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Pseudo-Exponential-Type Solutions of Wave Equations Depending on Several Variables

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Abstract. Using matrix identities, we construct explicit pseudo-exponential-type solutions of linear Dirac, Loewner and Schrödinger equations depending on two variables and of nonlinear wave equations depending on three variables.

Key words: Bäcklund–Darboux transformation; matrix identity; S-node; S-multinode; explicit solution; non-stationary Dirac equation; non-stationary Schrödinger equation; Loewner system; pseudo-exponential-type potential; integrable nonlinear equations

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1 Introduction

The term *pseudo-exponential potentials* was introduced in [20] (see Remark 1.2 on interrelations between pseudo-exponential-type potentials and multi-soliton solutions). Ordinary linear differential equations with the so called pseudo-exponential-type potentials were actively studied (see [14, 15, 20, 21, 22, 38, 41] and references therein), since their solutions could be constructed explicitly (and inverse problems to recover these equations from rational Weyl functions or reflection coefficients could be solved explicitly). Thus, pseudo-exponential-type potentials and solutions, that is, potentials and solutions, which, roughly speaking, rationally depend on matrix exponentials, are of a special interest. When matrices in the matrix exponentials (from the rational functions of matrix exponentials) are nilpotent, purely rational functions (potentials) appear as an important subcase of the pseudo-exponential-type potentials. For a more rigorous definition of the term *pseudo-exponential potential* see, for example, [15, 20].

Explicit solutions of linear and nonlinear wave equations are important both in theory and applications. The theory is well-developed for the case of linear equations depending on one variable and nonlinear integrable equations depending on two variables and includes, in particular, algebro-geometric methods and several versions of the commutation methods and of Bäcklund–Darboux transformations (BDTs), see some results and various references in [10, 12, 16, 17, 18, 23, 32, 41, 52]. In spite of numerous interesting results on the cases of more variables (see, e.g., [1, 5, 7, 8, 13, 31, 33, 34, 37, 46, 48, 49]), these cases are more complicated and contain also more open problems.

Matrix identities are actively used in this theory for the cases of one and several space variables starting from the seminal work [30]. By matrix (or operator) identities we mean an important subclass of so called Sylvester equations $AX - YB = Q$, which are considered, for instance, in control theory. Namely, matrix identities are equations of the form $AR - RB = Q$ or, more often, $AR - RB = \Pi_1\Pi_2^*$ (see, e.g., [35, 42, 44]) with Π_k of comparatively small rank. V.A. Marchenko [30] was the first to apply matrix and operator identities in this topic (see [46] and references therein for further developments of his approach). In another way (more precisely, for the construction of τ-functions) matrix identities were used in [25]. Our approach is based on the GBDT (generalized BDT) approach, which was introduced in [35, 36] (see further results and many references in [14, 15, 20, 39, 41]). Although the papers [35, 36] were initiated by [30], matrix identities in [30] and in GBDT are used in quite different ways. Moreover, solutions of the nonlinear equations are constructed in [30] as reductions of expressions of the form $\Gamma^{-1}\Gamma_x$ whereas GBDT is a kind of a binary Darboux transformation and solutions are expressed via matrix functions $\Phi_2^* S^{-1}\Phi_1$. (Here Γ, Φ_1 and Φ_2 satisfy some simple auxiliary linear systems.) See, for instance, (4.5) for solutions in terms of $\Phi_2^* S^{-1}\Phi_1$. Matrices of a much lesser order have to be inverted in GBDT when constructing, for instance, matrix solutions of nonlinear equations. In addition, Darboux matrices and wave functions are constructed explicitly using GBDT. The method develops during the last 20 years. Moreover, after the publication of [35, 36] a very close approach was used by M. Manas (see some comparative analysis in [10]) and related formulas are now successfully used by Mueller-Hoissen and coauthors (see, e.g., [13]).

In our paper we apply multidimensional versions of the GBDT. That is, we follow [37] (where S-nodes introduced in [42, 43, 44] were applied to matrix Kadomtsev–Petviashvili equations) and the S-multinodes approach from [40] in order to construct explicitly pseudo-exponential-type potentials and solutions of some important equations of mathematical physics depending on several variables. The transfer to S-multinodes is required in many examples because the same matrix should satisfy several matrix identities. S-multinodes first appeared in [40] as a certain generalization of the S-nodes on one hand and commutative colligations (introduced by M.S. Livšic [27]) on the other hand.

