS (Rovelli–Smolin) volume operator. We will restrict ourselves to the AL volume operator in this review[2]. The equation (3.1) describes the action of $\hat{H}_{T(\epsilon,\gamma)}[N]$ on $\mathcal{H}_\gamma$ for any $\gamma$. Whence we have a densely defined operator $\hat{H}_{T(\epsilon,\gamma)}[N]$ on the (dense domain of) entire Hilbert space $\mathcal{H}_{\rm kin}$.

So far the parameter space regularizing the Hamiltonian constraint is uncountably infinite dimensional. This is by any account quite huge. This itself is not an issue because it might happen that different regularization schemes lead to same continuum operator (or at least the the same physical spectrum of the theory).

One would also expect that putting physical requirements like diffeomorphism covariance will restrict the choice of triangulation. It turns out that the requirement of diffeomorphism covariance places the following restrictions on the allowed choices of triangulations. Given a pair of quintuples $(\gamma, v, \triangle \in T_\epsilon, \mathcal{S}_e, \mathcal{S}_L)$ and $(\gamma', v', \triangle' \in T(\epsilon'), \mathcal{S}'_e, \mathcal{S}'_L)$ such that $(\gamma, v)$ is diffeomorphic to $(\gamma', v')$, we require the entire quintuples to be related by some diffeomorphism for any choice of $\epsilon$, $\epsilon'$.

Let us introduce $\hat{U}(\phi)$ as the unitary operator corresponding to a diffeomorphism $\phi$ which acts on spin network states like $\hat{U}(\phi)T_{\mathbf{s}} = T_{\phi^{-1}\mathbf{s}}$. It can be easily shown that the above requirements imply that $\forall\, \phi \in \mathrm{Diff}(\Sigma)$

$$\hat{U}(\phi)\hat{H}_{T(\gamma,\epsilon)}[N]\hat{U}(\phi^{-1}) = \hat{U}(\phi')\hat{H}_{T(\phi^{-1}(\gamma),\epsilon)}[\phi^* N], \tag{3.2}$$

where $\phi'$ is a diffeomorphism not necessarily equal to $\phi$. As we will see in the next section, this would imply that the continuum Hamiltonian constraint in diffeomorphism covariant.

However the covariance requirement, though non-trivial, does not reduce the parameter space of allowed regulators significantly[3].

*We emphasize that this is the only physical requirement one places on regularization.*

### 3.1 Continuum limit

There are various operator topologies in which one can examine the convergence (or lack thereof) of $\hat{H}_{T(\epsilon,\gamma)}[N]$. The finest topology used in the literature so far is the so called URST topology[4], in which one defines the continuum limit on $\mathcal{H}_{\rm kin}$ itself. This is quite an important construction so let us review it a bit more.

[2] If one insists on defining continuum limit of $\hat{H}_{T(\delta)}[N]$ on $\mathcal{H}_{\rm kin}$ then it is essential to use AL volume operator. However if one instead defines continuum limit on some subspace of Cyl$^*$ then as shown in [81], RS volume operator is a perfectly viable choice.

[3] Although it should be emphasized that to the best of our knowledge the only explicit construction of a covariant family of regulators available in the literature is given in [106], but that of course does not mean that it is the only choice, in fact as explained in [8], there are infinitely many.

[4] URST is an abbreviation for Uniform by Rovelli, Smolin and Thiemann.

One corollary of requiring the one parameter family of regulators $\mathbf{R}_{\epsilon,\gamma}$ to be covariant is that for any two $\epsilon, \epsilon' > 0$ and for any vertex $v \in V(\gamma)$ the corresponding family of loops $\mathcal{S}_{L,v}(\epsilon)$, $\mathcal{S}_{L,v}(\epsilon')$ are diffeomorphism covariant. This implies that

$$(\Psi|\hat{H}_{T(\epsilon)}[N]|\mathbf{s}\rangle = (\Psi|\hat{H}_{T(\epsilon')}[N]|\mathbf{s}\rangle$$

$\forall\, \Psi \in \mathcal{H}_{\rm diff}$. One can now define the continuum limit of the Hamiltonian constraint (on $\mathcal{H}_{\rm kin}$) the following way. $\hat{H}[N]$ is a continuum limit of $\hat{H}_{T(\epsilon)}[N]$ if, given any $\delta > 0$, there exists $\epsilon_0(\delta) > 0$ such that forall $\epsilon < \epsilon_0$

$$|(\Psi|\hat{H}[N]|\mathbf{s}\rangle - (\Psi|\hat{H}_{T(\epsilon,\gamma(\mathbf{s}))}[N]|\mathbf{s}\rangle| < \delta. \tag{3.3}$$

This is trivially true because any $\hat{H}_{(\epsilon_1)}[N]|\mathbf{s}\rangle$ and $\hat{H}_{(\epsilon_2)}[N]|\mathbf{s}\rangle$ are related by a diffeomorphism, so that $(\Psi|\hat{H}_{T(\epsilon,\gamma(\mathbf{s}))}[N]|\mathbf{s}\rangle$ is actually independent of $\epsilon$.

Whence in the above mentioned topology, $\hat{H}_{T(\epsilon_0)}[N]$ is already a continuum operator for any $\epsilon_0$. Two properties of this construction are worth noting.

1. Since $\epsilon_0$ is independent of ($|\mathbf{s}\rangle$, $\Psi$ and $N$), the convergence is uniform.
2. If two operators when acting on any state in $\mathcal{H}_{\rm kin}$ give diffeomorphically related states, they are *identical*. This in conjunction with the equation (3.2) implies that the continuum operator is diffeomorphism covariant!

If we do not care about demanding uniform convergence (and it is not clear to us if there is any physical reason to care), then we can define the continuum limit on infinitely many subspaces of Cyl$^*$. That is, if we define continuum limit via,

$$[\hat{H}[N]'\Psi]|\mathbf{s}\rangle = \lim_{\epsilon\to 0} \Psi\big[\hat{H}_{T(\epsilon)}[N]|\mathbf{s}\rangle\big],$$

where the convergence is only point wise (the notion of continuity used is same as the one in (3.3) except that $\epsilon_0 = \epsilon_0(\delta, \Psi, \mathbf{s})$) then on a large class of states in Cyl$^*$ $\hat{H}[N]'$ could be well defined. Let us illustrate this point using a following schematic example. Let

$$\Psi = \sum_{s'\in\mathcal{A}} f(s')\langle s'|,$$

where $\mathcal{A}$ is some set of spin networks ($\mathcal{A}$ can be uncountable) and $f$ are functions of the spin networks defined on a suitable domain. (e.g. $f$ could be functions of vertices of $\mathbf{s}'$, or they could be functions of vertices and germs of edges incident on those vertices, or they could be functions of contractible loops contained in $\mathbf{s}'$ etc.). We refer to $f$ as vertex functions[5]. Schematically

$$\hat{H}_{T(\epsilon)}[N]|\mathbf{s}\rangle = \sum_{v\in V(\gamma(\mathbf{s}))} \sum_{I=1}^{m} N(v) a_{v,I} |\mathbf{s}^{\epsilon}_{vI}\rangle,$$

where due to diffeomorphism covariance of regulators, the coefficients $a_{v,I}$ are independent of $\epsilon$ and $|\mathbf{s}^{\epsilon}_{vI}\rangle = \hat{U}(\phi_\epsilon)|\mathbf{s}^0_{vI}\rangle$ for some reference spin network $|\mathbf{s}^0_{vI}\rangle$.

Now let the set A be such that if it contains some state then it necessarily contains all its diffeomorphic images. Furthermore let us for the sake of illustration assume that there exists precisely one vertex $v_0$ for which $|\mathbf{s}^{\epsilon}_{v_0 I}\rangle \in \mathcal{A}$. Whence

$$(\Psi|\hat{H}_{T(\epsilon)}[N]|\mathbf{s}\rangle = (\Psi| \sum_{v\in V(\gamma(\mathbf{s}))} \sum_{I=1}^{m} N(v) a_{v,I} |\mathbf{s}^{\epsilon}_{vI}\rangle$$

[5] See [64, 81] for more details.

$$= \sum_{v\in V(\gamma(\mathbf{s}))} \sum_{I=1}^{m} N(v) a_{v,I} \sum_{\mathbf{s}'\in\mathcal{A}} f(\mathbf{s}')\langle \mathbf{s}'|\mathbf{s}^{\epsilon}_{vI}\rangle = N(v_0)\sum_{I=1}^{m} a_{v_0,I} f(\mathbf{s}^{\epsilon}_{v_0,I}).$$

Thus

$$\hat{H}[N]'\Psi|\mathbf{s}\rangle = N(v_0)\sum_{I=1}^{m} a_{v_0,I} \lim_{\epsilon\to 0} f(\mathbf{s}^{\epsilon}_{v_0,I}).$$

Such subspaces of Cyl$^*$ on which continuum limit of the Hamiltonian constraint is well defined are known as the habitats. Our schematic computation illustrates that potentially there are infinitely many such habitats. One explicit construction of a habitat is given in the seminal paper [81]. Essentially it is characterized by the property that the function $f$ above is a (smooth) functions of vertices of the $\mathbf{s}$. This habitat turns out to be a sufficiently small extension of $\mathcal{H}_{\rm diff}$. We will have many occasions to use this habitat in this review and henceforth refer to it as Lewandowski–Marolf (LM) habitat.

This finishes the summary of the construction of candidate Hamiltonian constraints in LQG. Let us, for the benefit of the reader summarize some of the worrying aspects of the construction.

1. The parameter space underlying allowed choices of regulators is infinite dimensional. This would not be a problem if the continuum limit of the Hamiltonian constraint is independent of such choices, however this is far from being the case, e.g. in the URST topology, each pair of choices which is not diffeomorphically related to each other will give rise to a different operator.

2. It is a priori not clear what the domain of the Hamiltonian should be. It could either be a dense subspace of kinematical Hilbert space, or a habitat.

3. There are further ambiguities. In all the constructions available in the literature so far, Lapse is a multiplicative factor. However one could conceive type of regularizations where the lapse function is included in the definition of the plaquette one uses to define curvature [79].

4. There is a huge amount of choice available in defining curvature operators. One such choice in particular has received some attention in the literature. This has to do with the representation underlying the curvature approximant in the Hamiltonian constraint [93, 94]. The simplest choice for such a curvature approximant is

$$F^i_{ab}(v) \approx \frac{1}{\mathrm{Ar}(\alpha_{ab})}\big(\mathrm{Tr}(h_{\alpha_{ab}}\tau^i)\big),$$

where the trace is in the fundamental ($j=\frac{1}{2}$) representation. One could instead choose approximant to be

$$F^i_{ab}(v) \approx \frac{1}{\mathrm{Ar}(\alpha_{ab})d_j}\big(\mathrm{Tr}_j(h_{\alpha_{ab}}\tau^i)\big),$$

where $\mathrm{Tr}_j$ is the trace in some representation of spin $j$, and $d_j$ is the appropriate normalization factor. In fact as in LQG one often employs "state dependent" regularization schemes, this ambiguity could be generalized further where e.g.

$$\hat{F}^i_{ab}(v)\ \pi_{j_e}[h_e](A) = \frac{1}{\mathrm{Ar}(\alpha_{ab})d_{j_e}}\big(\mathrm{Tr}_{j_e}(h_{\alpha_{ab}}\tau^i)\big)\pi_{j_e}[h_e](A).$$

Here $\pi_{j_e}[h_e](A)$ is the gauge-variant spin network state based on a single edge $e$, and $v\in e$. whence the choice of representations used to define curvature approximants could themselves be state dependent (and in this sense dynamical).

5. Yet another ambiguity which seemed to have gone un-noticed in the literature until the arrival of improved dynamics in LQC is the following. Notice that the flux operator when acting on a spin network only depends on the intersections of the edges of the spin networks with the

underlying surface. That is, even if we scale the coordinate area of the surface arbitrarily, the action of flux operator on a spin network does not change as long as the intersection numbers remains the same. This means the following. Consider latticized classical curvature is given by

$$F^i_{ab}(v) \approx \frac{1}{\mathrm{Ar}\triangle} \mathrm{Tr}\left(h_{\triangle_{ab}}\tau^i\right).$$

However one could equally well consider a different approximant of the type

$$F^i_{ab}(v) \approx \frac{1}{\mathrm{Ar}\triangle}\left[\mathrm{Tr}\left(h_{\triangle_{ab}}\tau^i\right) + \mathrm{Tr}(h_{\triangle_{ab}})E^i(S_\epsilon)\right],$$

where $\triangle_{ab}$ is a plaquette based at $v$ of co-ordinate area $\epsilon^2$, and $S_\epsilon$ is a surface also of co-ordinate area $\epsilon^2$. Classically the second term is higher order in $\epsilon$ than the leading order term, however quantum mechanically the difference between states obtained by action of two approximants on a given spin network state will be a finite norm state. Whence this is a genuine ambiguity in the choice of curvature which can drastically affect the definition of the Hamiltonian constraint. In essence, it is this choice which was exploited in [9] to define a hamiltonian constraint in LQC which gave rise to better semi-classical properties of LQC.

Thus the situation may look rather worrying. We have infinitely many possible definitions of continuum Hamiltonian constraint and it is a priori not clear which if any of these choices are physically preferred (better semi-classical properties).

It should be mentioned that as explained in [105], it is conceivable that as the final spectrum of the theory consists of diffeomorphism invariant states, at least some of the choices which give rise to the kernel of the Hamiltonian constraint that are diffeomorphically related may not matter. Although this argument is intuitive and to the best of our knowledge not analyzed in detail, it is an important counter-point to the grim view that we have presented above.

Finally a worrisome feature of $\hat{H}[N]$ not necessarily related to the infinitude of ambiguities is its *ultra-local character* (see [93, 104] for details). Consider a spin network state $|\mathbf{s}\rangle$ and let $N$ have support in the neighborhood of a vertex $v_0 \in V(\gamma(\mathbf{s}))$. Then $\hat{H}[N]$ (assuming it is defined on $\mathcal{H}_{\mathrm{kin}}$) will give rise to a linear combination of states each of which is "different" from $\mathbf{s}$ only in the neighborhood of $v_0$ (which means only the edges, which are incident on $v_0$ will undergo any change under $\hat{H}[N]$ action.) This statement would remain true even if we worked with arbitrarily fine graphs and hence even when we consider say (kinematical) semi-classical states based on a single graph. Naively we expect the Quantum constraint equation in this case to "approximate" classical constraint equation which are elliptic and hence not ultra-local. Thus the semi-classical limit of such Hamiltonian constraint might turn out to be incorrect. One way around this issue suggested in [104] is to tailor the choice of loops underlying the curvature approximants in such a way that it intersects many vertices in a given graph. Such a construction has been done independently in the context of BF theory for 2+1 gravity, detailed in Section 6. Finally the criticism regarding ultra-locality has itself been criticized in [105] on the grounds that non-locality might emerge when one considers physical states which live in $\mathrm{Cyl}^*$.

### 3.2 Action of $\hat{H}[N]$ on $\mathcal{H}_{\mathrm{kin}}$

In the URST topology we have a continuum operator on $\mathcal{H}_{\mathrm{kin}}$,

$$\hat{H}[N]|\mathbf{s}\rangle = \hat{H}_{T(\epsilon_0)}[N]|\mathbf{s}\rangle = \sum_{I=1}^{N(\mathbf{s})} a_I |\mathbf{s_I}\rangle,$$

where $\epsilon_0$ is fixed once and for all[6]. Each $\mathbf{s}_I$ has two more vertices and one additional edge than $\mathbf{s}$. The two new vertices that are created are planar, and are in the kernel of AL Volume operator.

[6] There is a minor subtlety here. As the $\hat{H}_{T(\epsilon_0)}[N]$ when acting on $|\mathbf{s}\rangle$ must be finer then the graph $\gamma(\mathbf{s})$. $\epsilon_0$ implicitly depends on $\gamma(\mathbf{s})$.

This feature plays a key role in the "anomaly free" nature of the Hamiltonian constraint, *when it is defined on* $\mathcal{H}_{\rm kin}$.

### 3.3 The commutator and its unexpected triviality

One of the interesting properties of the Quantum Hamiltonian constraint defined by Thiemann was its "on-shell closure". That is the commutator of two continuum Hamiltonian constraints vanished. Although consistent, a closer look at the computation shows that the underlying mechanism for this commutativity is not too restrictive (as far as the regularization choices are concerned) and does not rely on any deep or non-trivial structural aspects of the Hamiltonian constraint.

This is because the commutativity only relies on two features of the regularization.

1. The density one nature of the constraints ensure that the corresponding operators are "ultra-local" (whether we define them on $\mathcal{H}_{\rm kin}$ or on Cyl$^*$) and can not give rise to one forms $N\nabla_a M - M\nabla_a N$ in the commutators.
2. The diffeomorphism covariance condition placed on the regularization scheme (which is the only physical requirement fed into the quantization so far) guarantees that the commutators at finite triangulation is anti-symmetric in the lapses. This can be seen as follows. As before, we will merely sketch the proof. Details can be found in [81, 105].

Let us schematically write the Hamiltonian constraint operator at finite triangulation as

$$\hat{H}_{T(\epsilon)}[N]|\mathbf{s}\rangle = \sum_{v\in V(\gamma)} N(v)\hat{O}_\epsilon(v)|\mathbf{s}\rangle.$$

Here $\hat{O}_\epsilon(v)$ is a non-local operator which depends on a set of loops (based at $v$), set of edges and the Volume operator.

The essence of diffeomorphism covariance of the regularization scheme is the statement that for any graph $\gamma$ and $\epsilon$, $\epsilon'$, $\exists$ a diffeomorphism $\phi_{\epsilon,\epsilon'}(\gamma)$ which preserves $\gamma$ and

$$\hat{O}_\epsilon(v)|\mathbf{s}\rangle = \hat{U}(\phi_{\epsilon,\epsilon',\gamma})\hat{O}_{\epsilon'}(v)|\mathbf{s}\rangle \tag{3.4}$$

$\forall\, v \in V(\gamma)$. Consider two lapse functions $N$, $M$ and as the Hamiltonian constraint operator is linear in lapse we can (without loss of generality) assume that given a graph $\gamma$ $N$, $M$ have support only in the neighborhood of vertices $v_N$, $v_M$ respectively. Clearly the commutator between $\hat{H}_{T(\epsilon)}[N]$, $\hat{H}_{T(\epsilon')}[M]$ is zero for all $\epsilon$, $\epsilon'$ if $v_N = v_M$,[7] whence the continuum limit of such a commutator (in URST topology on $\mathcal{H}_{\rm kin}$ or some weaker topology on a suitable habitat) will be zero.