A symmetric S-multinode (r-node) is a set of matrices

$$\{A_1, \dots, A_r; \nu_1, \dots, \nu_r; R; \widehat{C}\}$$

such that for $1 \le i,\, k \le r$ the relations

$$A_i A_k = A_k A_i, \qquad A_k R + R A_k^* = \widehat{C}\nu_k \widehat{C}^*, \qquad R = R^*, \qquad \nu_k = \nu_k^* \tag{1.1}$$

hold. Here we shall deal with the cases $r = 1, 2, 3$. In the case $r = 1$ we have the well-known symmetric S-node introduced by L.A. Sakhnovich (see, e.g., [41, 42, 43, 44, 45] for various applications). For $r > 1$ the situation is more complicated, since R in general position is defined already by one of the identities $A_k R + R A_k^* = \widehat{C}\nu_k\widehat{C}^*$. However, the construction of S-multinodes proves both possible and useful.

Remark 1.1. In our further considerations the matrices in the S-multinode or S-node (i.e., matrices in (1.1)) are constant and each S-multinode generates a potential and solution of a linear (or solution of a nonlinear) equation.

Remark 1.2. We note that pseudo-exponential-type solutions are close to multi-soliton solutions and their analogues. However, multi-soliton solutions are usually generated when matrices A_i are diagonal, whereas we do not require A_i to be necessarily diagonal. This correspondence for the solutions of sine-Gordon and sinh-Gordon equations was studied in [35, Section 4]. In particular, it was shown in [35] that solutions of sine-Gordon equation from [9, 24] are derived in this way (i.e., using S-nodes with diagonal matrices A_1).

Explicit solutions of linear equations (especially, of non-stationary Dirac and Schrödinger equations) are of wide interest, and in Section 2 we use 2-nodes in order to study the case of the non-stationary Dirac system

$$H\Psi = 0, \qquad H := \frac{\partial}{\partial t} + \sigma_2 \frac{\partial}{\partial y} - \mathrm{i}V(t,y), \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad V = V^*, \tag{1.2}$$

which presents more difficulties than the non-stationary (time-dependent) Schrödinger equation considered in [40]. Some new results for the non-stationary Schrödinger equation are derived in Section 3. Thus, we fill in the gap between papers [37] and [40], consider a class of solutions of the Schrödinger equation, which is wider than the one discussed in [1], and construct interesting examples.

Section 4 is dedicated to the nonlinear integrable equations. As examples we consider matrix Davey–Stewartson I (DS I) and generalized nonlinear optics equations. In particular, our approach allows to construct a wide class of rational solutions of matrix DS I (see Remark 4.3).

Remark 1.3. GBDT results for DS I and generalized nonlinear optics equation were obtained in [36, Section 3] but no examples were given. Here we construct wide classes of solutions using the S-node (S-multinode) approach, see Propositions 4.2 and 4.6. We note that GBDT results in [36, Section 3] include the case of nonzero background (in which situation auxiliary linear systems play a more essential role) and it would be very interesting to generalize S-multinode approach for that case.

As usual, $\mathbb{N}$ denotes the set of natural numbers, const stands for a constant (number or matrix), $\mathrm{Im}(A)$ stands for the image of the matrix A, $\sigma(D)$ stands for the spectrum of D, $[G,F]$ stands for the commutator $GF - FG$, $\otimes$ stands for Kronecker product, I_p is the $p \times p$ identity matrix, and $\Psi_{tx} := \frac{\partial}{\partial x}\big(\frac{\partial}{\partial t}\Psi\big) = \frac{\partial^2}{\partial x \partial t}\Psi$. By $\mathrm{diag}\{b_1, b_2, \dots, b_m\}$ we denote the diagonal matrix with the entries $b_1, b_2, \dots$ on the main diagonal.

2 Dirac and Loewner equations: explicit solutions

2.1 Non-stationary Dirac equation

We note that in the GBDT version of the Bäcklund–Darboux transformation the solution of the transformed equation is represented in the form $\Pi^* S^{-1}$, where Π^* is a matrix solution of the initial equation and the matrix function S is constructed using the S-node (see, e.g., [39, 41] and references therein). Here we construct solutions of (1.2) in the same form. Namely, we set

$$\Pi = C E_A(t,y) \widehat{C}, \qquad E_A = \exp\{tA_1 + yA_2\}, \qquad A_1 A_2 = A_2 A_1, \qquad \widehat{C} = \begin{bmatrix} g_1^* & g_2^* \end{bmatrix}, \tag{2.1}$$

where $\widehat{C}$ is an $N \times 2$ matrix, g_1^* and g_2^* are columns of $\widehat{C}$, A_1 and A_2 are $N \times N$ matrices and C is an $n \times N$ matrix ($n, N \in \mathbb{N}$). We emphasize that the matrices A_1, A_2, $\widehat{C}$ and C are constant (see also Remark 1.1). We assume that the equalities

$$g_1 A_1^* - \mathrm{i} g_2 A_2^* = 0, \qquad g_2 A_1^* + \mathrm{i} g_1 A_2^* = 0 \tag{2.2}$$

hold. From (2.1) and (2.2), we easily see that

$$H_0 \Pi^* = 0, \qquad H_0 := \frac{\partial}{\partial t} + \sigma_2 \frac{\partial}{\partial y}, \tag{2.3}$$

where H_0 is applied to Π^* columnwise.

Recall that matrices A_1, A_2, R, ν_1, ν_2 and $\widehat{C}$ form a symmetric 2-node if A_1 and A_2 commute and the following identities are valid:

$$A_k R + R A_k^* = \widehat{C} \nu_k \widehat{C}^*, \qquad k = 1, 2, \qquad R = R^*, \qquad \nu_k = \nu_k^*. \tag{2.4}$$

It is immediate that the matrix function

$$S(t,y) = S_0 + CE_A(t,y)RE_A(t,y)^*C^*, \qquad S_0 = S_0^* \equiv \text{const}, \tag{2.5}$$

satisfies equations $\frac{\partial}{\partial t}S = \Pi\nu_1\Pi^*$ and $\frac{\partial}{\partial y}S = \Pi\nu_2\Pi^*$. These equations and equation (2.3) yield the proposition below.

Proposition 2.1. *Let relations (2.1), (2.2), (2.4) and (2.5) hold and assume that $\nu_1 = \sigma_2$, $\nu_2 = -I_2$. Then, in the points of invertibility of S, we have*

$$H\big(\Pi(t,y)^*S(t,y)^{-1}\big) = 0,$$

where H has the form (1.2) with V defined by

$$V := \mathrm{i}\big(\Pi^*S^{-1}\Pi\sigma_2 - \sigma_2\Pi^*S^{-1}\Pi\big).$$

The important part of the problem is to find the cases where the conditions of Proposition 2.1 hold. Then we obtain families of explicitly constructed potentials V and solutions Π^*S^{-1} of the corresponding Dirac systems.

Example 2.2. Set $g_2 = -\mathrm{i}g_1j_n$, $A_1 = D = \mathrm{diag}\{D_1, D_2\}$ (where D_1 and D_2 are $n_1\times n_1$ and $n_2\times n_2$ diagonal blocks of the diagonal matrix D, $n_1+n_2 = n$, $\sigma(D_k)\cap\sigma(-D_k^*) = \varnothing$ for $k = 1,2$), $A_2 = Dj_n$ and

$$j_n := \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2}\end{bmatrix}, \qquad R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22}\end{bmatrix}.$$

We uniquely define R_{11} and R_{22} by the matrix identities

$$D_1R_{11} + R_{11}D_1^* = -g_1^*(I_n + j_n)g_1, \qquad D_2R_{22} + R_{22}D_2^* = g_1^*(I_n - j_n)g_1.$$

Then the conditions of Proposition 2.1 hold.

Thus, according to Proposition 2.1 and Example 2.2, each vector g_1 and diagonal matrix D (such that $\sigma(D_k)\cap\sigma(-D_k^*) = \varnothing$) determine a set (depending on the choice of C and S_0) of pseudo-exponential-type potentials and explicit solutions of (1.2).

2.2 Loewner's system

Loewner's system has the form

$$\Psi_x = \mathcal{L}(x,y)\Psi_y, \tag{2.6}$$

where $\mathcal{L}$ is an $m\times m$ matrix function. For the case $m = 2$, this system was studied by C. Loewner in the seminal paper [28] and applications to the hodograph equation were obtained. In [29], C. Loewner rewrote in this way the system $x_\eta - y_\xi = 0$, $(\rho x)_\xi + (\rho y)_\eta = 0$, which describes a steady compressible and irrotational flow of an ideal fluid. For the Loewner's system, its transformations, generalizations and applications, see also [47, 50] and references therein. (For some special kinds of similarity transformations of $\mathcal{L}$ see also [28, formulas (5.10a) and (5.27)].) Direct calculation proves the following proposition.

Proposition 2.3. *Let $m\times m$ and $m\times n$, respectively, matrix functions Λ_1 and Λ_2 satisfy a linear differential equation*

$$(\Lambda_i)_x = q_1(x,y)(\Lambda_i)_y + q_0(x,y)\Lambda_i, \qquad i = 1,2,$$

where the coefficients q_0 and q_1 are some $m\times m$ matrix functions. Then, in the points of invertibility of Λ_1, the matrix function $\Psi = \Lambda_1^{-1}\Lambda_2$ satisfies the Loewner equation (2.6), where

$$\mathcal{L} = \Lambda_1^{-1}q_1\Lambda_1.$$

Pseudo-exponential-type Ψ and $\mathcal{L}$ are constructed in the next proposition.