If $v_N \neq v_M$ then it can be easily shown that (essentially using the covariance of regulator as in (3.4)),

$$[\hat{H}_{T(\epsilon)}[N]\hat{H}_{T(\epsilon')}[M] - N \leftrightarrow M]\,|\mathbf{s}\rangle = N(v_N)M(v_M)\big(\hat{U}(\phi(\epsilon,\epsilon')) - 1\big)\hat{O}_\epsilon(v_N)\hat{O}_{\epsilon'}(v_M)|\mathbf{s}\rangle.$$

Clearly in the URST, this operator vanishes as

$$(\Psi|(\hat{U}(\phi) - 1)|\mathbf{s}\rangle = 0$$

$\forall\, (\Psi| \in \mathcal{H}_{\rm diff}$, $\phi \in {\rm Diff}(\Sigma)$, $|\mathbf{s}\rangle \in \mathcal{H}_{\rm kin}$. The argument can be extended for the continuum Hamiltonian constraint defined on Lewandowski–Marolf habitat (see [81] for details). Although

[7] This statement is true if we use AL volume operator. However as shown in [81], on the LM habitat, even if we use RS volume operator, even though this statement will no longer hold, in the continuum limit $[H[N], H[M]]$ will vanish if $v_N = v_M$.

plausible, we are not aware of any arguments showing that this result will continue to hold when the domain of the continuum Hamiltonian constraint is any suitable subspace of Cyl$^*$.

If one works on the LM habitat, this commutativity is puzzling at first. This is because states in the LM habitat have the following properties.

1. It carried a representation of the algebra of spatial diffeomorphisms.
2. By its very construction it contained the diffeomorphism invariant states.

Whence not all the states in the LM habitat are diffeomorphism invariant, and we do not expect the commutator of two Hamiltonian constraints to vanish on the habitat. In [64] it was shown that this apparent inconsistency could be avoided by a specific quantization of r.h.s. (Essentially what was shown in [64] was that there exists a quantization of $\hat{q}^{ab}$ which gave an identically zero operator on the habitat.) This happens because of the following simple reasons. Consider the r.h.s. of the Dirac algebra (for density one Hamiltonian constraint)

$$\text{r.h.s.} = \int_\Sigma d^3x\, q^{ab}\,(N\nabla_a M - M\nabla_a N)\, H_{(\text{diff})b}(x).$$

It turns out that one can quantize $q^{ab}H_b$ using identities introduced by Thiemann and the final operator at finite triangulation has the following form,

$$\widehat{\text{r.h.s.}_{T(\epsilon)}} = \sum_{\triangle\in T} \epsilon\,(N\nabla_a M - M\nabla_a N)\,, \hat{O}^a_{\triangle(\epsilon)}(v(\triangle)),$$

where $\hat{O}_{\triangle(\epsilon)}(v(\triangle))$ is a composite operator built out of holonomies and fluxes and hence it is well defined on $\mathcal{H}_{\text{kin}}$. Thus if the limit $\epsilon \to 0$ exists on some space (whether on $\mathcal{H}_{\text{kin}}$ or on LM habitat or some other subspace of Cyl$^*$) then it will trivially be zero.

Naively one would think that the chief culprit in the game is density one character of the constraint. That is, if we were to work with higher density constraints then at finite triangulation the operators will be of the type

$$\hat{\tilde{H}}_{T(\epsilon)}[N]|\mathbf{s}\rangle = \frac{1}{\epsilon^a} \sum_{v\in V(\gamma(\mathbf{s}))} N(v)\hat{O}_\epsilon(v)|\mathbf{s}\rangle,$$

where we have placed tilde on $H$ to emphasize on higher density nature. $a$ is a positive integer which depends on the density of $H$.

Due to the presence of $\epsilon$ in the denominator, it is certainly conceivable that commutator of two such Hamiltonians could give rise to factors of the type $\frac{N(v)M(v+\epsilon)-M(v)N(v+\epsilon)}{\epsilon}$. If the limit $\epsilon \to 0$ exists on some habitat, then the continuum limit of the commutator would contain factors of the type $M\nabla_a N - N\nabla_a M$. Whence higher density constraints have a chance to yield a faithful closure. However as argued in [64], this is not enough. Unless the new vertices created by the action of Hamiltonian are such that, they can themselves admit a non-trivial action of $\hat{\tilde{H}}[N]$, the commutator would *never* produce infinitesimal diffeomorphisms.

Thus the necessary conditions to obtain off-shell closure are, higher density weight of the classical constraint, and a choice of regularization which produces "non-trivial" vertices in the sense explained above. So far, no quantum Hamiltonian constraint satisfying these conditions has been found.

One could question the importance of off-shell closure in the formulation of LQG. After all, we have at our disposal the diffeomorphism invariant Hilbert space, and working within this arena, off-shell closure condition is super ceded. However the reasons for requiring off-shell closure are two fold.

1. As we argued above, the allowed choices of regulators which yield well defined continuum Hamiltonian constraint on either $\mathcal{H}_{\rm kin}$ or some habitat are infinite in number. It is important to reduce this ambiguity, not by appealing to subjective criteria, but to certain physical conditions. A key physical requirement would be local Lorentz invariance, which in canonical gravity is encoded in the Dirac algebra. From this point of view, demanding Dirac algebra to be faithfully represented in the quantum theory seems very appealing.
2. Moreover, the fact that commutator continues to vanish even on non diffeomorphism invariant states seems like an extremely unpleasant feature of the operator. The fact that one can choose to quantize r.h.s. in such a way that it also annihilates such states does not seem very comforting as it is tantamount to the inverse metric being zero operator.

It is sometimes argued, [105], that the LM-habitat is too small an extension of $\mathcal{D}^*_{\rm diff}$ to see any non-triviality or anomaly ($\mathcal{D}^*_{\rm diff}$ is a dense subspace of $\mathcal{H}^*_{\rm diff}$ which contains the diffeomorphism invariant distributions). However this is not true, as has been explicitly demonstrated in simpler contexts like Parametrized Field Theories (PFT) [77]. We will summarize results obtained in this simple model below in Section 4.

Whence as far as quantization of the Hamiltonian constraint in LQG is concerned, this is where things stand. One has a slue of regularization ambiguities each of which can potentially give rise to a different continuum operator. Non-trivial checks like the off-shell closure of the constraint algebra do not seem to go through and in simple models like Loop quantum cosmology and PFT, it has been shown that the regularization choices used in the full theory so far leads to either incorrect (in PFT case) or physically unsatisfactory (in LQC case) results.

One of the key lessons that LQC teaches us with regards to LQG is the need to write the curvature operator as a concomitant of holonomy and flux operators. Even though the reason as to why this curvature operator is selected in LQC comes from detailed semi-classical analysis of physical observables, it is interesting to note that the curvature operator used in that case does not "fall into" class of regularizations considered for curvature in the full theory.

We can now pose a following question. Is there a subset of choice of regularization ambiguities (choice of triangulation, choice of curvature, choice of operator ordering etc.) and a habitat in Cyl$^*$ such that quantization of (certain density weighted) Hamiltonian constraint *and* certain quantization of the diffeomorphism constraint gives a faithful representation of the Dirac algebra on the habitat. If we managed to find such regularization scheme, we could further ask if the choice of curvature operator had any (at least conceptual) similarity with the curvature operator used in LQC.

### 3.4 The master constraint program

Before going into the toy models at the core of the review, let us say a few words on the master constraint program.

Criticisms on Thiemann's construction appeared quite quickly after publication of his work. But due to the difficulty of the challenge, few modifications aiming at improving it have been proposed, and efforts have instead concentrated on developing the path integral spin foam formalism. In [111], Thiemann himself introduced the *master constraint* program, to revive the interest into the canonical quantization. He traced back the difficulties in the well-known peculiarities of the Dirac constraint algebra. In the latter, the scalar constraint does not generate a subalgebra, is not spatially diffeomorphism invariant, and the whole algebra closes with structure functions which are not polynomials in the metric components.

Hence, Thiemann introduced in [111] a re-writing of the scalar constraint as

$$M = \int_\Sigma d^3x \, \frac{|H(x)|^2}{\sqrt{\det(q(x))}},$$

The density-weight one Hamiltonian $H(x)$ is squared, rescaled to $\sqrt{\det(q(x))}$ to make it back density-wight one, and finally integrated over the canonical surface $\Sigma$. That stands for the infinite number of constraints $H(x) = 0$. $M$ commutes with the generator of spatial diffeomorphisms.

The price to pay for getting a single master equation is that a (Dirac) observable $O$ is now defined by a non-linear equation,

$$\{O, \{O, M\}\}_{|M=0} = 0. \tag{3.5}$$

The program has been successfully tested in various situations, [49, 50, 51, 52, 53] and an explicit comparison between the Dirac and path integral quantizations is presented in [71]. The loop quantum gravity case is developed in [110].

At the technical level, the master constraint program suffers from ambiguities just like the previous attempt. At the conceptual level, its point is to avoid the requirement of reproducing the Dirac algebra. It is appealing since the diffeomorphism algebra indeed encodes gauge, non-physical degrees of freedom. However, the generators of the algebra also form the full Hamiltonian of general relativity, which means that solving the dynamics is equivalent to finding the equivalence classes determined by different gauge orbits. It seems to us that this intuition becomes more complicated in the master constraint program, in particular since the equation for Dirac observables is not anymore invariance along the gauge orbits but (3.5) instead. Obviously, because the master constraint program is a re-writing, the usual notions from gauge systems have some equivalent in this formulation, but it is not clear to us whether hiding the difficulty of working with the Dirac algebra is fully satisfactory.

## 4 Parametrized field theory as an ideal toy model

The main question coming from our review of Thiemann's construction is whether there are some choices of regularization and a habitat such that the Dirac algebra can be faithfully represented. Needless to say, posing this question in LQG is going to be pretty hard, so as a warm-up we can pose it in some simplified model. The model has to be be a field theory so that it has spatial diffeomorphism covariance, and it has to be generally covariant, so that in the canonical theory one has Dirac algebra as it's constraint algebra.

Parametrized Field Theories (PFT) present us with just such a class of models. These are just free field theories on flat spacetime in $d \geq 2$ dimensions written in generally covariant guise by adding "fake" (pure gauge) degrees of freedom. For details we refer the reader to [70, 76].

Two dimensional parametrized massless field theories on a cylinder are the simplest toy models for canonical gravity as the constraint algebra just becomes the well known Witt algebra $(\mathrm{Diff}(S^1) \oplus \mathrm{Diff}(S^1))$, and hence is a true Lie algebra. The corresponding gauge group (which would not even be defined as an ordinary Lie group [26]) in canonical gravity is $\mathcal{G}_{\mathrm{PFT}} = \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. This is very nice as in LQG we know how to handle spatial diffeomorphisms very well. Thus once we quantize the kinematical Phase space of the theory using background independent techniques that LQG provides for us, we can average over the entire gauge group of the theory $\mathcal{G}_{\mathrm{PFT}}$ using the Refined Algebraic Quantization method as outlined in [66]. So on one hand we have Hamiltonian constraint and diffeomorphism constraint just as in canonical gravity, and on the other hand we could complete the entire quantization program without worrying about the constraints and obtain physical Hilbert space of the theory. So we could pose the problem we posed above for LQG Hamiltonian constraint with a further consistency check. If we instead of averaging over $\mathcal{G}_{\mathrm{PFT}}$, quantized Hamiltonian and diffeomorphism constraints, is there a choice of their quantization such that the kernel of the constraints yield the physical Hilbert space obtained via group averaging techniques. With this motivating monologue out of the way, we delve into more details.

### 4.1 Brief review of polymer PFT

Consider the following canonical field theory on a Lorentzian Cylinder ($\mathcal{M} = S^1 \times \mathbf{R}, \eta$). Cauchy slices are oriented circles coordinatized by the angular coordinate $x \in [0, 2\pi]$, with the direction of angular increase agreeing with the orientation of the circle. The theory has the following sets of fields. Two scalar fields $X^+$, $X^-$ which are known as embedding fields and their conjugate momenta, and a matter scalar field $f$ and it's conjugate momenta[8].

The Structure of the model can be summarized as follows:

**Canonically conjugate embedding variables**: $(X^+(x), \Pi_+(x))$, $(X^-(x), \Pi_-(x))$, $X^\pm(2\pi) = X^\pm(0) \pm 2\pi$.

**Matter variables**: $Y^\pm(x) := \pi_f \pm f'$, with the brackets $\{f(x), \pi_f(y)\} = \delta(x,y)$, $\{Y^+, Y^-\} = 0$, $\{Y^\pm(x), Y^\pm(y)\} = \pm(\partial_x \delta(x,y) - \partial_y \delta(y,x))$.

**Density weight 2 constraints**: $H_\pm(x) = [\Pi_\pm(x) X^{\pm\prime}(x) \pm \frac{1}{4} Y^\pm(x)^2]$.

The constraint algebra is isomorphic to the Lie Algebra of vector fields on the circle. This algebra is more commonly known as the Witt algebra.

Instead of working with the above constraints which form a true Lie algebra, we can also work with the "traditional" diffeomorphism and Hamiltonian constraints which form the Dirac algebra with structure functions.

**Diffeomorphism constraint**: $C_{\text{diff}}$ generates spatial diffeomorphisms

$$C_{\text{diff}}(x) = H_+ + H_- = \big[\Pi_+(x) X^{+\prime}(x) + \Pi_-(x) X^{-\prime}(x) + \pi_f(x) f'(x)\big].$$

**Hamiltonian constraint**: $C_{\text{ham}}$ generates evolution normal to the Cauchy slice,

$$\begin{aligned} C_{\text{ham}}(x) &= \frac{1}{\sqrt{X^{+\prime}(x) X^{-\prime}(x)}} (H_+ - H_-) \\ &= \frac{1}{\sqrt{X^{+\prime}(x) X^{-\prime}(x)}} \big[\Pi_+(x) X^{+\prime}(x) - \Pi_-(x) X^{-\prime}(x) + \frac{1}{4}\big(\pi_f^2 + f'^2\big)\big]. \end{aligned} \tag{4.1}$$

**Constraint algebra**: The Poisson algebra generated by $C_{\text{diff}}$ and $C_{\text{ham}}$ is the Dirac algebra,

$$\begin{aligned} &\{C_{\text{diff}}[\vec{N}], C_{\text{diff}}[\vec{M}]\} = C_{\text{diff}}[\vec{N}, \vec{M}], \qquad \{C_{\text{diff}}[\vec{N}], C_{\text{ham}}[M]\} = C_{\text{ham}}[L_{\vec{N}} M], \\ &\{C_{\text{ham}}[N], C_{\text{ham}}[M]\} = C_{\text{diff}}[\vec{\beta}(N, M)], \end{aligned} \tag{4.2}$$

wherein $\vec{N}$, $\vec{M}$ are shift vectors, $N$, $M$ are lapse functions and the structure function $\beta^a(N, M) := q^{ab}(N \nabla_b M - M \nabla_b N)$ in (4.2) is defined by the induced spatial metric $q_{ab}$,

$$q_{ab} dx^a dx^b = -X^{+\prime} X^{-\prime} (dx)^2.$$

Whence working with the density one constraints, we are led to a two dimensional avatar or constraint algebra of canonical gravity. On the other hand, working with density tow constraints, one obtains an algebra without the structure functions. The Lie algebraic nature of algebra generated by density two constraints is special to two dimensions. This happens essentially because in two dimensions (with one dimensional Cauchy slices), the structure function changes from $q^{ab}$ (for density one constraints) to $qq^{ab}$ for density two constraints, but $qq^{ab} = 1$. What we would like to check is if either of these two (or in principle any density constraints) admit an off-shell closure in quantum theory.

[8] We refrain from introducing the Lagrangian formulation of the theory from which the canonical structure is derived, and from where the fields receive their titles (embedding fields and matter fields), as it is not relevant for our purpose. It suffices to note that we have a field theory with constraint algebra which is isomorphic to Dirac algebra in two dimensions and which has no true Hamiltonian.

## 4.2 Quantum theory

A *charge network* $s$ is a finite collection, $\gamma(s)$, of colored, non-overlapping (except at vertices) edges, $e$, which span the range of the angular coordinate $x$ (i.e. $[0, 2\pi]$), the colors being referred to as charges, and the collection of edges being referred to as a graph. Charge network states are in correspondence with charge networks and constitute an orthonormal basis similar to spin network states in LQG.