Proposition 2.4. *Introduce $m \times m$ and $m \times n$, respectively, matrix functions Λ_1 and Λ_2 by the equalities*

$$\Lambda_i = \mathcal{C}_i E_A(x,y,i)\widehat{\mathcal{C}}_i, \qquad i = 1,2, \tag{2.7}$$
$$E_A(x,y,i) := \exp\{x\breve{A}_i + y\widetilde{A}_i\}, \qquad \breve{A}_i := D \otimes A_i, \qquad \widetilde{A}_i := I_m \otimes A_i,$$
$$D = \mathrm{diag}\{d_1, \dots, d_m\}, \qquad \mathcal{C}_i := \sum_{k=1}^{m} (e_k e_k^*) \otimes (e_k^* c_i),$$

where A_i are $l_i \times l_i$ matrices, c_i are $m \times l_i$ matrices, $\widehat{\mathcal{C}}_1$ is an $N_1 \times m$ matrix, $\widehat{\mathcal{C}}_2$ is an $N_2 \times n$ matrix, $N_i = ml_i$ and $l_i \in \mathbb{N}$. Here $\otimes$ is Kronecker product, e_k is a column vector given by $e_k = \{\delta_{jk}\}_{j=1}^m$ and δ_{jk} is Kronecker's delta.

Then, in the points of invertibility of Λ_1, the matrix functions

$$\Psi = \Lambda_1^{-1}\Lambda_2 \qquad \text{and} \qquad \mathcal{L} = \Lambda_1^{-1} D \Lambda_1$$

satisfy (2.6).

Proof. It is easy to see that Λ_1 and Λ_2 given by (2.7) satisfy equation $(\Lambda_i)_x = D(\Lambda_i)_y$. Now, Proposition 2.4 follows from Proposition 2.3. ■

In a similar (to the construction of Λ_i in the proposition above) way, matrix functions Π satisfying (4.19) are constructed in (4.21)–(4.23).

3 Non-stationary Schrödinger equation: explicit solutions and examples

We consider the subcase of [40, Theorem 3.2], where $\mathcal{S}_0 = \mathcal{S}_0^*$, and use notations Π instead of Ψ_0, S instead of $\mathcal{S}$ and S_0 instead of $\mathcal{S}_0$. We substitute

$$\alpha = \mathrm{i}, \qquad k = 1, \qquad A_1 = A, \qquad B_1 = -A^*, \qquad \nu_1 = I_p,$$
$$C_\Phi = \widehat{C}, \qquad C_\Psi = \widehat{C}^*, \qquad \widehat{C}_\Phi = C, \qquad \widehat{C}_\Psi = C^*$$

into [40, formula (3.1) and Theorem 3.2]. For this particular case, Theorem 3.2 from [40] takes the following form.

Proposition 3.1. *Fix some $p, n, N \in \mathbb{N}$, an $N \times N$ matrix A, an $n \times N$ matrix C, an $N \times p$ matrix $\widehat{C}$ and an $n \times n$ matrix $S_0 = S_0^*$. Let $R = R^*$ satisfy the matrix identity*

$$AR + RA^* = \widehat{C}\widehat{C}^*, \tag{3.1}$$

and put

$$\Pi(x,t) = Ce_A(x,t)\widehat{C}, \qquad e_A(x,t) := \exp\{xA - \mathrm{i}tA^2\}, \tag{3.2}$$
$$S(x,t) = S_0 + Ce_A(x,t)Re_A(x,t)^*C^*. \tag{3.3}$$

Then, the matrix function $\widetilde{\Pi}^* := \Pi^* S^{-1}$ satisfies the vector non-stationary Schrödinger equation

$$H(\widetilde{\Pi}^*) = 0, \qquad H := \mathrm{i}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - \widetilde{q}(x,t), \tag{3.4}$$

where H is applied to $\widetilde{\Pi}^*$ columnwise and $\widetilde{q}$ is the $p \times p$ matrix function:

$$\widetilde{q}(x,t) = -2\big(\Pi(x,t)^* S(x,t)^{-1}\Pi(x,t)\big)_x. \tag{3.5}$$

Our approach allows to consider the cases of non-diagonal matrices A, and we adduce below several examples, where A is a 2×2 Jordan cell. Using some simple calculations, we easily construct e_A, Π, S and, finally, solution $\widetilde{\Pi}^*$ and potential $\widetilde{q}$ in the following example of a scalar Schrödinger equation.

Example 3.2. Let us put

$$p = 1, \qquad N = n = 2, \qquad A = \begin{bmatrix} \mu_0 & 1 \\ 0 & \mu_0 \end{bmatrix}, \qquad \widehat{C} = \begin{bmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{bmatrix}, \qquad S_0 = \begin{bmatrix} 0 & b \\ \overline{b} & d \end{bmatrix}. \tag{3.6}$$

Formulas (3.1) and (3.6) yield (for $R = \{r_{ij}\}_{i,j=1}^2$) the equality

$$AR + RA^* = \varkappa R + \begin{bmatrix} r_{12} + r_{21} & r_{22} \\ r_{22} & 0 \end{bmatrix}, \qquad \varkappa := \mu_0 + \overline{\mu}_0. \tag{3.7}$$

From the definition of A we also obtain

$$e_A(x,t) = \mathrm{e}^{\mu_0 x - \mathrm{i}\mu_0^2 t} \left(I_2 + \begin{bmatrix} 0 & x - 2\mathrm{i}\mu_0 t \\ 0 & 0 \end{bmatrix} \right). \tag{3.8}$$

Assume (in addition to (3.6)) that

$$\varkappa := \mu_0 + \overline{\mu}_0 = 0, \qquad \widehat{c}_1 = 1, \qquad \widehat{c}_2 = 0, \qquad C = I_2. \tag{3.9}$$

Taking into account (3.7) and the first three equalities in (3.9), we see that the relations $R = R^*$ and (3.1) are equivalent to the equalities

$$r_{11} = \overline{r_{11}}, \qquad r_{21} = \overline{r_{12}}, \qquad r_{12} + \overline{r_{12}} = 1, \qquad r_{22} = 0. \tag{3.10}$$

In view of (3.2), (3.3), (3.8) and (3.9), we have

$$\Pi(x,t) = \mathrm{e}^{\mu_0 x - \mathrm{i}\mu_0^2 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad S(x,t) = S_0 + \begin{bmatrix} 1 & x - 2\mathrm{i}\mu_0 t \\ 0 & 1 \end{bmatrix} R \begin{bmatrix} 1 & 0 \\ x - 2\mathrm{i}\mu_0 t & 1 \end{bmatrix}. \tag{3.11}$$

Here we took into account that $\varkappa = 0$ yields $|\mathrm{e}^{\mu_0 x - \mathrm{i}\mu_0^2 t}| = 1$. From (3.5), (3.6), (3.10) and (3.11), after some simple calculations we derive

$$\begin{aligned} &\widetilde{\Pi}(x,t)^* = \Pi(x,t)^* S(x,t)^{-1} = \big(c + d(x - 2\mathrm{i}\mu_0 t)\big)^{-1} \mathrm{e}^{\mathrm{i}\mu_0^2 t - \mu_0 x} \begin{bmatrix} d & -r_{12} - b \end{bmatrix}, \\ &c := d r_{11} - |r_{12} + b|^2, \\ &\Pi(x,t)^* S(x,t)^{-1} \Pi(x,t) = d\big(c + d(x - 2\mathrm{i}\mu_0 t)\big)^{-1}, \\ &\widetilde{q}(x,t) = 2d^2 \big(c + d(x - 2\mathrm{i}\mu_0 t)\big)^{-2}. \end{aligned} \tag{3.12}$$

Clearly, this potential $\widetilde{q}$ is rational, depends on one variable $x - 2\mathrm{i}\mu_0 t$ and has singularity at certain values of $x, t \in \mathbb{R}$. According to Proposition 3.1, each entry of $\widetilde{\Pi}^*$ of the form (3.12) (in our case these entries are collinear) satisfies the Schrödinger equation with the potential $\widetilde{q}$, which is given above.

In the following example, the potential $\widetilde{q}$ is rational and depends on two real-valued variables x and t or, equivalently, on one complex-valued variable $P := x - \mathrm{i}\mu_0 t$ (and its complex conjugate $\overline{P}$).

Example 3.3. Put $p = n = 1$, $N = 2$, $S_0 = 0$,

$$A = \begin{bmatrix} \mu_0 & 1 \\ 0 & \mu_0 \end{bmatrix}, \qquad \varkappa := \mu_0 + \overline{\mu}_0 > 0, \qquad \widehat{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{3.13}$$

Using (3.7), we immediately check that

$$R = \varkappa^{-1} \begin{bmatrix} 2\varkappa^{-2} & -\varkappa^{-1} \\ -\varkappa^{-1} & 1 \end{bmatrix}. \tag{3.14}$$

Taking into account (3.3), (3.8), (3.13) and (3.14), we easily calculate

$$S(x,t) = \varkappa^{-1}\big|e^{\mu_0 P(x,t)}\big|^2\big(2\varkappa^{-2} - \varkappa^{-1}(P(x,t) + \overline{P}(x,t) + 2) + |P(x,t)+1|^2\big). \tag{3.15}$$

We sometimes omit the variables x, t in our further formulas. In view of (3.2), (3.8), (3.13) and (3.15) we derive

$$\Pi^* S^{-1} \Pi = \frac{\varkappa |P+1|^2}{2\varkappa^{-2} - \varkappa^{-1}(P + \overline{P} + 2) + |P+1|^2}.$$