### 4.2.1 Embedding sector

- Charge network: $s^\pm = \{\gamma(s^\pm), (k^\pm_{e^\pm_1}, \ldots, k^\pm_{e^\pm_n})\}$ where $k^\pm_{e^\pm_I}$ are embedding charges whose range is specified by $k^\pm_{e^\pm_I} \in \frac{2\pi L}{\hbar A}\mathbf{Z}$, for all $I = 1, \ldots, n^\pm$. Here $A$ is a fixed, positive, integer-valued Barbero–Immirizi-like parameter. It is useful to define the 'minimum length increment', as $a := \frac{2\pi L}{A}$.
- Elementary variables: $X^\pm(x)$, $T_{s^\pm}[\Pi_\pm] := \exp\big[-i \sum_{e\in\gamma(s^+)} k^\pm_{e^\pm} \int_{e^\pm} \Pi_\pm\big]$.
- Representation: $T_{s^\pm_1}$ denotes an embedding charge network state. $\hat{X}^\pm(x), \hat{T}_{s^\pm}$ denote the operators corresponding to the classical quantities $X^\pm(x)$, $T_{s^\pm}[\Pi_\pm]$. Their action is given by

$$\hat{T}_{s^\pm} T_{s^\pm_1} = T_{s^\pm + s^\pm_1} \qquad \text{and} \qquad \hat{X}^\pm(x) T_{s^\pm} := \lambda_{x,s^\pm} T_{s^\pm},$$

where, for $\gamma(s^\pm)$ with $n^\pm$ edges,

$$\lambda_{x,s^\pm} := \begin{cases} \hbar\, k^\pm_{e^\pm_{I^\pm}} & \text{if } x \in \text{Interior}(e^\pm_{I^\pm}) \text{ for } 1 \le I^\pm \le n^\pm, \\ \frac{1}{2}\hbar\, \big(k^\pm_{e^\pm_{I^\pm}} + k^\pm_{e^\pm_{(I^\pm+1)}}\big) & \text{if } x \in e^\pm_{I^\pm} \cap e^\pm_{(I^\pm+1)} \text{ for } 1 \le I^\pm \le (n^\pm - 1), \\ \frac{1}{2}\hbar\big(k^\pm_{e^\pm_{n^\pm}} \mp \frac{2\pi L}{\hbar} + k^\pm_{e^\pm_1}\big) & \text{if } x = 0, \\ \frac{1}{2}\hbar\big(k^\pm_{e^\pm_1} \pm \frac{2\pi L}{\hbar} + k^\pm_{e^\pm_{n^\pm}}\big) & \text{if } x = 2\pi. \end{cases}$$

### 4.2.2 Matter sector

- Charge-network: $s^\pm = \{\gamma(s^\pm), (l^\pm_{e^\pm_1}, \ldots, l^\pm_{e^\pm_n})\}$, with the technical condition[9] $\sum_{I=1}^{n^\pm} l^\pm_{e^\pm_I} = 0$ and $l^\pm_{e^\pm_I} \in \epsilon\mathbf{Z}$ for all $I = 1, \ldots, n^\pm$. Here $\epsilon$ is a fixed real positive parameter with dimensions $(ML)^{-\frac{1}{2}}$. $\epsilon$ is also a Barbero–Immirizi-like parameter.
- Elementary variables: $W_{s^\pm_{\lambda^\pm}}[Y^\pm] = \exp[i \sum_{e^\pm\in\gamma(s^\pm)} l^\pm_{e^\pm} \int_{e^\pm} Y^\pm]$.
- Weyl algebra[10] of operators, $\hat{W}(s^\pm)\hat{W}(s'^\pm) = \exp[-i\frac{\hbar}{2}\alpha(s^\pm, s'^\pm)]\hat{W}(s^\pm + s'^\pm)$. Here the exponent in the phase-factor $\alpha(s^\pm, s'^\pm)$ is given by

$$\alpha(s^\pm, s'^\pm) := \sum_{e^\pm\in\gamma(s^\pm)} \sum_{e'^\pm\in\gamma(s'^\pm)} l^\pm_{e^\pm} l^\pm_{e'^\pm}\ \alpha(e^\pm, e'^\pm).$$

[9] The zero sum condition on the matter charges stems from technicalities related to the scalar field zero mode [78].

[10] The definition of the Weyl algebra follows in the standard way from the Poisson brackets between $Y^\pm(x), Y^\pm(y)$ and an application of the Baker–Campbell–Hausdorff lemma.

Here $\alpha(e^\pm, e'^\pm) = (\kappa_{e'^\pm}(f(e^\pm)) - \kappa_{e'^\pm}(b(e^\pm))) - (\kappa_{e^\pm}(f(e'^\pm)) - \kappa_{e^\pm}(b(e'^\pm)))$. Here $f(e)$, $b(e)$ are the final and initial points of the edge $e$ respectively. $\kappa_e$ is defined as

$$\kappa_e(x) = \begin{cases} 1 & \text{if } x \text{ is in the interior of } e, \\ \frac{1}{2} & \text{if } x \text{ is a boundary point of } e. \end{cases}$$

- Representation: $\hat{W}(s^\pm)W(s'^\pm) = \exp(\frac{-i\hbar}{2}\alpha(s^\pm, s'^\pm))W(s^\pm + s'^\pm)$.

### 4.2.3 Kinematic Hilbert space

The kinematic Hilbert space $\mathcal{H}_{\rm kin}$ is the product of the plus and minus sectors, $\mathcal{H}^\pm_{\rm kin}$, each of which is a product of the appropriate embedding and matter sectors. $\mathcal{H}^\pm_{\rm kin}$ is spanned by an orthornormal basis of charge network states.

A charge network state in $\mathcal{H}^\pm_{\rm kin}$ is denoted by $|\mathbf{s}^\pm\rangle := T_{s^\pm} \otimes W(s'^\pm)$.

The label $\mathbf{s}^\pm$ is specified by $\mathbf{s}^\pm_{\lambda^\pm} := \{\gamma(\mathbf{s}^\pm), (k^\pm_{e^\pm_1}, l^\pm_{e^\pm_1}), \ldots, (k^\pm_{e^\pm_{n^\pm}}, l^\pm_{e^\pm_{n^\pm}})\}$. Here we have used the equivalence of charge networks to set $\gamma(\mathbf{s}^\pm) := \gamma(s^\pm) = \gamma(s'^\pm)$ so that each edge of the charge network is labeled by an embedding charge and a matter charge.

We refer to the quantum theory as Polymer Parametrized field theory (PPFT).

### 4.2.4 Unitary representation of gauge transformations

Finite gauge transformations generated by the density 2 constraints act, essentially, as two *independent* diffeomorphisms of the spatial manifold, one which acts only on the '+' fields and one which acts only on the '−' fields. Consequently, in analogy to spatial diffeomorphisms in LQG, their action on charge networks is to appropriately 'drag' them around the circle[11].

Then a unitary representation of the gauge group is given by

$$\hat{U}^\pm(\phi^\pm)T_{s^\pm} := T_{\phi^\pm(s^\pm)}\hat{U}^\mp(\phi^\mp)T_{s^\pm} := T_{s^\pm},$$
$$\hat{U}^\pm(\phi^\pm)W(s'^\pm) := W((\phi^\pm)(s'^\pm), \qquad \hat{U}^\mp(\phi^\mp)W(s'^\pm) := W(s'^\pm).$$

Denoting $T_{s^\pm} \otimes W(s'^\pm)$ by $|\mathbf{s}^\pm\rangle$, the above equations can be written in a compact form as

$$|\mathbf{s}^\pm_{\phi^\pm}\rangle := \hat{U}^\pm(\phi^\pm)|\mathbf{s}^\pm\rangle.$$

### 4.2.5 Physical Hilbert space

Physical states are obtained by group averaging the action of the finite gauge transformations discussed in the previous section. Henceforth we restrict attention to a physically relevant superselected sector of the physical Hilbert space. This sector is obtained by group averaging a superselected subspace, $\mathcal{D}_{ss}$ of $\mathcal{H}_{\rm kin}$, $\mathcal{D}_{ss} = \mathcal{D}^+_{ss} \otimes \mathcal{D}^-_{ss}$.

$\mathcal{D}^\pm_{ss}$ is defined as follows. Fix a pair of graphs $\gamma^\pm$ with $A$ edges. Place the embedding charges $\vec{k}^\pm$ such that $k^\pm_{e^\pm_{I^\pm}} - k^\pm_{e^\pm_{I^\pm-1}} = \frac{2\pi}{A\hbar}$, for all $I^\pm$. Consider the set of all charge-network states $\{|\mathbf{s}^\pm\rangle = |\gamma^\pm, \vec{k}^\pm, (l^\pm_{e^\pm_1}, \ldots, l^\pm_{e^\pm_A})\rangle\}$, where $l^\pm_{e^\pm_I} \in \mathbf{Z}\epsilon$ are allowed to take all possible values subject to the zero sum condition $\sum_I l^\pm_{e^\pm_I} = 0$. Let $\mathcal{D}^\pm_{ss}$ be finite span of charge network states of the type $\{|\mathbf{s}^\pm_{\lambda^\pm\phi^\pm}\rangle, \forall\, \phi^\pm\}$.

[11] Due to the quasi periodic nature of $X^\pm$ it is more appropriate to think of these diffeomorphisms as being periodic diffeormorphisms of the real line. Consequently the action of these gauge transformations in quantum theory also keeps track of 'factors of $2\pi$' when embedding charge edges 'go past $x = 2\pi$'. We refrain from going into such technicalities in this review.

The action of the group averaging map $\eta^{\pm}$ on a charge network state in $\mathcal{D}^{\pm}_{ss}$ yields the distribution

$$\eta^{\pm}(|\mathbf{s}^{\pm}\rangle) = \sum_{\mathbf{s}^{\pm}\in[\mathbf{s}^{\pm}]} \langle \mathbf{s}'^{\pm}| = \sum_{\phi^{\pm}\in \mathrm{Diff}^{P}_{[\mathbf{s}^{\pm}]}\mathbf{R}} \langle \mathbf{s}^{\pm}_{\phi^{\pm}}|.$$

Here $[\mathbf{s}^{\pm}]$ is the equivalence class defined by $[\mathbf{s}^{\pm}] := \{\mathbf{s}'^{\pm} | \mathbf{s}'^{\pm} = \mathbf{s}^{\pm}_{\phi^{\pm}}$ for some $\phi^{\pm}\}$, and $\mathrm{Diff}^{P}_{[\mathbf{s}^{\pm}]}\mathbf{R}$ is a set of gauge transformations such that for each $\mathbf{s}'^{\pm} \in [\mathbf{s}^{\pm}]$ there is precisely one gauge transformation in the set which maps $\mathbf{s}^{\pm}$ to $\mathbf{s}'^{\pm}$. The space of such gauge invariant distributions comes equipped with the inner product

$$\langle \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle), \eta^{\pm}(|\mathbf{s}^{\pm}_2\rangle)\rangle_{\mathrm{phys}} = \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle)[|\mathbf{s}^{\pm}_2\rangle],$$

which can be used to complete $\eta^{\pm}(\mathcal{D}^{\pm}_{ss})$ to the Hilbert space $\mathcal{H}^{ss\pm}_{\mathrm{phys}}$. We shall restrict attention to $\mathcal{H}^{ss}_{\mathrm{phys}} := \mathcal{H}^{ss+}_{\mathrm{phys}} \otimes \mathcal{H}^{ss-}_{\mathrm{phys}}$.

### 4.3 The construction of $\mathcal{H}_{\mathrm{diff}}$

Spatial diffeormorphism invariant distributions are constructed by the action of the group averaging map, $\eta_{\mathrm{diff}}$ on the dense space of finite linear combinations of charge network states as follows. Let $[\mathbf{s}^{+}, \mathbf{s}^{-}]$ be the orbit of $\mathbf{s}^{+}$, $\mathbf{s}^{-}$ under all spatial diffeomorphisms so that $[\mathbf{s}^{+}, \mathbf{s}^{-}]$ is the set of all distinct charge network labels obtained by the action of spatial diffeomorphisms on $\mathbf{s}^{+}$, $\mathbf{s}^{-}$. Then

$$\eta_{\mathrm{diff}}(|\mathbf{s}^{+}\rangle \otimes |\mathbf{s}^{-}\rangle) = \eta_{[\mathbf{s}^{+},\mathbf{s}^{-}]} \sum_{\mathbf{s}'^{+},\mathbf{s}'^{-}\in[\mathbf{s}^{+},\mathbf{s}^{-}]} \langle \mathbf{s}'^{+}| \otimes \langle \mathbf{s}'^{-}|. \tag{4.3}$$

Here $\eta_{[\mathbf{s}^{+},\mathbf{s}^{-}]}$ is a constant which depends only on the orbit of $\mathbf{s}^{+}$, $\mathbf{s}^{-}$.[12]

### 4.4 The Hamiltonian constraint operator in PPFT

This section is devoted to the construction of the Hamiltonian constraint as an operator on the space of diffeomorphism invariant states. We follow the strategy used in LQG. Our aim is to first define a discrete approximant to the Hamiltonian constraint on a triangulation of the spatial manifold, promote the expression to an operator on the kinematic Hilbert space and then show that its dual action on diffeomorphism invariant distributions, admits a well defined continuum limit.

Quantization of Hamiltonian constraints suffers from similar type of regularization dilemmas as the Hamiltonian constraint of LQG. We would like to see if there exists any choice of regularization, such that

(a) The quantum operators carry a faithful representation of the Dirac algebra (or at least the Witt algebra).

(b) The kernel of the quantum constraints (or more precisely the kernel of the Hamiltonian constraint plus group averaging over spatial diffeomorphisms) contains (as a vector space) $\mathcal{H}_{\mathrm{phys}}$.

From equation (4.1) the smeared Hamiltonian constraint with lapse $N(x)$ is

$$C_{\mathrm{ham}}[N] = \int N(x) \left[\Pi_{+}(x)X^{+\prime}(x) - \Pi_{-}(x)X^{-\prime}(x) + \frac{1}{4}(\pi_f^2 + f'^2)\right] \frac{1}{\sqrt{X^{+\prime}(x)X^{-\prime}(x)}}.$$

[12] The arbitrariness in the choice of this constant can be reduced by requiring that $\eta_{\mathrm{diff}}$ commute with all diffeomorphism invariant observables.

On a triangulation $T$, a discrete approximant to the above expression is given by

$$
\begin{aligned}
C_{\mathrm{ham},T}[N] = \sum_{\triangle\in T} |\triangle|\ N(b(\triangle))\frac{1}{\sqrt{X^{+\prime}X^{-\prime}}}(b(\triangle)) \\
\times \Bigg[\Pi_+(b(\triangle))\left(\frac{X^+(m(\triangle)) - X^+(m(\triangle-1) + L\delta_{b(\triangle),0})}{|\triangle|}\right) \\
- \Pi_-(b(\triangle))\left(\frac{X^-(m(\triangle)) - X^-(m(\triangle-1) - L\delta_{b(\triangle),0})}{|\triangle|}\right) \\
+ \frac{1}{4}(Y^+)^2(b(\triangle)) + \frac{1}{4}(Y^-)^2(b(\triangle))\Bigg],
\end{aligned}
\tag{4.4}
$$

where $b(\triangle)$ is the beginning vertex of simplex $\triangle$, $|\triangle|$ is its length and $m(\triangle)$ its midpoint. The symbol $\triangle-1$ denotes the simplex to the left of $\triangle$, and it is understood that if $\triangle_1$ is the left-most simplex with $b(\triangle_1)=0$ then $m(\triangle_1-1)=m(\triangle_N)$, with $\triangle_N$ being the right-most simplex such that $f(\triangle_N)=2\pi$ where $f(\triangle)$ is the ending vertex of $\triangle$.

Since only the holonomies of $\Pi_\pm$, $Y^\pm$ are well defined operators on $\mathcal{H}_{\mathrm{kin}}$, the local fields $\Pi_\pm$, $(Y^\pm)^2$ need to be approximated on $T$ by appropriate combinations of holonomies. The equation (4.4) is a discrete approximant to the continuum expression and we may modify it by terms which vanish in the continuum limit, $|\triangle|\to 0$. It is straightforward to see that the following expression is one such modification,

$$
\begin{aligned}
C_{\mathrm{ham},T}[N] = \sum_{\triangle\in T} \frac{-i\hbar N(b(\triangle))}{|\triangle|\sqrt{X^{+\prime}X^{-\prime}}}\Bigg(\left[1+\frac{|\triangle|^2}{-i\hbar}\Pi_+(b(\triangle))X^{+\prime}(b(\triangle))\right] \\
\times\left[1+\frac{|\triangle|^2}{i\hbar}\Pi_-(b(\triangle))(b(\triangle))\right]\left[1+\frac{|\triangle|^2}{-4i\hbar}(Y^+)^2(b(\triangle))\right]\left[1+\frac{|\triangle|^2}{-4i\hbar}(Y^-)^2(b(\triangle))\right]\Bigg).
\end{aligned}
\tag{4.5}
$$

In a nutshell our aim is to choose a quantization scheme for $C_{\mathrm{ham},T}[N]$ such that the quantum constraints are a faithful representation of the classical constraint algebra, and the kernel of Hamiltonian constraint (after averaging over spatial diffeomorphisms) leads to $\mathcal{H}_{\mathrm{phys}}$. One way to anticipate what kind of regularization choices one should make is to consider action of classical Hamiltonian vector field of the hamiltonian constraint on cylindrical functions. If we start with the density one constraint, this action is quite involved due to the presence of $\frac{1}{\sqrt{q}}$ factor. However as we will see below, in quantum theory the spin-network states are eigenstates of inverse metric operator, and the non-triviality of quantum action basically comes from $\Pi_+X^+ - \Pi_-X^- + (Y^+)^2 + (Y^-)^2$. Whence let us focus on that part, which is really just the density two constraint.

It is a simple computation to show that the action of Hamiltonian vector field of (density two) constraints on cylindrical functions can be written as

$$
\begin{aligned}
\mathcal{L}_{\tilde{\tilde{H}}[N]}\ T_{s^+}[\Pi_+] &= \lim_{\delta\to 0}\frac{1}{\delta}\big(T_{s^+}[(\phi_N^\delta)^*\Pi_+] - T_{s^+}[\Pi_+]\big), \\
\mathcal{L}_{\tilde{\tilde{H}}[N]}T_{s^-}[\Pi_-] &= \lim_{\delta\to 0}\frac{1}{\delta}\big(T_{s^-}[(\phi_N^{-\delta})^*\Pi_-] - T_{s^+}[\Pi_-]\big).
\end{aligned}
$$

Whence we seek finite triangulation holonomy approximants to the various local fields of interest in such a way that $C_{\mathrm{ham},T}[N]$ is proportional to a combination of finite gauge transformations minus the identity, with the finite gauge transformations being parametrized by $|\triangle|$ so that at $|\triangle|=0$ the gauge transformations are just identity.

We display exactly such approximants to $\Pi_\pm$, $(Y^\pm)^2$ in next section. The holonomies turn out to be state dependent[13], the approximants yield $\Pi_\pm(b(\triangle))$ and $(Y^\pm)^2(b(\triangle))$ to leading order in $|\triangle|$ and the resulting quantum constraint is a linear combination of operators localised on vertices, each of which is proportional to a the difference of a finite gauge transformation and identity.

### 4.5 Suitable "curvature approximants"

We shall focus on the left moving (+) variables. Given a state $T_{s^+}$, let $V_E(\gamma(s^+))$ be the set of all vertices $v$ such that $k^+_{e_v} - k^+_{e^v} \neq 0$ (where $e_v(e^v)$ refer to edges which terminate (originate) at $v$. Similarly let $V_M(\gamma(s^+))$ be the set of all vertices $v$ such that $l^+_{e_v} - l^+_{e^v} \neq 0$. Then as shown in [77], the quantization of $\Pi_+(v)$ which defines the Hamiltonian constraint whose kernel contains states in $\mathcal{H}_{\rm phys}$ is

$$\hat{\Pi}^\triangle_+(v)T_{s^+} = \begin{cases} \dfrac{i}{|\triangle|(k^+_{e_v} - k^+_{e^v})}\big(\hat{h}^{(k^+_{e_v}-k^+_{e^v})} - 1\big)T_{s^+} & \text{if } v \in V_E(\gamma(s^+)), \\ \dfrac{i}{|\triangle|}(\hat{h}^+_\triangle - 1)T_{s^+} & \text{if } v \notin V_E(\gamma(s^+)). \end{cases} \tag{4.6}$$

Next, we show that the above choice (4.6) directly leads to an operator action of the '+' embedding part of the $C_{{\rm ham},T}$ which is a finite diffeomorphism on the + part of the embedding state. As we shall see, this will finally lead to a satisfactory definition of the Hamiltonian constraint.