The rational potential $\widetilde{q}$, which is given by (3.5), takes the form

$$\widetilde{q} = \frac{2\big((P+1)^2 + (\overline{P}+1)^2 - 2\varkappa^{-1}(P+\overline{P}+2)\big)}{(2\varkappa^{-2} - \varkappa^{-1}(P+\overline{P}+2) + |P+1|^2)^2}. \tag{3.16}$$

Finally, the solution $\widetilde{\Pi}^* = \Pi^* S^{-1}$ of the Schrödinger equation, where the potential $\widetilde{q}$ has the form (3.16), is given by the formula:

$$\widetilde{\Pi}^* = \frac{\varkappa e^{-\mu_0 P(x,t)}(x + 2i\overline{\mu}_0 t + 1)}{2\varkappa^{-2} - \varkappa^{-1}(P(x,t) + \overline{P}(x,t) + 2) + |P(x,t)+1|^2}.$$

It was shown in [37] that if $\sigma(iA) \subset \mathbb{C}_+$ and the pair A, $\widehat{C}$ is full range, i.e.,

$$\operatorname{span} \bigcup_{\ell=0}^{N-1} \operatorname{Im}\big(A^\ell \widehat{C}\big) = \mathbb{C}^N,$$

then the solution R of (3.1) is unique and positive-definite, that is, $R > 0$. Hence, we obtain our next proposition.

Proposition 3.4. *Assume that $\sigma(iA) \subset \mathbb{C}_+$, the pair A, $\widehat{C}$ is full range,* rank $C = n$ *and $S_0 \geq 0$. Then we have $S(x,t) > 0$. Therefore, $S(x,t)$ is invertible and the potential $\widetilde{q}$ is nonsingular.*

In our next example we deal with a nonsingular pseudo-exponential potential depending on two variables.

Example 3.5. Let the parameter matrices A, $\widehat{C}$ and S_0 have the form (3.6). Instead of the relations (3.9), we assume now that

$$\varkappa := \mu_0 + \overline{\mu}_0 > 0, \qquad \widehat{c}_1 = 0, \qquad \widehat{c}_2 = 1, \qquad b = 0, \qquad d > 0, \qquad C = I_2. \tag{3.17}$$

Like in Example 3.3, formula (3.7) again yields (3.14). Taking into account (3.2), (3.3), (3.6), (3.8), (3.14) and (3.17) we calculate

$$\Pi^* S^{-1} \Pi = Z_1/Z_2, \qquad Z_1 = 2\varkappa^{-3} + d\big|e^{\mu_0 P}\big|^{-2}|P|^2,$$
$$Z_2 = \varkappa^{-4} + \varkappa^{-1} d \big|e^{\mu_0 P}\big|^{-2}\big(|P|^2 - \varkappa^{-1}(P + \overline{P}) + 2\varkappa^{-2}\big), \qquad P := x - i\mu_0 t.$$

Next, one easily obtains the derivatives of Z_1 and Z_2 with respect to x:

$$(Z_1)_x = -\varkappa(Z_1 - 2\varkappa^{-3}) + d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}(P + \overline{P}),$$
$$(Z_2)_x = -\varkappa(Z_2 - \varkappa^{-4}) + \varkappa^{-1}d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}\big(P + \overline{P} - 2\varkappa^{-1}\big).$$

Hence, in view of (3.5) and formulas for Z_k and $(Z_k)_x$ above, we have

$$\begin{aligned}\widetilde{q} &= -2\big(\Pi^* S^{-1}\Pi\big)_x \\ &= 2\varkappa^{-2}Z_2 - \varkappa^{-3}Z_1 + 2\varkappa^{-2}d|e(\mu_0)|^{-2}Z_1 + d|e(\mu_0)|^{-2}(P+\overline{P})\big(Z_2 - \varkappa^{-1}Z_1\big) \\ &= -\frac{2d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}}{Z_2^2}\big(8\varkappa^{-5} - 3\varkappa^{-4}(P+\overline{P}) \\ &\quad + \varkappa^{-3}\big(|P|^2 + 2d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}(P+\overline{P})\big) + \varkappa^{-2}d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}\big(2|P|^2 - (P+\overline{P})^2\big)\big).\end{aligned}$$

The solution $\widetilde{\Pi}^* = \Pi^* S^{-1}$ of (3.4) is given (in our case) by the formula

$$\widetilde{\Pi}^* = \big(\mathrm{e}^{-\mu_0 P}/Z_2\big)\left[\varkappa^{-2} + d\left|\mathrm{e}^{\mu_0 P}\right|^{-2}\overline{P} \quad 2\varkappa^{-3} - \varkappa^{-2}P\right].$$

4 Nonlinear integrable equations

Among $(2+1)$-dimensional integrable equations, Kadomtsev–Petviashvili, Davey–Stewartson (DS) and generalized nonlinear optics (also called N-wave) equations are, perhaps, the most actively studied systems. S-nodes were applied to the construction and study of the pseudo-exponential, rational and nonsingular rational (so called multi-lump) solutions of the Kadomtsev–Petviashvili equations in [37]. Here we investigate the remaining two equations from the three above.