With the above definition, the approximant to the $\Pi_+ X^{+\prime}$ term is

$$\begin{aligned} \hat{\Pi}_+\hat{X}^{+\prime}(b(\triangle))|_T T_{s^+} &:= \frac{1}{|\triangle|}\hat{\Pi}^\triangle_+(b(\triangle))\big(\hat{X}^+(m(\triangle)) - \hat{X}^+(m(\triangle-1))\big)T_{s^+} \\ &= \begin{cases} \dfrac{-i\hbar}{|\triangle|^2}(\hat{h}^{k^+_{e_v}-k^+_{e^v}}_\triangle - 1)T_{s^+} & \text{if } b(\triangle) \in V_E(\gamma^+), \\ 0 & \text{if } b(\triangle) \notin V_E(\gamma^+). \end{cases} \end{aligned}$$

Whence for $x \in V_E(\gamma^+)$,

$$\begin{aligned} \hat{\Pi}_+\hat{X}^{+\prime}(b(\triangle))|_T T_{s^+} &= \frac{-i\hbar}{|\triangle|^2}(\hat{h}^{k^+_{e_v}-k^+_{e^v}}_\triangle - 1)T_{s^+} \\ &= \frac{-i\hbar}{|\triangle|^2}(T_{s^+_{\phi_\triangle}} - T_{s^+}) = \frac{-i\hbar}{|\triangle|^2}(\hat{U}^{+,E}(\phi_\triangle) - 1)T_{s^+}. \end{aligned}$$

Here, $\phi_\triangle$ is a diffeomorphism of the circle (more precisely, $\phi_\triangle$ is a periodic diffeomorphism of the real line) which is identity in the neighborhood of all the vertices of $T$ except $b(\triangle)$, $f(\triangle)$.[14] Further, $\phi_\triangle$ maps $b(\triangle)$ to $f(\triangle)$ and its action on the charge network label $s^+$ is denoted by $s^+_{\phi_\triangle}$. $\hat{U}^{+,E}(\phi_\triangle)$ is the restriction of the unitary action of the finite gauge transformation $\hat{U}^+(\phi^+ = \phi_\triangle)$ to the left-moving embedding Hilbert space.

Similarly the quantization of $\widehat{(Y^+)^2})$ that yields a satisfactory definition of the Hamiltonian constraint is

$$\widehat{(Y^+)^2}(b(\triangle))_T W_{s^+} = \frac{-4i\hbar}{|\triangle|^2}\left[\exp\left(-i\frac{\hbar}{2}\big(l^+_{e^+_I} - l^+_{e^+_{I+1}}\big)^2\right)\hat{h}^{\left(l^+_{e^+_I} - l^+_{e^+_{I+1}}\right)}_\triangle - 1\right] W_{s^+},$$

[13] As we will see shortly, this dependence involves the eigenvalues of the embedding operators $\hat{X}^\pm$. Since $\hat{X}^\pm$ are the analogs of the LQG densitized triad operators, this feature is reminiscent of the $\bar{\mu}$ scheme employed in the improved dynamics in LQC [9].

[14] In addition, $\phi_\triangle$ also differs from identity at $x = 2\pi$ (or $x = 0$) if $b(\triangle) = 0$ (or $f(\triangle) = 2\pi$); this is just a consequence of the circular topology of space.

which can also be rewritten as

$$\widehat{(Y^+)^2}(b(\triangle))_T = \frac{-4i\hbar}{|\triangle|^2}\big(\hat{U}^{+,M}(\phi_\triangle) - 1\big),$$

where $\phi_\triangle$ has been defined above and $\hat{U}^{+,M}(\phi_\triangle)$ is the restriction of the finite gauge transformation operator $\hat{U}^+(\phi^+ = \phi_\triangle)$ to the matter Hilbert space. The analysis of the right-moving $(-)$ mode proceeds in a similar way.

Substituting the above "curvature approximants" into (4.5) leads to the following definition of $\hat{C}_{\mathrm{ham},T}[N]$ on $\mathcal{H}_{\mathrm{kin}}$,

$$\hat{C}_{\mathrm{ham},T}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle = \sum_{\substack{\triangle\in T\\ b(\triangle)\in V_E(s^+)\cup V_E(s^-)}} N(b(\triangle))$$
$$\times\Big[\hat{U}^{+,E}(\phi_\triangle)\otimes\hat{U}^{-,E}\big(\phi^{-1}_{\triangle-1}\big)\otimes\hat{U}^{+,M}(\phi_\triangle)\otimes\hat{U}^{-,M}\big(\phi^{-1}_{\triangle-1}\big) - 1\Big]\frac{-i\hbar}{|\triangle|\sqrt{\hat{X^{+\prime}X^{-\prime}}}}|\mathbf{s}^+,\mathbf{s}^-\rangle.$$

Using the fact that the unitary operators in the above equation are just restricted actions of unitary operators associated with finite gauge transformations, we obtain

$$\hat{C}_{\mathrm{ham},T}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle = \sum_{\substack{\triangle\in T\\ b(\triangle)\in V_E(s^+)\cup V_E(s^-)}} N(b(\triangle))\left(\frac{-i\hbar}{a^2}\right)\lambda(s^+,s^-,b(\triangle))$$
$$\times\Big[\hat{U}^{+}(\phi_\triangle)\otimes\hat{U}^{-}\big(\phi^{-1}_{\triangle-1}\big) - 1\Big]|\mathbf{s}^+,\mathbf{s}^-\rangle.$$

This action precisely mimics the action of the Hamiltonian vector field of $C_{\mathrm{ham}}[N]$. We could now consider the continuum limit of $\hat{C}_{\mathrm{ham},T}[N]$ on $\mathcal{H}_{\mathrm{kin}}$ in the URST topology. As shown in [77], just like in LQG, this operator converges to a continuum operator $\hat{C}_{\mathrm{ham}}[N]$ on $\mathcal{H}_{\mathrm{kin}}$.

Specifically, in the notation used above, we may choose the limit of the one-parameter family $\hat{C}_{\mathrm{ham},T(\gamma,\delta)}[N]$ to be the operator $\hat{C}_{\mathrm{ham}}[N]$ where

$$\hat{C}_{\mathrm{ham}}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle := \hat{C}_{\mathrm{ham},T(\gamma,\delta_0)}[N]|\mathbf{s}^+,\mathbf{s}^-\rangle.$$

On one hand this is a very promising candidate for the Hamiltonian constraint of PFT. A moment of thought reveals that given any $(\Psi|$ in $\mathcal{H}_{\mathrm{phys}}$

$$(\Psi|\hat{C}_{\mathrm{ham}}[N])|\mathbf{s}^+,\mathbf{s}^-\rangle = 0,$$

for all $N$ and $|\mathbf{s}^+,\mathbf{s}^-\rangle$. On the other hand, this Hamiltonian exhibits the same triviality properties that the LQG Hamiltonian does. Instead of going into technical details, we summarise the findings of [77] in qualitative terms.

### 4.6 Commutator of Hamiltonian constraints in PPFT

(1) Precisely due to the reasons we elaborated upon in the earlier sections, the commutator of density one constraints $\hat{C}_{\mathrm{ham}}[N]$ vanishes on $\mathcal{H}_{\mathrm{kin}}$.

(2) One can define an analog of the LM habitat, on which both the r.h.s. and the l.h.s. of constraint algebra can be quantized. Due to the density one character of the constraints, the product of the two Hamiltonians is ultra-local in the lapses and hence vanishes under anti-symmetrization. It turns out that if we use the same curvature approximants that we used to define $\hat{C}_{\mathrm{ham},T}[N]$ to define r.h.s., then two things happen. On one hand the operator at finite triangulation $\widehat{C}_{\mathrm{diff},T}[q^{-1}[\vec{N},M]]$ mimics the action of corresponding classical Hamiltonian vector

field, and on the other hand $\widehat{C}_{\mathrm{diff},T}[q^{-1}[\vec{N}, M]]$ is proportional to $|\triangle|$, whence on LM habitat, its continuum limit is zero.

(3) As we had noted earlier, in their analysis of the LM habitat in LQG, the authors of [64] noted certain necessary conditions for the commutator to be non-trivial. One of them was the density $\neq 1$ character of the constraint, and the other was, a definition of curvature approximant such that the newly created vertices where themselves not in the "kernel" of the Hamiltonian. The choice of curvature approximants in PFT ensure the second condition is satisfied. So one would hope that appropriately changing the density weight (which is dictated by how many powers of $|\triangle|$ we need in the denominator, so that the continuum limit contains correct number of derivatives) would give us a non-trivial constraint algebra.

In two dimensions, in order to get the right powers of $|\triangle|$ in the denominator, one needs to work with density two constraints. But when we work with the density two constraints, several remarkable things happen.

(3$i$) The continuum limit of $\hat{\tilde{C}}_{\mathrm{ham},T}[N]$ (we use tilde to denote the density two character) is not well defined on $\mathcal{H}_{\mathrm{kin}}$ or the LM habitat. However, the commutator of two such Hamiltonians $i$s well defined on the LM habitat and has an anomaly!

(3$ii$) There exists a different habitat, on which the density one constraints admit a continuum limit and the constraint algebra closes off-shell without any anomaly! and all the states in $\mathcal{H}_{\mathrm{phys}}$ are contained in this new habitat.

Whence we see that at least in the case of our toy model, everything nicely fits into a coherent picture. The requirement that constraints close off-shell is a very stringent requirement. It not only pins down the curvature approximants and the density weights, but also dictates that the right home for the constraints might be something quite different then $\mathcal{H}_{\mathrm{kin}}$ or even $\mathcal{H}_{\mathrm{diff}}$. In the case of PPFT, it turns out that the kernel of such constraints rather remarkably contains the solution obtained via completely different methods.

## 5 Diffeomorphism constraint in LQG

If the lessons learnt above in the case of PPFT could be applied to LQG, one could in principle obtain a quantization of the Dirac algebra. The key lessons one learnt in PPFT were

1. Higher density objects could be well defined in the theory if one worked on suitable habitats.
2. In the case that the constraints generate a true Lie algebra, the constraint operators at finite triangulation (with triangulation parametrized by $\delta$) should be related to the action of finite gauge transformation if the finite gauge transformations were unitarily represented in the theory.

It is unclear at this stage whether these lessons could be applied to the Hamiltonian constraint but as a warm-up one could apply these to the diffeomorphism constraint.

Recall that in the Dirac algebra, there is a Lie subalgebra generated by spatial diffeomorphisms,

$$\left\{H_{\mathrm{diff}}[\vec{N}], H_{\mathrm{diff}}[\vec{M}]\right\} = -H_{\mathrm{diff}}[\vec{N}, \vec{M}]. \tag{5.1}$$

We can now ask if there exists a quantization of $H_{\mathrm{diff}}[\vec{N}]$ on some subspace of $\mathrm{Cyl}^*$ such that

($a$) The operator $\hat{H}_{\mathrm{diff}}[\vec{N}]$ is a faithful representation of (5.1) on that subspace.

($b$) The kernel of $\hat{H}_{\mathrm{diff}}[\vec{N}]$ matches precisely (or at least contains) the vector space of diffeomorphism invariant states obtained via Rigging map techniques.

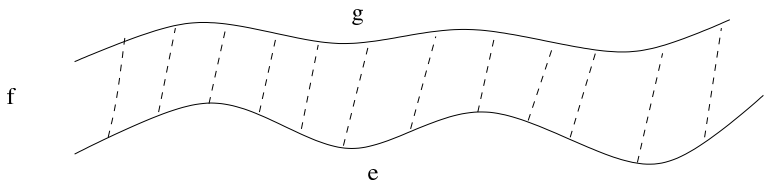


**Figure 2.**

It is important to note that at the level of finite triangulation, $\hat{H}_{T\mathrm{diff}}[\vec{N}]$ is well defined on $\mathcal{H}_{\mathrm{kin}}$ and also admits infinite number of quantization choices (choice of triangulation, choice of loops to define operator for curvature, choice of surfaces to define operator for triads, choice of representation for the curvature operator, etc). Whence it is interesting to see if requiring the diffeomorphism constraint operator to satisfy $(a)$ and $(b)$ restricts these choices in anyway. Also as the r.h.s. of Poisson bracket of two Hamiltonian constraints has a diffeomorphism constraint, it might be essential to have an operator correspondent of $H_{\mathrm{diff}}[\vec{N}]$.

As shown in [79], there does exist a quantization of $H_{\mathrm{diff}}[\vec{N}]$ which precisely generates infinitesimal diffeomorphisms on LM habitat, and hence satisfies (5.1)[15].

The ideas underlying the quantization are sketched below. As always we set $G = \hbar = c = 1$. We shall also set the Barbero–Immirizi parameter to unity. The diffeomorphism constraint $H_{\mathrm{diff}}(\vec{N})$ is

$$H_{\mathrm{diff}}(\vec{N}) = \int_\Sigma \mathcal{L}_{\vec{N}} A^i_a \tilde{E}^a_i = V(\vec{N}) - \mathcal{G}(N^c A^i_c),$$

where $V(\vec{N}) = \int_\Sigma N^a F^i_{ab} \tilde{E}^b_i$ and $\mathcal{G}(N^c A^i_c) = \int_\Sigma N^c A^i_c \mathcal{D}_a \tilde{E}^a_i$.

Let $T(\delta)$ be a one-parameter family of triangulations of $\Sigma$ with the continuum limit being $\delta \to 0$ and let $H_{\mathrm{diff},T(\delta)}$, $V_{T(\delta)}$, $\mathcal{G}_{T(\delta)}$ be finite triangulation approximants to the quantities $H_{\mathrm{diff}}$, $V$, $\mathcal{G}$ of the above equations. Thus $D_{T(\delta)}$, $V_{T(\delta)}$, $\mathcal{G}_{T(\delta)}$ are expressions which yield $H_{\mathrm{diff}}$, $V$, $\mathcal{G}$ in the continuum limit.

For simplicity consider a (non- gauge invariant) spin network state consisting of a single edge $e$ with spin label $j$ so that the state is just the $m$, $n$ component of an edge holonomy $h^{(j)}_{e\,m}{}^{n}$ of the (generalized) connection along the edge $e$ in the representation $j$, the indices $m$, $n$ taking values in the set $1, \ldots, 2j+1$. In what follows we shall suppress some of these labels and denote the state simply by $h_e$.

As we want the operator at finite triangulation to equal finite diffeomorphism minus the identity, our first goal is

$$\big(1 + i\delta \hat{H}_{\mathrm{diff},T(\delta)}\big) h_e = h_{\phi(\delta,\vec{N})\circ e},$$

where $\phi(\delta, \vec{N}) \circ e$ is the image of $e$ by the diffeomorphism $\phi(\delta, \vec{N})$ which translates $e$ by an amount $\delta$ along the integral curves of the shift vector field $N^a$ (see Fig. 2).

As shown in [79] the desired result is obtained in the following steps.

$(i)$ First we set

$$(1 + i\delta \hat{H}_{\mathrm{diff},T(\delta)}) := (1 - i\delta \hat{\mathcal{G}}_{T(\delta)})(1 + i\delta \hat{V}_{T(\delta)}).$$

[15] Recall that on $\mathcal{H}_{\mathrm{kin}}$ only finite diffeomorphisms are unitarily represented and the generator of such diffeomorphisms does not exist. However such generators could and indeed they do exist on certain distributional spaces.

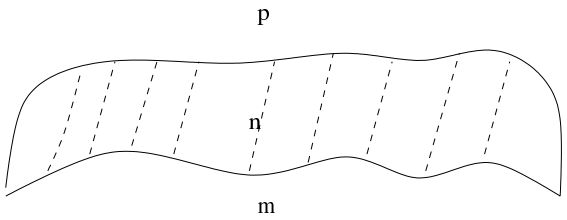


**Figure 3.**

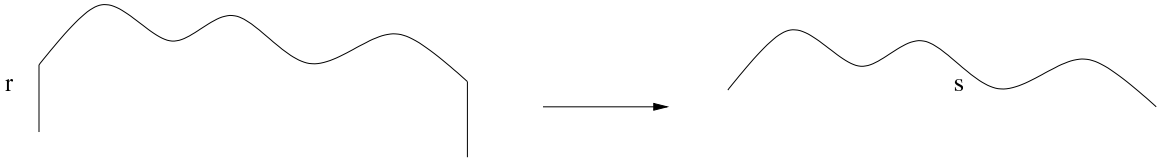


**Figure 4.**

(*ii*) Next, one shows that

$$(1 + i\delta\hat{V}_{T(\delta)})h_e = h_{\bar{e}(\vec{N},\delta)}. \tag{5.2}$$

Here $\bar{e}(\vec{N},\delta)$ has the same end points as $e$ (as it must by virtue of the gauge invariance of $V$) and is obtained by joining the end points of $\phi(\delta,\vec{N})\circ e$ to those of $e$ by a pair of segments which are aligned with integral curves of $N^a$ as shown in Fig. 3.

(*iii*) Finally, one shows that the Gauss Law term, $(1-i\delta\hat{\mathcal{G}}_{T(\delta)})$ removes these two extra segments (see Fig. 4).

The major part of the analysis concerns the derivation of the identity (5.2) in step (ii) above. $V_{T(\delta)}$ is written as a sum over contributions $V_\triangle$ where $\triangle$ denotes a 3-cell of the triangulation *dual* to $T(\delta)$, and $V_\triangle$ is a finite triangulation approximant to the integral $\int_\triangle N^a F^i_{ab}\tilde{E}^b_i$.

**Choice of operator ordering.** Order the triad operator to the right in $\hat{V}_\triangle$ so that only those 3-cells contribute which intersect $e$.

**Choice of triangulation.** The triangulation $T(\delta)$ is adapted to the edge $e$ so that its restriction to $e$ defines a triangulation of $e$. Thus, there is a triangulation of $e$ by 1-cells and vertices of $T(\delta)$ so that each of these vertices $v_I$, $I = 1,\ldots,N$ is located at the center of some 3-cell $\triangle = \triangle_I$.