4.1 Davey–Stewartson equations

The Davey–Stewartson equations are well-known in wave theory (see, e.g., [6, 11, 23, 26] and references therein). Since Davey–Stewartson equations (DS I and DS II) are natural multidimensional generalizations of the nonlinear Schrödinger equations (NLS), their matrix versions should also be of interest (similar to matrix versions of NLS, see, e.g., [4]).

1. The matrix DS I has the form

$$\mathrm{i}u_t - (u_{xx} + u_{yy})/2 = uq_1 - q_2 u, \tag{4.1}$$
$$(q_1)_x - (q_1)_y = \frac{1}{2}\big((u^*u)_y + (u^*u)_x\big), \qquad (q_2)_x + (q_2)_y = \frac{1}{2}\big((uu^*)_y - (uu^*)_x\big), \tag{4.2}$$

where u, q_1 and q_2 are $m_2 \times m_1$, $m_1 \times m_1$ and $m_2 \times m_2$ matrix functions, respectively ($m_1 \geq 1$, $m_2 \geq 1$). We note that another matrix version of the Davey–Stewartson equation, where $m_1 = m_2$, was dealt with in [26]. It is easy to see that in the scalar case $m_1 = m_2 = 1$ equations (4.1) and (4.2) are equivalent, for instance, to [23, p. 70, system (2.23)] (after setting in (2.23) $\varepsilon = \alpha = 1$).

GBDT version of the Bäcklund–Darboux transformation for the matrix DS I was constructed in [36]. When the initial DS I equation (in GBDT for DS I, see [36, Theorem 5]) is trivial, that is, when we set (in [36]) $u_0 \equiv 0$ and $Q_0 \equiv 0$, Theorem 5 from [36] takes the form:

Proposition 4.1. *Let an $n \times m$ ($n \in \mathbb{N}$, $m = m_1 + m_2$) matrix function Π and an $n \times n$ matrix function S satisfy equations*

$$\Pi_x = \Pi_y j, \qquad \Pi_t = -\mathrm{i}\Pi_{yy} j, \qquad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2}\end{bmatrix}, \tag{4.3}$$

$$S_y = -\Pi\Pi^*, \qquad S_x = -\Pi j \Pi^*, \qquad S_t = \mathrm{i}(\Pi_y j \Pi^* - \Pi j \Pi_y^*). \tag{4.4}$$

Partition Π into $n \times m_1$ and $n \times m_2$, respectively, blocks Φ_1 and Φ_2 (i.e., set $\Pi =: \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$). Then, the matrix functions

$$u = 2\Phi_2^* S^{-1}\Phi_1, \qquad q_1 = \frac{1}{2}u^*u - 2\big(\Phi_1^* S^{-1}\Phi_1\big)_y, \qquad q_2 = -\frac{1}{2}uu^* + 2\big(\Phi_2^* S^{-1}\Phi_2\big)_y \tag{4.5}$$

satisfy (in the points of invertibility of S) DS I system (4.1), (4.2).

Introduce Φ_1, Φ_2 and S via relations

$$\Phi_1(x,t,y) = C_1 E_1(x,t,y)\widehat{C}_1, \qquad E_1(x,t,y) := \exp\big\{(x+y)A_1 - \mathrm{i}tA_1^2\big\}, \tag{4.6}$$

$$\Phi_2(x,t,y) = C_2 E_2(x,t,y)\widehat{C}_2, \qquad E_2(x,t,y) := \exp\big\{(x-y)A_2 + \mathrm{i}tA_2^2\big\}, \tag{4.7}$$

$$\begin{aligned} S(x,t,y) = S_0 &+ C_1E_1(x,t,y)R_1E_1(x,t,y)^*C_1^* \\ &- C_2E_2(x,t,y)R_2E_2(x,t,y)^*C_2^*, \qquad S_0 = S_0^*, \end{aligned} \tag{4.8}$$

where C_1 and C_2 are $n \times N$ matrices, $A_1, A_2, R_1 = R_1^*$ and $R_2 = R_2^*$ are $N \times N$ matrices, $\widehat{C}_1$ and $\widehat{C}_2$ are $N \times m_1$ and $N \times m_2$, respectively, matrices, S_0 is an $n \times n$ matrix and the following identities hold:

$$A_1R_1 + R_1A_1^* = -\widehat{C}_1\widehat{C}_1^*, \qquad A_2R_2 + R_2A_2^* = -\widehat{C}_2\widehat{C}_2^*. \tag{4.9}$$