**Choice of curvature approximant.** We define a finite triangulation approximant to $F^i_{ab}$ in a such a way that the following identity holds:

$$\big(1 + i\delta\hat{V}_{\triangle_I}\big)h_e = h_{\bar{e}(\triangle_I)}$$

$\forall\, I \in \{1,\ldots,N-1\}$. Here $\bar{e}(\triangle_I)$ is obtained by moving the segment of $e$ between $v_I$ and $v_{I+1}$ along the integral curves of $N^a$ by an amount $\delta$ and joining this segment to the rest of $e$ at the points $v_I$, $v_{I+1}$ by a pair of segments which run along the integral curves of $N^a$ as shown in Fig. 5.

This can be accomplished by using the full quantization ambiguity that is available to us when defining curvature operator in LQG. More precisely not only does one have to tune the spin (in which the trace of holonomy is evaluated) to the spin of the underlying state, but one also has to add the "higher-order operators" obtained by involving the fluxes. For more details we refer the reader to [79].

**Sum to product reformulation**: Finally the contributions from all the $\hat{V}_{\triangle_I}$ yield the edge $\bar{e}(\vec{N},\delta)$ of Fig. 3. Recall that $V_{T(\delta)}$ is obtained by summing over all the cell contributions $V_\triangle$.

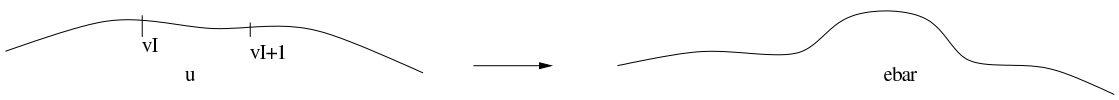


**Figure 5.**

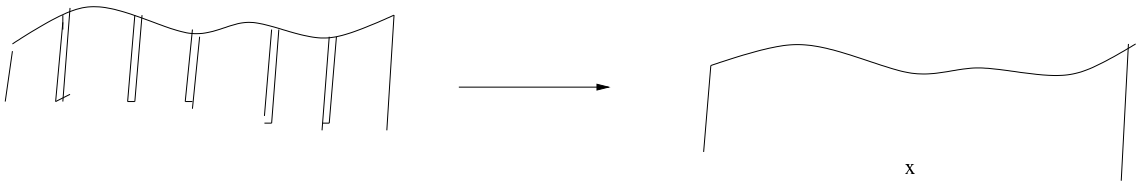


**Figure 6.** The double lines indicate retraced paths.

However, summing over the action of all the $\hat{V}_{\triangle_I}$ on $h_e$ only yields a sum over states of the type $h_{\bar{e}(\triangle_I)}$. In order to obtain the desired result, $h_{\bar{e}(\vec{N},\delta)}$, the sum over $\triangle$ is first converted to a *product* over $\triangle$, i.e. to leading order in $\delta$, we show that

$$1 + i\delta \sum_{\triangle} V_{\triangle} \sim \prod_{\triangle} (1 + i\delta V_{\triangle}).$$

Hence, replacing the sum over the corresponding operators by the product provides an equally legitimate definition of $\hat{V}_{T(\delta)}$. The replacement then leads, to the following identity

$$\prod_{\triangle} (1 + i\delta \hat{V}_{\triangle}) h_e = \prod_{I=1}^{N-1} (1 + i\delta V_{\triangle_I}) h_e.$$

The definition of $\hat{V}_{\triangle_I}$ is such that each factor in the product acts independently, the $I$th factor acting only on the part of $e$ between $v_I$ and $v_{I+1}$. We are then able to show that the result (5.2) follows essentially through the mechanism which is illustrated schematically in Fig. 6.

Thus we see that by making some very specific choices of triangulation, curvature and operator ordering, one obtains

$$\hat{H}_{\mathrm{diff},T(\delta)}[\vec{N}]|\gamma, \vec{j}, \vec{c}\rangle = \frac{1}{\delta}\big(\hat{U}(\phi_\delta^{\vec{N}}) - 1\big)|\gamma, \vec{j}, \vec{c}\rangle.$$

It can be easily shown that $\hat{H}_{\mathrm{diff},T(\delta)}[\vec{N}]$ admits a continuum limit on LM habitat where it precisely generates Lie action by $\vec{N}$.

## 6 New progress and insights from the BF topological model

We now present a different family of toy-models, namely the topological BF field theories in arbitrary dimensions. It is a field theory which has the same unreduced phase space as Yang–Mills or general relativity with Ashtekar–Barbero variables. But as a topological theory it is an exactly solvable theory and it has been considered as the main testing ground for the loop quantum gravity program since its early days, and even more after the birth of the spin foam program. Just like in the previous sections, the idea is to quantize *à la loop* the unreduced phase space, and then impose the topological symmetries at the quantum level. One expects to reproduce in this simplified situation a quantum algebra of constraints, and the resulting quantization should be equivalent to the one which has been performed using standard path integral methods.

This section first reviews the BF theory itself, in its standard quantization, then the most recent works in the loop framework, emphasizing the geometric aspects which are expected to show up similarly in four-dimensional quantum gravity. A nice review from the loop and spin foam perspective is [10]. The present section should be considered as a new review on the state of the art for the loop approach to this topological model, ten years after.

We focus on the symmetry group SU(2), but arbitrary spacetime dimension $n$. The BF field theory was introduced by Blau and Thompson in [25] and by Horowitz in [72]. It also appears in a number of interesting field theories. When $n = 2$ [114] and $n = 4$ [40], it appears as the zero coupling limit of Yang–Mills theory, an approach known as BF Yang–Mills (and as such it was observed early in lattice gauge theory). In three dimensions, it describes pure Riemannian gravity (with degenerate metrics), and remarkably can be recast in a Chern–Simons formulation [113]. In four dimensions, it is also possible to establish a link with gravity (which is crucial to spin foam models for quantum gravity) through the BF Plebanski formalism [43], or the BF MacDowell–Mansouri formulation [62].

It is also an interesting model exhibiting a relationship between field theory and topological invariants. The partition function produces the Ray–Singer torsion [24], and expectation values of observables are studied in [39].

## 6.1 Transition amplitudes

### 6.1.1 Field equations and flat connections

Let $P$ be a principal SU(2) bundle, over a $n$-dimensional manifold $M$. We consider a connection $A$ on $P$, locally seen as a $\mathfrak{su}(2)$-valued 1-form over $M$ with specific transformation rules. We also introduce a $(n-2)$-form taking values in the adjoint bundle, i.e. $B$ is $\mathfrak{su}(2)$-valued and transform under the adjoint action of SU(2). When $d = 3$, $B$ physically corresponds to the (Riemannian) triad, usually denoted $e$, and the spacetime metric is obtained from $g_{\mu\nu} = \delta_{ij} e^i_\mu e^j_\nu$.

To write the BF action, we consider the non-degenerated bilinear invariant form 'tr' on the algebra, and

$$S_{\mathrm{BF}}(B, A) = \int_M \mathrm{tr}(B \wedge F_A),$$

with $F_A = dA + \frac{1}{2}[A, A]$ being the curvature of $A$. The group $\mathcal{G}$ of usual gauge transformations consists in $G$-valued fields, which generate the following transformations,

$$A \mapsto \mathrm{Ad}_g A + g\, dg^{-1}, \qquad B \mapsto \mathrm{Ad}_g B.$$

In matrix notation the adjoint action reads $\mathrm{Ad}_g X = gXg^{-1}$. Since $S_{\mathrm{BF}}$ is the integral of a differential form, it is naturally invariant under diffeomorphisms of $M$.

The equations of motion are particularly simple,

$$d_A B = 0, \qquad F_A = 0,$$

where $d_A = d + [A, \cdot]$ is the covariant derivative. It turns out that all solutions are locally identical up to gauge transformations. Notice that the second equations comes from the fact that $B$ is a Lagrange multiplier for the curvature, so that the connection has to be flat. We know then that all flat connections are locally gauge equivalent. We also get that $B$ is locally exact, $B = d_A \psi$. Now it is time to state that the action has an extra invariance under

$$A \mapsto A, \qquad B \mapsto B + d_A \eta,$$

for any $(n-3)$-form $\eta$ over $M$ taking values in $\mathfrak{su}(2)$. That additional gauge symmetry is often called translation symmetry. Denoting $\mathcal{T}$ the group of such translations, the full group of gauge

symmetries is the semi-direct product $\mathcal{G} \ltimes \mathcal{T}$. It is then possible to transform any solution $B$ into the trivial solution. The physical consequence is that the theory is deprived of local degrees of freedom.

Those equations admit a useful cohomological interpretation. Squaring the covariant derivative gives $d_A^2\omega = [F_A, \omega]$ which vanishes on-shell. So $d_A$ defines cohomology classes. The set of non-gauge-equivalent solutions to the classical equations of motion consists in non-gauge-equivalent flat connections with elements of the $(n-2)$th cohomology group, the latter being isomorphic to the second cohomology group.

It is interesting to note that general relativity and the BF model share some essential properties which enable to make geometry dynamical. They are both formulated using differential forms, without any spacetime metric, and are diffeomorphism invariant. Hence, it should be clearly understood that what makes the BF model special is the translation symmetry, so that diffeomorphisms are actually encoded into the full set $\mathcal{G} \ltimes \mathcal{T}$. Suppose $\xi$ is a vector field on $M$, then using the inner product $i_\xi$, we can form the function $\phi_\xi = i_\xi A$ and the $(n-3)$-form $\eta_\xi = i_\xi B$. The translation symmetry and diffeomorphisms are related in the following way

$$\delta_\xi A = \mathcal{L}_\xi A = d_A\, \phi_\xi + i_\xi F, \qquad \delta_\xi B = \mathcal{L}_\xi B = d_A\, \eta_\xi + [\phi_\xi, B] + i_\xi\, d_A B,$$

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi$. When written in this way, infinitesimal diffeomorphisms appear as gauge transformations up to terms proportional to the variations of the action $\delta S/\delta B = F$, so that the two symmetries are *de facto* equivalent. Obviously in general relativity, it becomes impossible to rewrite diffeomorphisms as gauge symmetries plus terms which vanish on-shell.

In the context of the BRST (or Batalin–Vilkovisky) formalism, the symmetry of the BF model has been interpreted as a vector supersymmetry [65, 86], which turns out to be crucial to solve the local cohomology of the BRST operator, and in the end to prove the perturbative renormalizability [85]! An important aspect in that algebraic approach is that the BRST algebra does not close *off-shell*. The BRST operator is nilpotent only on-shell when $n \geq 4$, due to the fact that the translation symmetry is on-shell reducible. Indeed, the transformation of $B$ is the same for $\eta$ and $\eta + d_A\, \psi$. As a consequence, care is due when trying to quantize it in dimension four and higher.

### 6.1.2 The Hamiltonian analysis and flat connections again

The Hamiltonian analysis is straightforward. One finds that one canonical variable is the pull-back of the connection to the canonical surface $\Sigma$, denoted $A_a^i$. The other canonical variable is the analog to the electric field, $E_i^a$, transforming in the adjoint representation. Those variables satisfy the canonical bracket

$$\big\{E_i^a(x), A_b^j(y)\big\} = \delta_b^a \delta_i^j \delta^{(3)}(x-y).$$

The phase space is actually identified as the cotangent space $T^*\mathcal{A}$ to the space of connections $\mathcal{A}$, exactly like in Yang–Mills theory.

However, the Hamiltonian is different since it is a combination of constraints imposed with Lagrange multipliers. The constraints are simply the restrictions of the equations of motion to $\Sigma$, which do not involve any time derivatives. We integrate the local constraints as follows

$$\mathcal{C}_G(\Lambda) = \int dx\ \mathrm{tr}\big(\Lambda(x) D_a E^a(x)\big), \qquad \mathcal{C}_F(N) = \int dx\, \epsilon^{abc}\, \mathrm{tr}\big(N_c(x)\, F_{ab}(x)\big),$$

where $D$ is the covariant derivative with respect to $A_a^i$. Through the Poisson brackets, those constraints generate the whole set of gauge transformations on the canonical variables. Hence

they satisfy the expected algebra,

$$\{\mathcal{C}_G(\Lambda), \mathcal{C}_G(\Lambda')\} = \mathcal{C}_G([\Lambda, \Lambda']), \qquad \{\mathcal{C}_G(\Lambda), \mathcal{C}_F(N)\} = \mathcal{C}_F([\Lambda, N]),$$
$$\{\mathcal{C}_F(N), \mathcal{C}_F(N')\} = 0,$$

which is recognized as the Poincaré algebra ISU(2). The algebra is first class and the local number of degrees of freedom is easily evaluated to zero, recovering the result of the covariant analysis. $\mathcal{C}_G$ is the Gauß law which generates local SU(2) rotations, while $\mathcal{C}_F$ generates translations as expected since curvature is zero.

To completely solve the theory classically, one can first reduce the phase space by the Gauß law $\mathcal{C}_G$, just requiring invariance under the standard SU(2) gauge transformations. That leads to the reduced Yang–Mills phase space $T^*(\mathcal{A}/\mathcal{G})$. Reducing with respect to the constraint $\mathcal{C}_F$ implies that we consider only the set $\mathcal{A}_0$ of flat connections. It is also necessary to identify any two momenta which differ by a transformation generated by $\mathcal{C}_F$, i.e. $E^a_i \sim E^a_i + \epsilon^{abc} D_b \eta_c$. The relevant set of momenta is thus the second cohomology group of $D$ (which squares to zero on $\mathcal{A}_0$). In conclusion, the reduced phase space is the set of flat connections modulo the $\mathcal{G}$-action together with the second cohomology group over each flat connection. This is the set $T^*(\mathcal{A}_0/\mathcal{G})$, as described in [72, 115].

There exists a description of the set of flat connections which shows that the reduced phase space is often finite dimensional and which proves useful in practice. Holonomies of a flat connection only depend on the homotopy types of its paths. Hence a flat connection defines a morphism from the fundamental group $\pi_1(\Sigma)$ into the structure group SU(2). Under a gauge transformation, such a morphism gets conjugated by the value of the transformation at the reference point of $\pi_1(\Sigma)$. We thus get

$$\mathcal{A}_0/\mathcal{G} \subseteq \mathrm{Hom}\big(\pi_1(\Sigma), G\big)/G,$$

where $G$ acts by conjugation. The set $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is known as the moduli space of flat SU(2)-bundles. When $\pi_1(\Sigma)$ has a finite presentation, then it is a real algebraic variety. It also usually has singularities which prevent it from being a smooth manifold. Later, we will ignore those singularities and consider the case where $\mathcal{A}_0/\mathcal{G}$ really coincides with the moduli space of flat bundles.

This space has been mainly studied when $\Sigma$ is a two-dimensional closed surface, where a symplectic structure is available. That has been quite efficient to solve Yang–Mills theory in $2d$ [114], Chern–Simons theory in the canonical formalism [113]. More recently [22, 35, 36, 63].

### 6.1.3 Quantization

**The path integral.** Formally, without gauge fixing, it reads

$$\text{“}Z(M) = \int DADB \exp\Big(i \int_M \mathrm{tr}\big(B \wedge F_A\big)\Big) = \int DA \prod_{x \in M} \delta\big(F_A(x)\big)\text{”},$$

from which we learn two things.

- The functional integral is exactly peaked on flat connections over $M$.
- Gauge fixing is crucial to get a topological invariant (if any). Indeed, if we naively continue the above calculation, we find (after some regularization) the determinant of $\delta F_A/\delta A$ which is the the covariant derivative $d_A$ acting on 1-forms and which is not a topological invariant. The intuitive reason is that $d_A$ here only probes the topology going from 1-forms to 2-forms. The final topological invariant should instead probe all dimensional forms (or cells) over $M$.

The path integral can be evaluated using the usual Faddeev–Popov gauge-fixing in three dimensions, and the Schwarz' resolvent in higher dimensions (to deal with the reducibility), see [24, 25].

Let us sketch the three-dimensional situation. We assume for simplicity that the moduli space of flat connections is finite and that for any flat connection $A_{(\alpha)}$ the cohomology of the operator $d_{A_{(\alpha)}}$ is trivial. We pick up a background metric $g$ and select a representative $A_{(\alpha)}$ of each equivalence class of flat connections up to gauge which satisfies $d_{A_{(\alpha)}} * A_{(\alpha)} = 0$, where $*$ is the Hodge operator. Setting $A = A_{(\alpha)} + a$ in the neighborhood of $A_{(\alpha)}$, we get

$$Z(M) = \sum_{\alpha} \mu(A_{(\alpha)}),$$

with

$$\mu(A_{(\alpha)}) = \int DaDBD\omega D\bar{\omega}D\chi D\bar{\chi}DuDv$$
$$\times \exp\left( i \int \mathrm{tr}\left( B \wedge d_{A_{(\alpha)}} a + u \wedge d_{A_{(\alpha)}} * a + v \wedge d_{A_{(\alpha)}} * B + \bar{\omega} \wedge \Delta_{A_{(\alpha)}} \omega + \bar{\chi} \wedge \Delta_{A_{(\alpha)}} \chi \right) \right),$$

where $\Delta_{A_{(\alpha)}} = d_{A_{(\alpha)}} * d_{A_{(\alpha)}}$. $u$, $v$ are Lagrange multipliers imposing gauge conditions on $a$, $B$, and $\omega$, $\chi$ and $\bar{\omega}$, $\bar{\chi}$ are the corresponding ghosts and anti-ghosts. We refer to [24, 25] for details. The main point is that $\mu(A_{(\alpha)})$ can be evaluated to be the Ray–Singer analytic torsion. It is a topological invariant, which does not depend on the choice of the background metric.

**The wave-functions.** From the canonical analysis on $M = \Sigma \times [0, 1]$, it is clear that they are square-integrable functions over the set of flat connections modulo gauge transformations [72].

**The measure.** The inner product is obtained by considering the path integral on $M = \Sigma \times [0, 1]$. The measure is given by the Ray–Singer torsion on $M$ which in this case descents on $\Sigma$, so that the measure is well defined on the canonical surface.

**A physical IR divergence.** The partition function is well-defined for topologies where the cohomology $H^*(d_A)$ is trivial. At least in three dimensions, there is a nice physical interpretation, noticed by Witten [115]. There the partition function diverges after gauge fixing if the moduli space of flat connections is not a finite set. Notice that the equation for the tangent vectors to that space is $\delta F = d_A \delta A = 0$. This is exactly the equation of motion for the triad $d_A e = 0$. Hence, the partition function is finite on those topologies where there are no non-trivial (equivalence classes of) solutions for the triad. The intuitive picture of the quantum mechanical consequence is that there are only some quantum fluctuations around the trivial solution. When restoring the units, it means that lengths are fluctuating around the Planck scale, explaining the finiteness of $Z(M)$. However, if there are some tangent directions to the moduli space of flat connections, there are some non-trivial (equivalence classes of) triads satisfying $d_A e = 0$. The integrals along those directions are non-compact, hence the divergence. Physically, that corresponds to an IR divergence, since the main contributions to the path integral come from large triads, hence spacetimes much larger than the Planck scale. The result is that a macroscopic physics can exist only on topologies with non-trivial cohomology and a divergent partition function.

### 6.1.4 Loop quantization

Since the loop quantization relies on cylindrical functions supported on graphs and since the measure on the set of flat connections is the torsion, it would be convenient to evaluate the torsion using cell decomposition. That notion exists and is known as the Franz–Reidemeister torsion. This actually shows that the BF model admits an *exact* quantum formulation in a discrete setting. Some details can be found in [37].

**Discrete flat connections and covariant derivative.** The main idea is to find an equivalent on cells to the covariant derivative. Let us describe its parts which act on 0-cells and 1-cells. Further details can be found in [22, 35, 36], in [63] in the context of knot invariants, and also in the Chern–Simons literature (where the measure is rather the square-root of the torsion) such as [73, 101].

Let $\Delta$ be a cell decomposition of $\Sigma$, and $\Delta_i$ the set of $i$-cells. Since we focus on holonomies, we can forget that they come from a connection, and consider instead the notion of discrete connection which is the assignment of SU(2) elements to edges of $\Delta$, denoted $A = (g_e)_{e\in\Delta_1}$.

Gauge transformations induce a $\mathrm{SU}(2)^{|\Delta_0|}$-action. A discrete gauge transformation $h$ is therefore a set of group elements $(h_v)_{v\in\Delta_0}$ acting at the vertices of $\Delta$ according to

$$h \cdot A = \left(h_{t(e)} g_e h_{s(e)}^{-1}\right)_{e\in\Delta_1}.$$

To evaluate the effect of a small gauge transformation, we consider the differential at $h = \mathbb{1}$ of the map $\gamma_A$ which sends $h$ to $h \cdot A$,

$$d\gamma_{A|\mathbb{1}} : \mathfrak{su}(2)^{|\Delta_0|} \to T_A \,\mathrm{SU}(2)^{|\Delta_1|}, \qquad v \mapsto \left(R_{g_e *}\big(\mathrm{id} - \mathrm{Ad}_{g_e}\big)v\right)_{e\in\Delta_1},$$

where Ad stands for the adjoint representation of the group on its algebra, and $R_{g*}$ for the right translation. In matrix notation: $\mathrm{Ad}_g\, v = gvg^{-1}$ and $R_{g*}v = vg$. The kernel of $d\gamma_{A|\mathbb{1}}$ is the algebra of the isotropy group $\zeta(A)$ of the connection $A$, while its image corresponds to the directions along which $A$ is changed by the group action. If $\mathcal{O}_A$ denotes the orbit through $A$, then we know that

$$\mathcal{O}_A \simeq \mathrm{SU}(2)/\zeta(A), \qquad \text{and} \qquad T_A\mathcal{O}_A = \mathrm{im}\, d\gamma_{A|\mathbb{1}}.$$

The curvature of a connection is encoded in the holonomies along the boundary of 2-cells. It is represented by the map

$$H : \mathrm{SU}(2)^{|\Delta_1|} \to \mathrm{SU}(2)^{|\Delta_2|}, \qquad A \mapsto \left(H_f(A) = \prod_{e\in\partial f} g_e^{\epsilon_{fe}}\right)_{f\in\Delta_2},$$

where $\epsilon_{fe}$ is the relative orientation of the face $f$ on the edge $e$, and $H_f(A)$ is the 'holonomy' of the connection $A$ around the face $f$. This provides a notion of flatness on $\Delta$: the connection is *flat* if

$$H(A) = \mathbb{1}.$$

For a discrete flat connection $\phi$, the tangent space to the moduli space of flat SU(2)-bundles $\hom(\pi_1(\Sigma), \mathrm{SU}(2))$ is

$$T_\phi\big(\hom(\pi_1(\Sigma), \mathrm{SU}(2))/\,\mathrm{SU}(2)\big) = \ker dH_\phi / \,\mathrm{im}\, d\gamma_{\phi|\mathbb{1}}.$$

Moreover, gauge invariance implies that $dH_\phi \circ d\gamma_{\phi|\mathbb{1}} = 0$. That equation means that cohomology classes can be defined. The analogy with the continuum is that $d\gamma_{\phi|\mathbb{1}}$ is the covariant derivative on $\mathfrak{su}(2)$-valued 0-form, and $dH_\phi$ the covariant derivative on $\mathfrak{su}(2)$-valued 1-form. Hence, the tangent space $T_\phi \hom(\pi_1(\Sigma), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ is the first cohomology group. That provides with an algebraic description of the local properties of the moduli space of flat connections.

**Reidemeister torsion.** The above description gives a (co)chain complex which includes 0-, 1- and 2-cells. If $\Delta$ is a cell decomposition of dimension $d$ greater than two, the complex can be extended for any discrete flat connection $\phi$,

$$0 \longleftarrow C_\phi^d(\Delta) \xleftarrow{\delta_\phi^{d-1}(\Delta)} \cdots \longleftarrow C_\phi^2(\Delta) \xleftarrow{\delta_\phi^1(\Delta)} C_\phi^1(\Delta) \xleftarrow{\delta_\phi^0(\Delta)} C_\phi^0(\Delta) \longleftarrow 0, \tag{6.1}$$

where each cochain group basically contains assignments of a $\mathfrak{su}(2)$ element to each cells, $C^i_\phi(\Delta) \simeq \mathfrak{su}(2)^{|\Delta_i|}$. We have also relabel the previous $0,1$-cell covariant derivatives, $\delta^0_\phi = d\gamma_{\phi|\mathbb{1}}$ and $\delta^1_\phi = dH_\phi$.

Assuming the cohomology of the above complex is trivial, one can define the Reidemister torsion as follows

$$\mathrm{tor}_\phi(\Sigma) = \prod_{j=0}^{d} \big(\det\big(\delta^{j\dagger}_\phi \delta^j_\phi\big)\big)^{(-1)^j/2}.$$

Here $\dagger$ denotes the adjoint with respect to the natural $\mathfrak{su}(2)$ inner product. Obviously the operators $\delta^{j\dagger}_\phi \delta^j_\phi$ have some kernels so one must take care of removing them by restricting the operators to the orthocomplements of their kernels. Then the torsion only depends on the equivalence class $[\phi]$ of the flat connection modulo gauge and is a topological invariant in the sense that it is independent[16] of the choice of cell decomposition $\Delta$.

**Transition amplitudes.** That combinatorial torsion provides a natural analog to the Ray–Singer torsion which is well adapted to cylindrical functions. The latter depend on a finite number of holonomies, just like those used to define the torsion. Moreover, provided we consider gauge invariant wave-functions, their value on a flat discrete connection $\phi$ will actually only depend on the equivalence class $[\phi]$.

Hence, a natural candidate for BF transition amplitudes is built by gathering the torsion to get the measure and cylindrical functions evaluated on equivalence classes of flat connections as quantum states. Consider two graphs $\Gamma_{1,2}$ supporting two cylindrical functions $\psi_{\Gamma_{1,2},f_{1,2}}$. Choose a cell decomposition of $\Delta$ which contains $\Gamma_{1,2}$. Then the natural extension of the formal path integral is

$$\langle \psi_{\Gamma_1,f_1} | \psi_{\Gamma_2,f_2} \rangle = \sum_{[\phi]} \bar{f}_1([\phi])\, \mathrm{tor}_{[\phi]}(\Sigma) f_2([\phi]).$$

Notice that the dependence of $f_{1,2}([\phi])$ on their graphs is only through their homotopy classes. It implies for instance that different knotting classes of a graph embedding cannot be distinguished by that invariant.

Such a formula defines the physical inner product on the reduced Hilbert space, or equivalently the matrix elements of the projector onto those states which satisfy the flatness condition at the quantum level.

Obviously that proposal should be made more precise by building the full path integral on spacetime directly from a cell decomposition, identifying the gauge symmetries in that setting and gauge-fixing them appropriately. This has been done at the covariant level when spacetime is three-dimensional (with boundaries): the full set of gauge symmetries is described in [59] while the relationship to torsion can be found in [22]. At the canonical level, i.e. on a two-dimensional surface $\Sigma$, the transition amplitudes have been evaluated in [88] by defining the projector onto physical states which satisfy flatness. In that situation, there is no translation gauge symmetry to take care of, and the torsion is trivial, as well-known in $2d$ Yang–Mills theory [114].

**Coupling to extended matter.** As a topological field theory, the BF model is known to feature exotic statistics [12, 23], which depending on the dimension involve particles, strings and/or branes. The dynamics and quantization of BF theory coupled to such extended matter are presented in [11, 56].

[16] The full statement is that torsion is invariant under simple homotopy transformations of $\Delta$. The notion of simple homotopy is somehow intermediate between homeomorphisms and homotopy.

### 6.2 The Wheeler–DeWitt equation for quantum flat space

We showed in the previous section that the BF field theory admits an *exact* discretization, which is well-adapted to cylindrical functions, and hence to the loop quantization. If $\Delta$ is a cell decomposition of the canonical surface $\Sigma$, the projector on physical states, satisfying the flatness constraint $\mathcal{C}_f$, reads formally

$$P_\Delta = \prod_{f\in\Delta_2} \delta(H_f).$$

Let us say a few words on that projector. When written in this way, it only sees the 2-skeleton of $\Delta$, so that one can try to study it on a arbitrary 2-complex, which may or may not be the 2-skeleton of a higher dimensional cell complex. Technically, divergences arise because they are too many delta functions in the above expression. If only a 2-complex is given, the complex (6.1) is truncated to cells of dimension lower than two. Since gauge symmetries come from the action of dimensions three and higher onto the faces, divergences cannot be understood for the presence of gauge symmetries. However, the structure of the divergences are still encoded into the cohomology twisted by flat connections (truncated at dimension 2), and the rate of divergence can be extracted as exactly the dimension of the cohomology group $H^2_\phi(\Delta)$ (see [35, 36] and [34] for a summary and a comparison with other approaches to those divergences). This is an important result since the divergences are now well understood even when there cannot be interpreted as due to gauge symmetries. That may be helpful for the flat sector of full quantum gravity, since the gauge symmetries of BF are generically broken in the full theory.

We now restrict to situations where this projector is well-defined. That means either the gauge-fixing is trivial, like when $\Delta$ is a decomposition of a closed orientable two-dimensional surface $\Sigma_g$ of genus $g$ greater than two, or it is almost trivial in the sense that it is sufficient to remove a given number of faces which carry redundant delta functions. That is the case of the spherical topology, and such a gauge-fixing is described in [59].

The projector $P_\Delta$ on $\Sigma_g$ has been constructed in [88]. Its matrix elements provide a definition of the physical inner product. The projector can be unfolded in spacetime, using for instance the spin network basis. That mimics a time evolution for spin networks which actually corresponds to a spin foam model, namely the Ponzano–Regge model.

#### 6.2.1 The new Hamiltonian for 2+1 gravity

Though the result of [88] was a major progress, it was still unclear whether a Wheeler–DeWitt equation for 2+1 gravity in the spin network basis can be written. Since the BF model describes in (2+1)-dimensions pure gravity (with degenerate metrics), the canonical formalism can be performed differently than in Section 6.1.2. It can be done like in general relativity, by introducing the shift and lapse. They are respectively Lagrange multipliers for the vector constraint which generates diffeomorphisms of the canonical surface and for the scalar constraint $H$ which generates time reparametrizations,

$$V_a = E^b_i F^i_{ab}, \qquad H = \epsilon_{ij}{}^k F^k_{ab} E^a_i E^b_j.$$

So we can trade the constraint $\mathcal{C}_F$ for those two. Physical states have to satisfy the vector constraint $\widehat{V}_a|\psi\rangle = 0$ and the Wheeler–DeWitt equation

$$\widehat{H}|\psi\rangle = 0,$$

for some regularization of $V_a$ and $H$. There are a few things we know about the Hamiltonian,

- from the previous section, we know we can work on a single cell decomposition of spacetime, because $H$ should not change the graphs of cylindrical functions,

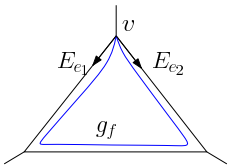


**Figure 7.** One gets the new Hamiltonian by contracting the Wilson loop $g_f$ around the face with the fluxes meeting at the starting node of $g_f$.

- but $H$ should change spins in the spin network basis, so that it probably should generate difference equations.

Note that on a two-dimensional surface $\Sigma$ the curvature tensor $F^i_{ab}$ is just a vector $\vec{F}_{12}$. Since $B^i_\mu$ is the co-triad, the vector $\vec{n} = \vec{B}_1 \times \vec{B}_2$ is the normal to the canonical surface. Then $H = \vec{n} \cdot \vec{F}_{12}/|\vec{n}|$. That is the origin of most difficulties to regularize $H$ on cylindrical functions, as we explain.

Assume that $\Delta$ is a complex dual to a triangulation. The natural variables are the Wilson lines along edges of $\Delta$ and their canonical momenta $E^i_e$ which are fluxes through those edges. The geometric picture is that the fluxes represent the edges of each triangle of the triangulation embedded into $\mathbb{R}^3$. The curvature at a node of the triangulation is regularized by the Wilson loop around the face dual to the node. However, there no clear definition of the normal $\vec{n}$ to the node.

The way-out which has been proposed in [29] is the following. If the node of the triangulation is for instance 3-valent, it is shared by three triangles, and one can define three regularized quantities looking like $H$ using the normal to each triangle. Since the three normals are generically linearly independent, that gives three constraints which are enough to impose the Wilson loop to be trivial, $H_f = \mathbb{1}$. Hence that proposal enforces flatness around the node, and thus implicitly takes into account both the scalar and vector constraints.

Let $f$ be a face and $v$ a node of $\Delta$ on the boundary of $f$ where $e_1$, $e_2$ meet. The curvature tensor is replaced with

$$\epsilon^{ij}{}_k F^k_{ab} \longrightarrow \delta^{ij} - \big(R(g_f)\big)^{ij},$$

where $g_f$ is the Wilson loop around $f$ based at $v$, and $R$ the vector representation. The products of fields $E^a_i E^b_j$ becomes the products of fluxes through $e_1$ and $e_2$, and we define (see Fig. 7)

$$H_{v,f} = E^i_{e_1}\big(\delta_{ij} - R(g_f)_{ij}\big)E^j_{e_2}.$$

For a face with $k$ vertices, there are $k$ such constraints $(H_{v_n,f})_{n=1,\dots,k}$ around $f$. Since they probe the matrix elements of $R(g_f)$ which has three independent degrees of freedom, we expect only three such constraints to be independent. That was proved in [29], and the key point is that the differentials of these constraints around the solution $g_f = \mathbb{1}$ is a system of rank 3. That result was used to prove that the constraint algebra is first class, hence it generates a gauge symmetry, on the 2-sphere triangulated by the boundary of a tetrahedron (the generic case being unclear so far).

### 6.2.2 Euclidean flat geometry

The kinematical Hilbert space gives a quantization of the intrinsic geometry. Quantizing the Gauß law (2.1) (i.e. imposing local SU(2) invariance) is indeed quantizing the law of cosines (2.2) (the Euclidean formula which evaluates angles of triangles in terms of their lengths). We now discuss the extrinsic curvature which is actually computed from the flatness constraint.

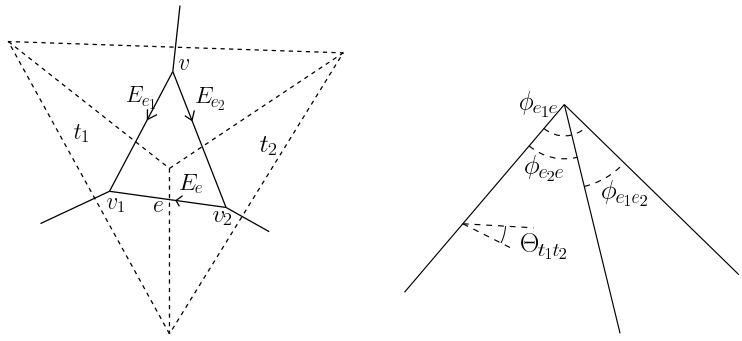


**Figure 8.** On the left, the dashed lines represent three triangles meeting at a node, and the dual graph, in normal lines, depicts a face with three vertices. On the right, the geometry of the three triangles is pictured. Flat geometry is characterized by the fact that the dihedral angle $\Theta_{t_1t_2}$ between the triangles $t_1$, $t_2$ is evaluated from the three angles $\phi_{ee'}$ at the apex, through the Euclidean formula (6.2).

Physically, what we expect from the classical flatness constraint $F^i_{ab} = 0$ is that $\Sigma$ with its geometry can be locally embedded into flat 3-space. For the triangulation dual to $\Delta$, it means that the $(3d)$ dihedral angles $\Theta_{t_1t_2}$ between two adjacent triangles $t_1$, $t_2$ should be given as a standard function of the dihedral $2d$ angles $(\phi_{ee'})$. A typical situation is depicted at Fig. 8, for a trivalent node of the triangulation. The triangles $t_1$, $t_2$ are dual to the vertices $v_1$, $v_2$ of $\Delta$, which are linked by the edge $e$. The classical angle between the two triangles $t_1$, $t_2$ can be computed from the three $2d$ angles $\phi_{ee'}$ around the node which is dual to the face $f$, using the formula

$$\cos\Theta_{t_1t_2}(E,\widetilde{E}) = \frac{\cos\phi_{e_1e_2} - \cos\phi_{e_1e}\cos\phi_{e_2e}}{\sin\phi_{e_1e}\sin\phi_{e_2e}}. \tag{6.2}$$

We have written it as a function of the flux variables since all angles $\phi_{ee'}$ can be evaluated from them, without writing down holonomies[17].

There is another natural notion of dihedral angles which makes use of flux variables and holonomies. Since the fluxes represent the directions of the edges of the triangulation, the normal $N_t$ to a triangle is given by the cross product of two of them. Following Fig. 8, $N^i_{t_1} = \epsilon^i{}_{jk}\widetilde{X}^j_e\widetilde{X}^k_{e_1}$, and $N^i_{t_2} = \epsilon^i{}_{jk}\widetilde{X}^j_{e_2}X^k_e$. It suggests to look at the dot product of these normals as being the cosine of the dihedral angle between $t_1$ and $t_2$. But for this, it is necessary to transport them in a common frame, say $N_{t_2}$ along the edge $e$ to the vertex $v_1$. So we define the angle $\theta_{t_1t_2}$ as

$$\cos\theta_{t_1t_2}(E,\widetilde{E},g) = -\frac{N_{t_1}\cdot R(g_e)N_{t_2}}{|N_{t_1}||N_{t_2}|}.$$

This is a function of the holonomy $g_e$ since one has to compare the local embedding of $t_1$ to that of $t_2$. The result can be partially evaluated in terms of the $2d$ dihedral angles, because it is a dot product of two vector products in $\mathbb{R}^3$,

$$\cos\theta_{t_1t_2}(E,\widetilde{E},g) = \frac{\widetilde{E}_1\cdot R(g_e)\widetilde{E}_2 - \cos\phi_{e_1e}\cos\phi_{e_2e}}{\sin\phi_{e_1e}\sin\phi_{e_2e}}.$$

Obviously, one would like to this notion of dihedral angle to coincide with the standard formula (6.2). This is what the Hamiltonian does [29]

$$H_{v,f} = \sin\phi_{e_1e}\sin\phi_{e_2e}\big(\cos\Theta_{t_1t_2} - \cos\theta_{t_1t_2}\big). \tag{6.3}$$

[17] In addition, when the closure relation (2.1) holds, the angles $\phi_{ee'}$ are determined as functions of the lengths $(\ell_e = |E_e|)$, and hence so are the angles $\Theta_{t_1t_2}$.

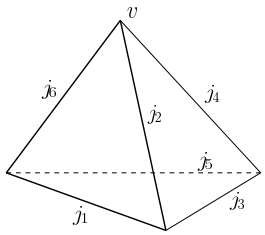


**Figure 9.** The tetrahedral graph on the 2-sphere.

Hence, dihedral angles are indeed computed from the $2d$ angles with the usual formula of flat Euclidean geometry.

### 6.2.3 The Wheeler–DeWitt equation and the pentagon identity

Let $\Delta$ be the dual graph to a triangulation of the 2-sphere by the boundary of a tetrahedron. It has four vertices (dual to the triangles), six links (dual to the six edges), and four triangular faces (dual to the nodes). A spin network function has six group elements as arguments, and six spins labeling the edges. Since the vertices are three-valent, the intertwiners are completely determined by the three spins meeting at each node, up to normalization. A common choice is to write the spin network function like

$$s_{\text{tet}}^{\{j_e\}}(g_1,\dots,g_6) = \sum_{\substack{a_1,\dots,a_6\\ b_1,\dots,b_6}} \begin{pmatrix} j_1 & j_2 & j_3 \\ b_1 & -a_2 & -a_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -a_1 & b_5 & -a_6 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ b_3 & -a_4 & -a_5 \end{pmatrix} \times \begin{pmatrix} j_2 & j_6 & j_4 \\ b_2 & b_6 & b_4 \end{pmatrix} \left[\prod_{e=1}^{6} (-1)^{j_e - a_e} \langle j_e, a_e | g_e | j_e, b_e \rangle\right].$$

The range of summation is $-j_e \leq a_e, b_e \leq j_e$ for the magnetic numbers on each link. The quantities into brackets are the components of the intertwiners in the usual spin and magnetic number basis, and are known as Wigner $3jm$-symbols [112], while $\langle j, a|g|j, b\rangle$ denotes the matrix elements of the Wigner $D$-matrix with spin $j$. Any state $\psi$ can be seen as a wave function and decomposed in that basis,

$$\psi(g_1,\dots,g_6) = \sum_{j_1,\dots,j_6} \left[\prod_{e=1}^{6} (2j_e+1)\right] \psi(j_1,\dots,j_6) s_{\text{tet}}^{\{j_e\}}(g_1,\dots,g_6).$$

We now consider the node $v$ where the edges 2 and 6 meet in the tetrahedral graph of Fig. 9 and the face $f$ with 1, 2, 6 on its boundary. The associated $H_{26} = E_2 \cdot E_6 - E_2 \cdot R(g_f)E_6$ has been quantized in [29]. $E_2 \cdot E_6$ is diagonal on spin network functions, but $E_2 \cdot R(g_f)E_6$ produces some shifts on the spin $j_1$ due to the presence of matrix elements of $g_1$ in $R(g_f)$. The latter acts by multiplication on spin networks, so that shifts are due to re-coupling between the spin $j_1$ and the spin 1 carried by the vector representation $R$. In the end the Wheeler–DeWitt equation $H_{12}|\psi\rangle = 0$ is in the spin network representation

$$A_{+1}(j_1)\psi(j_1+1,\dots,j_6) + A_0(j_1)\psi(j_1,\dots,j_6) + A_{-1}(j_1)\psi(j_1-1,\dots,j_6) = 0. \tag{6.4}$$

The coefficients are $A_+(j_1) = j_1 E(j_1+1)$ and $A_-(j_1) = (j_1-1)E(j_1)$, with:

$$E(j_1) = \Big[\big((j_2+j_3+1)^2 - j_1^2\big)\big(j_1^2 - (j_2-j_3)^2\big)\big((j_5+j_6+1)^2 - j_1^2\big)\big(j_1^2 - (j_5-j_6)^2\big)\Big]^{\frac{1}{2}},$$

and

$$A_0(j) = (2j_1+1)\Big\{2\big[j_2(j_2+1)j_5(j_5+1)+j_6(j_6+1)j_3(j_3+1)-j_1(j_1+1)j_4(j_4+1)\big] \\ -\big[j_2(j_2+1)+j_3(j_3+1)-j_1(j_1+1)\big]\big[j_5(j_5+1)+j_6(j_6+1)-j_1(j_1+1)\big]\Big\}.$$

Hence the Wheeler–DeWitt equation is a second order difference equation, which can be solved from a single initial condition[18].

This equation is sometimes taken as the definition of the Wigner $6j$-symbol, $\psi(j_1,\dots,j_6) = \left\{\begin{smallmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{smallmatrix}\right\}$. It is a basic re-coupling coefficient of quantum angular momenta theory. If one wants to find a basis of the tensor product $j_1 \otimes j_2 \otimes j_4 \otimes j_5$, one can choose a spin $j_3$ which lies in both tensor products $j_1 \otimes j_2$ and $j_4 \otimes j_5$. Another basis is obtained by choosing a spin $j_6$ shared by $j_1 \otimes j_5$ and $j_2 \otimes j_4$. The $6j$-symbol gives the corresponding change of basis.

The difference equation is thus the standard recursion relation on the $6j$-symbol. The latter comes from the Biedenharn–Elliott (pentagon) identity [112], which is the key to re-coupling SU(2) angular momenta. The above result is actually the achievement of some efforts to prove a conjecture relating the Biedenharn–Elliott identity to the Wheeler–DeWitt equation (see [19]). The main idea behind the conjecture was that there exists a state-sum model, the Ponzano–Regge model [96], which is known to give the partition function of 2+1 gravity on a 3-manifold [59, 88, 90]. It is built on labeling edges of a triangulation with spins, assigning a $6j$-symbol to each tetrahedron and summing over all labelings. When finite, it enjoys an invariance under some changes of the triangulation. Those changes are generated by 3-2 Pachner moves. The elementary move is the following. Take three tetrahedra glued along a common edge, each pair sharing a triangle, and remove those three triangles together with the common edge, and instead glue a new triangle dual to the removed edge. One ends up with only two tetrahedra glued along a triangle. This move preserves the topology, and the corresponding invariance of the Ponzano–Regge model is just due to the Biedenharn–Elliott identity. The latter thus appears as the algebraic translation of the Pachner move. As the 3-2 Pachner move is part of the topological invariance[19], a similar relation was expected in the Hamiltonian formulation.

The relationship to flat connections on the tetrahedral graph is most conveniently seen by going back to wave functions of the holonomies,

$$\psi(g_1,\dots,g_6) = \sum_{j_1,\dots,j_6}\left[\prod_{e=1}^{6}(2j_e+1)\right]\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} s_{\mathrm{tet}}^{\{j_e\}}(g_1,\dots,g_6) \\ = \delta\big(g_4^{-1}g_5g_6\big)\delta\big(g_2^{-1}g_1^{-1}g_6\big)\delta\big(g_2^{-1}g_3g_4\big).$$

[18] It is an interesting feature that the solution is determined by one initial condition and instead of two. Looking at the lowest possible value of the spin $j_1$, that is: $j_1^{\min} = \max(|j_2-j_3|,|j_5-j_6|)$, it turns out that the lowering coefficient $A_{-1}(j_1^{\min})$ evaluated on this spin is zero. Thus, the recurrence can be implemented starting from the initial value on this lowest spin. This property is important with respect to the asymptotic limit. There, it is known [97, 103, 112] that the $6j$-symbol oscillates like the cosine of the Regge action $S_{\mathrm{R}}(j_e) = \sum_e (j_e+\frac{1}{2})\theta_e$, for a tetrahedron with lengths $(j_e+\frac{1}{2})$,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \approx \frac{1}{\sqrt{12\pi V(j_e)}}\frac{1}{2}\left(e^{i(S_{\mathrm{R}}(j_e)+\frac{\pi}{4})}+e^{-i(S_{\mathrm{R}}(j_e)+\frac{\pi}{4})}\right).$$

Having a second order equation, it may have been expected to be solved in the asymptotics by both the positive and negative exponentials of the action. But the model is fully independent from the orientation: there is only one required initial condition, because of the vanishing of $A_{-1}(j_1^{\min})$, leading in the large spin regime to the cosine of the action.

[19] The full topological invariance needs the 4-1 Pachner move, but unfortunately it is divergent, because the model as itself does not gauge fix the non-compact gauge symmetry at the root of the topological invariance. Gauge-fixing in the Ponzano–Regge model is described in [59] and finiteness in [22].

The above state is exactly the projector onto flat connections for the chosen graph and topology[20].

As a remark, notice that the naive state would have a delta function on each of the four faces of the graph. However, that would be ill-defined since one delta function would be trivially satisfied thanks to the other. So it is usually removed by hand. Here, the state is well-defined as a distribution and that is an outcome of our approach.

### 6.2.4 Extra remarks

**Symmetry in the Boulatov model for 3*d* gravity.** The latter is a group field theory, i.e. a non-local field theory on a group, whose Feynman expansion generates a sum over simplicial complexes supporting spin foam amplitudes, see [18], and more remarks on Section 6.3.3. Interestingly, a global symmetry of the model has been found in [18] and there argued to be the translation of the diffeomorphism symmetry of $3d$ gravity in group field theory. The symmetry involves an action of the Drinfeld double of SU(2), and there is indeed a quite well established approach to quantize $3d$ gravity by going from the classical group ISU(2) (which takes into account both the standard gauge symmetry and the translation of the co-triad), to the quantum group $\mathcal{D}\,\mathrm{SU}(2)$ at the quantum level. We do not comment further as we are not expert, but refer to [18] instead.

A direct connection with the quantization of the scalar constraint we have just presented is obtained from the way the global symmetry of [18] is expressed in the spin basis. It is a recursion relation on the $6j$-symbol, which can be proved from (6.4). It also derives directly from an operator acting on wave-functions, with the corresponding quantum constraint,

$$\big[\mathrm{tr}_j(g_f) - (2j+1)\big]\psi = 0,$$

where $g_f$ is the Wilson loop around a plaquette. In the spin basis, for a plaquette with $n$ links labeled $e = 1, \dots, n$, it gives

$$\sum_{j_1,\dots,j_n} (-1)^{j+\sum_{e=1}^{n} j_e+k_e+l_e} \left[\prod_{e=1}^{n} d_{j_e}\right] \begin{Bmatrix} k_2 & j_2 & j \\ j_1 & k_1 & l_1 \end{Bmatrix} \begin{Bmatrix} k_3 & j_3 & j \\ j_2 & k_2 & l_2 \end{Bmatrix} \dots \begin{Bmatrix} k_1 & j_1 & j \\ j_n & k_n & l_n \end{Bmatrix} \psi(j_e) = (2j+1)\psi(k_e). \tag{6.5}$$

That equation is obviously solved by the evaluation of the spin network on the identity.

There is an appealing geometric interpretation of that equation, as generating a quantum tent move evolution. We use the Ponzano–Regge interpretation [59] of the $6j$-symbol: it is a weight associated a tetrahedron whose edge lengths are given by $(j_e+1/2)$. A Ponzano–Regge amplitude is then obtained by taking products of $6j$-symbols corresponding to a gluing of tetrahedra, and summing over the spins of the internal dual edges, keeping those on the boundary fixed.

That interpretation shows that the equation (6.5) is actually computing the Ponzano–Regge amplitude for a piece of a $3d$ triangulation built as follows. The plaquette is dual to a vertex $s$ of a $2d$ triangulation. A new triangulation, identical to the first, is obtained by evolving $s$ into another vertex $s^*$. The piece of $3d$ triangulation is contained between the two triangulations, and is made from $n$ tetrahedra all sharing the edge $(ss^*)$.

This evolution process in background independent approaches to lattice gravity is presented in [13] at the classical level, and further details are given in Section 6.3.1. In the equation (6.5), the evolved state $\psi(k_e)$ is obtained by summing over all admissible values of the spins $(j_e)$ on the initial surface, and the constraint just imposes that the physical state must be invariant under tent moves weighted by their Ponzano–Regge amplitudes.

[20] If one gauge-fixes, using the SU(2) action at each node, the group elements $g_1$, $g_2$, $g_3$ to the unit, the delta functions impose $g_4 = g_5 = g_6 = \mathbb{1}$, which is the unique flat connection up to gauge on the sphere

Notice that since we are dealing with a flat spacetime, the quantum length of $(ss^*)$, i.e. $j+1/2$, is a gauge degree of freedom, which simply corresponds to the lapse. It can be chosen arbitrarily, and naively summing over all values of this length would result in a divergence[21].

**Thiemann's Hamiltonian in 3*d* gravity.** The regularization and quantization of the scalar constraint has been performed by Thiemann [109] following the prescriptions he used in 3+1 gravity. The construction is mathematically well-defined, as the continuum limit exists and there are no anomalies. The aim was, like in the new proposal we have presented, to use 2+1 gravity as a testing ground, to be able to compare with the already known quantization. It turns out that the solutions to Thiemann's quantum constraints contain the usual solutions of [115], but also other, unusual solutions.

**Non-zero cosmological constant.** The major remaining challenge for the loop quantization of pure three-dimensional gravity is to include the cosmological constant $\Lambda \neq 0$. The partition function and expectation values of observables are known in the case $\Lambda > 0$ using the Chern–Simons formulation [113] which makes it clear it is still topological. In particular the partition function is given by the Turaev–Viro model, which looks similar to the Ponzano–Regge model, expect that all objects from SU(2) representation theory are changed to their counterpart for the *quantum group* $U_q(\mathrm{SU}(2))$, with $q = e^{i\sqrt{\Lambda}}$. Hence, it seems that from the loop quantization view, one has to trade off Clebsch–Gordan coefficients and $6j$-symbols (and maybe spin networks themselves) for their $q$-deformed versions. It is still unclear why and how that should be taken care of precisely by the constraints in the loop quantization. Some recent progress have been obtained in [89], where the Kauffman bracket is derived by combining standard SU(2) spin networks with the Chern–Simons (non-commutative) connection. Interestingly, this comes out just from kinematical considerations. While this strengthens the robustness of the loop quantization itself, it also leads to some puzzle. To extend the Kauffman bracket, which is defined for the spin 1/2, one has to tensor several spins 1/2. However, it must be followed by some symmetrization process, and the usual choice in LQG is the standard symmetrizer of SU(2). But to get the correct extension of the Kauffman bracket, one would have to use the $q$-symmetrizer instead. There is no reason at the kinematical level to do so, and it is further unclear why the dynamics would change the symmetrizer.

In the continuum, turning on $\Lambda$ implies a deformation of the constraint algebra where the Gauß law is unchanged but the curvature is obviously non-zero anymore,

$$F^i_{ab} + \Lambda \epsilon^i{}_{jk} e^j_a e^k_b = 0,$$

where $e^i_a$ is the pullback of the triad to the canonical surface. The algebra is first class, it is a Lie algebra corresponding to SO(4), where the Gauß law generates rotations and the constraint on the curvature generates the Euclidean boosts.

The most natural regularizations of the constraints in the Hilbert space of the loop quantization produce some anomalies in the algebra [89, 95]. It seems to indicate that since the algebra itself is changed, the usual regularizations (used for $\Lambda = 0$) must be amended to reproduce a first class algebra.

A hint may come from representation theory again. When $\Lambda = 0$, we have two types of constraints, the Gauß law and the flatness, whose geometric contents are respectively the law of cosines (the formula which evaluates angles of triangles from lengths) (2.2) and a formula to evaluate dihedral angles between triangles from the angles within triangles (6.3). At the quantum level, those constraints generate difference equations solved by classical SU(2) recoupling objects. It is quite clear for the closure relation: if $e_1$, $e_2$, $e_3$ meet at a node, the constraint $(\widehat{E}_1 + \widehat{E}_2 + \widehat{E}_3)|\psi\rangle = 0$ produces recursion relations satisfied by Clebsch–Gordan

[21] For a triangular plaquette, this is nothing but the well-known divergence of the 1-4 Pachner move already observed by Ponzano and Regge. The quantum tent move presented here is a gauge-fixed version.

coefficients (or $3mj$-symbols). Moreover, we have described above a regularization of the scalar constraint which leads to a difference equation on the $6j$-symbol.

Geometrically, the Gauß law regulated as the closure relation for the triangle (2.1) only makes sense for a flat triangle. If geometry is homogeneously positively curved, we would expect the regulated Gauß law to describe spherical triangles instead. In terms of lengths and angles that should be the spherical law of cosines,

$$\cos\phi_{e_1e_2} = \frac{\cos\ell_{e_3} - \cos\ell_{e_1}\cos\ell_{e_2}}{\sin\ell_{e_1}\sin\ell_{e_2}},$$

where $\ell_e$ is the spherical length of the edge $e$. From that view, the second constraint, that relating dihedral angles to angles of triangles, is actually unchanged, (6.2),

$$\cos\Theta_{t_1t_2} = \frac{\cos\phi_{e_1e_2} - \cos\phi_{e_1e}\cos\phi_{e_2e}}{\sin\phi_{e_1e}\sin\phi_{e_2e}}.$$

To make those ideas more precise, in particular to quantize, it is necessary to know the expression of the above geometric relations in terms of holonomies and fluxes, or to have the symplectic structure which involves spherical lengths.

**Conjecture.** *Based on the results we have reviewed in the case $\Lambda = 0$, one can conjecture the following. Recursions satisfied by q-deformed Clebsch–Gordan coefficients and the q-deformed $6j$-symbol are quantizations of the above formulae for spherical triangles and tetrahedra.*

Finding a regulated Gauß law whose geometric content is curved triangles is certainly the way to get an anomaly-free regularization. This is actually what comes from the covariant analysis of discretized gravity by Dittrich and collaborators which we describe in Section 6.3. In a nutshell, the discrete action with $\Lambda \neq 0$ has no gauge symmetry when flat simplices are used. To restore the exact symmetry at the discrete level while turning on $\Lambda > 0$, it is possible to use a discretization based on homogeneously curved simplices. Geometrically, that is very natural and quite obvious: that means that translating the nodes of the lattice is a symmetry if it is done by moving on the curved manifold.

**Four-dimensional extension.** A direct extension of the new Hamiltonian is presented in [27]. Its form is the same, built on holonomies and fluxes. It still acts on cycles independently, and thus necessarily related to the topological model which imposes flatness of the connection. On the boundary of a 4-simplex, triangulating the 3-sphere, the Wheeler–DeWitt equation is a difference equation solved by Wigner $15j$-symbols.

More generally it is easily seen that the Hamiltonian we have discussed produces difference equations which are solved by Wigner symbols. It is actually not surprising, but it is instead a confirmation that the formalism is under control. Indeed, Wigner symbols are evaluation of spin networks on the trivial connection, and as such they naturally arise in the quantization of the BF model.

**Large spin asymptotics and loop quantum gravity geometries.** The large spin limit of the recursion (6.4) makes the geometric picture quite clear,

$$\big[\Delta_e + 2(1 - \cos\Theta_e)\big]\psi(j_1, \dots, j_6) = 0, \tag{6.6}$$

where $\Delta_e$ is the discrete Laplacian on the edge $e$, and $\Theta_e$ the dihedral angle at $e$, computed from the lengths. Such an equation can be straightforwardly derived from a simple dynamics in Regge calculus. The canonical variables are the edge lengths $l_e$ and momenta $\theta_e$, and the constraint reads

$$\cos\theta_e - \cos\Theta_e(l) = 0,$$

which means that the momenta are fully determined as dihedral angles between triangles for *flat* space.

A similar constraint was proposed for flat space in 3+1 dimensions in [47], with lengths replaced with triangle areas, and it was argued that they form a first class system. As a convergence of ideas and approaches, it turns out that the BF model provides a realization at the quantum level of such a constraint for flat space. Indeed, the large spin behavior of the difference equation coming from $H_{v,f}$ on the boundary of the 4-simplex, using coherent state methods, takes exactly the form (6.6), where spins label triangles and geometrically stand for quantum areas instead of quantum lengths [27].

However, we would like to stress that the Hamiltonian on holonomies and fluxes goes beyond Regge calculus in the 3+1 dimensional case, in a sense we now explain. In three dimensions, all flux configurations which satisfy the closure equation are Regge geometries, determined by length variables. We simply say that loop quantum gravity geometries are Regge. But the situation is different in four dimensions, because a tetrahedron is not determined by its areas. The result of the full analysis [47] is that SU(2) loop quantum gravity geometries are discontinuous: each tetrahedron unambiguously describes a flat Regge geometry, but different tetrahedra will generically describe their common triangles with different lengths. Such geometries were later called *twisted geometries* [61].

Instead of going further on the kinematical aspects (see [47, 100]), we simply want to emphasize that, as far as flat space is concerned, the Hamiltonian is well-defined on the *whole* phase space of loop quantum gravity, while the large spin behavior (6.6) only describes the geometric sector, where geometries are really of the Regge type. For non-Regge configurations, the exact difference equation takes a different asymptotic form, which is consistent with asymptotics of different symbols in that regime [20]. In the geometric sector, (6.6) can be taken as a criterion for a quantum geometry model to be approximated with some quantum Regge calculus in the large spin limit.

**More on recursion relations.** The study of $H_{v,f}$ and other recursion relations on Wigner symbols has suggested a nice feature of quantum BF theory, that geometric properties of quantum flat space are encoded into recursion relations coming from group representation theory. An example, discussed in [27, 31, 33] is the closure of the simplex. In the asymptotics, the idea is to combine (6.6) with the classical closure which reads $\det(\cos\Theta_e(l)) = 0$, i.e. the vanishing of the Gram matrix. Remarkably, the $10j$-symbol at the core of the four-dimensional Barrett–Crane model is known to satisfy such an equation, but in an exact way [33]. This leads to expect that in the geometric sector of full quantum gravity, geometric properties are still probed through difference equations which arise from representation theory.

### 6.2.5 What we learn

In addition to its own interest as a new way of quantizing 2+1 gravity using the scalar Hamiltonian, this analysis gives the following insights.

- The spin 1 is the natural spin to regularize the curvature. It makes it possible to solve the Wheeler–DeWitt equation for all values of the state on integers. Hence, that solves the model for SO(3). By contrast, working with a spin 2 regularization produces a difference equation with shifts $\pm1$, $\pm2$, so that more initial conditions are needed to get the state on all integers [29]. Solving the model for SU(2) instead of SO(3) however requires a spin 1/2 regularization of the curvature, which is accessible in the spinor formalism [84].
- The splitting between spatial diffeomorphisms and the scalar component finds an unexpected realization in this exact model. Indeed, the usual tension is that the curvature at a node of a triangulation is regularized by Wilson loop around it, while the scalar Hamiltonian is defined by contracting the curvature with the normal to the surface, $H \sim \vec{n} \cdot \vec{F}$.

The way-out which turns out to be successful is to take a different normal for each triangle meeting at a node, three of them being linearly independent. From that view the above construction is more something which mimics the scalar constraint on spin networks.

- The asymptotic equation (6.6) is a key feature of the quantum simplex in the large quantum number regime, and can used as a criterion for a model to reproduce Regge calculus there.
- Classical geometric properties of flat space can be implemented at the quantum level as difference equations coming from group representation theory. This strengthens the core of spin foam models where spacetime dynamics is expressed with objects from group representations.
- The loop quantization supports a non-trivial Hamiltonian, not only in the asymptotics but also for twisted geometries in four dimensions.
- No divergences are present, because the generator of the gauge symmetry is quantized and not the projector. That improves the spherical case with respect to [88], and it should be a good framework to study perturbations with respect to the topological order.

## 6.3 Beyond the topological model

### 6.3.1 Broken symmetries

The above BF model is topological and enjoys an exact lattice realization which allows the evaluation of transition amplitudes between spin network states using a single cell decomposition of spacetime (containing the graphs). But generically in a theory which does have local degrees of freedom, a lattice formulation is a truncation which misses some degrees of freedom and hence breaks the symmetries. In a series a papers Bahr and Dittrich and collaborators [13, 14, 15, 17], and Dittrich and Höhn [46], studied the discretization of reparametrization invariant systems, especially the Regge approximation to general relativity. Main issues are how to restore broken symmetries, and how to define a canonical discrete formulation which reproduces the dynamics of the discrete action. Here we sum up some ideas of [13].

Starting from a triangulation of the initial canonical hypersurface, a local evolution with discrete time steps can be defined in terms of *tent moves*. Let $\Delta_{(d-1)}$ be the initial triangulation, and $s$ a node. A tent move on $s$ creates a new node $s^*$, and new links which join $s^*$ to all the nodes $s$ was connected to. That defines a new canonical triangulation $\Delta^*_{(d-1)}$, on which the number of variables is unchanged. Equivalently, the move glues a piece $T$ of a $d$-dimensional triangulation between $\Delta_{(d-1)}$ and $\Delta^*_{(d-1)}$, and the link $(ss^*)$ is called the tent pole.

The independent variables on $T$ are the lengths. To get momenta and equations of motion on $\Delta_{(d-1)}$ and $\Delta^*_{(d-1)}$, one uses the action on $T$ as a generating function. When $d=3$, it reads

$$S_T = -\sum_{e\subset \mathrm{int}T} l_e\varepsilon_e - \sum_{e\subset\partial T} l_e\psi_e^T + \Lambda\sum_v V_v.$$

The deficit angles around the edges $e$ in the bulk of $T$ and on the boundary are

$$\varepsilon_e = \left(2\pi - \sum_{v\supset e}\vartheta_{ev}\right), \qquad \text{and} \qquad \psi_e^T = \left(\pi - \sum_{v\supset e}\vartheta_{ev}\right),$$

where $\vartheta_{ev}$ is the dihedral angle hinged at $e$ in the tetrahedron $v$. $V_v$ is the volume of $v$, and $\Lambda$ the cosmological constant. Denote $s_i$ the nodes connected to $s$, with lengths $l_i$ and $l^*$ the tent pole length. The momenta are then

$$p_i = -\frac{\partial S_T}{\partial l_i}, \qquad p_i^* = \frac{\partial S_T}{\partial l_i^*}, \qquad p = -\frac{\partial S_T}{\partial l^*}.$$

From the equations of motion, we get $p = 0$. Moreover, performing a second tent move, it can be seen that there is a single well-defined momentum per edge.

Symmetries on a solution can be observed from the existence of zero eigenvalues of the Hessian matrix evaluated on that solution. In the case of vanishing cosmological constant, a symmetry is found in [13] around the flat solution. It corresponds to moving a internal vertex in flat 3-space (after several tent moves are performed), and those degrees of freedom are interpreted as the lapse and the shift. That translates into constraints on the canonical data on $\Delta_{(d-1)}$ only,

$$p_i = -\pi + \vartheta_{ss_i}(l_i, l_{ij}), \tag{6.7}$$

in agreement with the fact that $3d$ gravity supports a discretization with exact symmetries. Those constraints state that the momenta are identified with the dihedral angles determined by the lengths. This is analogous to the constraints (6.3) considered in the previous section. We thus see a convergence of idea coming from different approaches to the same issue.

However, when turning on the cosmological constant, it was observed that there are no symmetry around homogeneously curved solutions. Therefore, there are no constraints anymore, but instead pseudo-constraints. That means that the equations of motion from $T$ translates in the canonical setting into equations which depend on the data living on two consecutive time slices, and not a single one like in (6.7).

### 6.3.2 Restoring symmetries and the continuum limit

In that example (homogeneous curvature), it is nevertheless quite clear how one can discretize in an exact way. The correct symmetry should be translation of a vertex not in flat space but following the curved solution. Hence, one should use curved simplices instead of the flat ones, so as to get the constraint

$$p_i = -\pi + \vartheta^{(\Lambda)}_{ss_i}(l_i, l_{ij}),$$

where the dihedral angle $\vartheta^{(\Lambda)}_{ss_i}$ is evaluated within the curved tetrahedron.

The main result of this approach is a canonical framework which reproduces the discrete action. The presence of symmetries is therefore determined by the choice of the discrete action. That view was further expanded in [46] by studying the expansion of Regge calculus around a background metric, and in [14]. In the latter, the key idea is to restore the symmetries which are broken upon discretization by going to the *perfect action*. That perfect action is reached, at least formally, in the continuum limit. Hence, it can be approached by a coarse-graining process at the classical level. Let $\Delta_2$ be a triangulation whose edges are denoted $E$, and $\Delta_1$ a finer triangulation with edges $e$. We will obtained from the Regge action on $\Delta_1$ with flat simplices, an *improved action* on $\Delta_2$. Let us fix the lengths $L_E$, and solve the equations of motion for $l_e$ with a homogeneously curved geometry determined by $\Lambda$. The action on the coarse-grained triangulation will be obtained thanks to the constraint $\sum_{e\subset E} l_e = L_E$. The improved action is then

$$S_{\Delta_2}(L_E) = S_{\Delta_1}(l^*_e)_{|\sum_{e\subset E} l^*_e = L_E},$$

where the lengths $l^*_e$ are solutions of the curved geometry. The action for $\Delta_2$ consequently takes into account the dynamics of the finer triangulation. Repeating the process many times, the internal geometry of the simplices of the most coarse triangulation receives more and more contributions from the curvature of the finer ones, and thus in the appropriate limit, each simplex becomes homogeneously curved. That leads to the Regge discretization which has the exact symmetries.

While the topological BF model supports an exact discrete version, gauge symmetries are generically broken. Restoring them (and hence getting the correct number of degrees of freedom, in particular of propagating degrees of freedom) necessitates to go to the continuous limit. It can be reached as above by coarse-graining (recursive integration of the equations of motion) to yield at least formally a discrete action with exact symmetries, or by taking the number of simplices to infinity.

The work of Dittrich and collaborators was recently re-interpreted by Rovelli [99]. The desired reparametrization invariance (diffeomorphisms) to be recovered was there called Ditt-invariance. One key feature is that the continuum limit is reached by sending the number of discrete steps of the parameters of the theory to infinity, while the physical time step is sent to zero in a dynamical way. In the case of the harmonic oscillator, portions where the trajectory can be approximated by a free particle need very little time steps. On the contrary portions where the potential is important need more points to describe the trajectory accurately. It turns out that the discretization in the arbitrary time parameter together with energy conservation leads to that picture in terms of the physical time parameter.

The discretization and coarse-graining process is discussed for reparametrization invariant systems at the quantum level in [17].

### 6.3.3 The continuum limit through matrix and tensor models

The notion of continuum limit at the quantum (or statistical) level has been given a sense in matrix models (for two-dimensional surfaces) [44] and tensor models (in higher dimensions) [4, 67, 102]. These are field theories which generate sums over triangulations, and as such the latter can be controlled. The free energy of a typical model reads

$$F = \sum_{n\geq 0} e^{-\mu n} \sum_{\substack{\text{triangulations } T_n \\ \text{with } n \text{ simplices}}} s(T_n) A(T_n).$$

Here $T_n$ are triangulations of the sphere[22] with $n$ simplices, $s(T_n)$ is a combinatorial factor, and $A(T_n)$ the amplitude assigned by the model to a given triangulation. The feature we want to stress here comes from the chemical potential $\mu$. When it is very large, only small $n$ contribute. However, when it decreases, the sum loses its summability. This is the regime where $F$ is dominated by an infinite number of simplices, and it is obtained for a (non-zero) critical value of $\mu \to \mu_c$. This continuum limit is approached like $F \simeq \left(e^{-\mu_c} - e^{-\mu}\right)^{2-\gamma}$. In the simplest models, where $A(T_n) = 1$, the sum at fixed $n$ just counts the number of triangulations with a given number of vertices, so that $\gamma$ is known as the entropy exponent. In the matrix model, it is $\gamma = -1/2$ and in higher dimensions, $\gamma = 1/2$.

Such models have a natural interpretation as *dynamical triangulations*, in the regime of small Newton constant and large cosmological constant. It is also possible to modify the amplitudes $A(G)$ to accommodate loop quantum gravity, or rather spin foam, amplitudes. That is the purpose of group field theories, which re-sums spin foam amplitudes, and much of the recent progress in tensor models have been motivated by the group field theory program.

## 7 Conclusion

Since the original definition of Hamiltonian constraint by Thiemann, several new approaches to tackling quantum dynamics in LQG have emerged, and the quantum constraint, still the most

[22] They are isolated from the full sums over triangulations of all topologies using the large $N$ limit [42, 68, 74], where $N$ is the typical size of the matrices or tensors. The triangulations to be summed over may moreover be a specific subset of all triangulations of the sphere [30].

popular of all its rivals in mini-superspace models, has taken a backseat in the full theory. The infinite number of ambiguities, lack of spacetime covariance at quantum level, and ultra-locality are some prime reasons why it is even now considered to be a highly unsatisfactory definition.

We have argued that the quantum spacetime covariance, i.e. the off-shell closure of the constraint algebra, could be achieved if one was ready to broaden one's perspective on all possible choices that are available in defining composite operators, thanks to the discontinuity of representations and its inherent non-local nature. Already in loop quantum cosmology and in parametrized field theory, one sees the fruits of such endeavours and ends up with regularization schemes which remarkably enough have never been considered in the full theory.

Applying these lessons to the diffeomorphism constraint bears some real fruits. A precise definition of the quantum constraint which generates infinitesimal diffeomorphisms on states in the LM habitat, and whose kernel matches with the well-known diffeomorphism invariant space $\mathcal{H}_{\rm diff}$. Once again the choices one has to make in order to get such an operator are very different then the ones which have been looked at in the literature so far.

We have then presented a different road based on the loop quantization of the BF topological model. It includes pure 2+1 gravity, where a recent Hamiltonian, [27, 29], was found to mimic the scalar constraint while reproducing the standard results. That shows that the framework supports non-trivial operators, with a nice geometric content: the result is a quantization of flat (Euclidean) geometry. More generally, that model shows that classical properties of flat geometry take the form of difference equations from group representation theory, in the spin network quantization scheme. It also shed some light on the quantization ambiguities, such as curvature regularization and its spin, and is close to Smolin's proposal [104] to avoid ultra-locality. However, to go further, the continuum limit should be investigated, beyond the topological model. We have presented some directions in the context of discrete gravity, where symmetries are broken, but can be restored upon coarse-graining, leading to the perfect action.

The remaining issues in 2+1 gravity are mainly to write explicitly the quantum algebra of the new Hamiltonian. It should be interesting since it does not implement the splitting between spatial diffeomorphisms and the scalar component in the usually expected way. That was a surprise and it should certainly be considered seriously also in 3+1 general relativity.

This unexpected splitting of the algebra observed in 2+1 gravity speaks in favor of the spin foam formalism, because it is a space-time formulation on cell decompositions, and hence naturally compatible with the geometry arising from the new Hamiltonian. The difference equations coming from group representation to describe quantum flat geometry also point towards spin foam models, since they are built from group Fourier transforms. In particular, the asymptotic form of the Wheeler–DeWitt equation derived through the new Hamiltonian (for the topological model) is satisfied on the quantum 4-simplex by some spin foam models [27]. However, in 3+1 dimensions, exhausting the gauge symmetries is subtler (due to reducibility). Hence, in spite of those quantum difference equations which match with the spin foam model for 3+1 BF theory [91], the latter is not well-defined yet: it is not known how to gauge-fix it properly. We think it is a serious issue which should be resolved if one wants to understand what the diffeomorphism symmetry becomes in the flat or almost flat sector of the full theory, and how the internal gauge symmetry of BF theory is broken on non-flat solutions.

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