It is immediate from (4.6)–(4.9) that $\Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$ and S satisfy relations (4.3) and the first two relations in (4.4). In order to prove the third equality in (4.4), we note that

$$\begin{aligned} \big(C_1E_1R_1E_1^*C_1^*\big)_t &= -\mathrm{i}C_1E_1\big(A_1^2R_1 - R_1\big(A_1^2\big)^*\big)E_1^*C_1^* \\ &= -\mathrm{i}C_1E_1\big(A_1(A_1R_1 + R_1A_1^*) - (A_1R_1 + R_1A_1^*)A_1^*\big)E_1^*C_1^* \\ &= \mathrm{i}\big((\Phi_1)_y\Phi_1^* - \Phi_1(\Phi_1^*)_y\big). \end{aligned} \tag{4.10}$$

Here we used (4.6) and the first identity in (4.9).

In a similar way we show that

$$\big(C_2E_2R_2E_2^*C_2^*\big)_t = \mathrm{i}\big((\Phi_2)_y\Phi_2^* - \Phi_2(\Phi_2^*)_y\big). \tag{4.11}$$

Equalities (4.8), (4.10) and (4.11) yield the last equality in (4.4). Hence, the conditions of Proposition 4.1 are valid, and so we proved the following proposition.

Proposition 4.2. *Let Φ_1, Φ_2 and S be given by the formulas (4.6)–(4.8) and assume that (4.9) holds. Then, the matrix functions u, q_1 and q_2 given by (4.5) satisfy (in the points of invertibility of S) DS I system (4.1), (4.2).*

Remark 4.3. It is easy to see that if $\sigma(A_1) = \sigma(A_2) = 0$, then Φ_1, Φ_2 and S are rational matrix functions. Thus, if $\sigma(A_1) = \sigma(A_2) = 0$, the solutions u, q_1 and q_2 of the DS I system, which are constructed in Proposition 4.2, are also rational matrix functions.

Remark 4.4. Note that matrices considered in (4.9) form two separate S-nodes or, equivalently, an S-node, where R is a block diagonal matrix and the matrix identity

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} R + R \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix} = -\begin{bmatrix} \widehat{C}_1\widehat{C}_1^* & 0 \\ 0 & \widehat{C}_2\widehat{C}_2^* \end{bmatrix}, \qquad R := \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

is valid. Another example of a block diagonal matrix R is dealt with in Subsection 4.2. It would also be of interest to compare solutions of the same system constructed using r_1-nodes and r_2-nodes ($r_1 \neq r_2$).

2. The compatibility condition $w_{tx} = w_{xt}$ of the auxiliary systems

$$w_x = \pm \mathrm{i} j w_y + jVw, \qquad w_t = 2\mathrm{i} j w_{yy} \pm 2jVw_y \pm jQw, \tag{4.12}$$

where

$$V = \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} q_1 & u_y \mp \mathrm{i} u_x \\ u_y^* \pm \mathrm{i} u_x^* & -q_2 \end{bmatrix}, \tag{4.13}$$

$$q_k(x,t) = -q_k(x,t)^*, \qquad k = 1,2, \tag{4.14}$$

is equivalent (for the case that the solution w is a non-degenerate matrix function) to the matrix DS II equation

$$u_t + \mathrm{i}(u_{xx} - u_{yy}) = \pm(q_1 u - u q_2), \tag{4.15}$$

$$(q_1)_x \mp \mathrm{i}(q_1)_y = (uu^*)_y \mp \mathrm{i}(uu^*)_x, \qquad (q_2)_x \pm \mathrm{i}(q_2)_y = (u^*u)_y \pm \mathrm{i}(u^*u)_x. \tag{4.16}$$

As we see from (4.12)–(4.16), there are two versions of auxiliary systems and corresponding DS II equations. After setting $m_1 = m_2 = 1$ (and setting also $\varepsilon = 1$, $\alpha = \mp \mathrm{i}$ in [23, p. 70, system (2.23)]), like for the scalar DS I case, equations (4.15) and (4.16) are equivalent to [23, p. 70, (2.23)].

Open problem. Use the approach from Proposition 4.1 in order to construct explicit pseudo-exponential solutions of the matrix DS II.

We note that various results on DS II, including BDT results, are not quite analogous to the results on DS I (see, e.g., [23]). A quasi-determinant approach to explicit solution of noncommutative DS equations is presented in [19].

4.2 Generalized nonlinear optics equation

The integrability of the generalized nonlinear optics equation (GNOE)

$$[D, \xi_t] - [\widetilde{D}, \xi_x] = \big[[D,\xi],[\widetilde{D},\xi]\big] + D\xi_y\widetilde{D} - \widetilde{D}\xi_y D, \tag{4.17}$$

$$\xi(x,t,y)^* = B\xi(x,t,y)B, \qquad B = \operatorname{diag}\{b_1, b_2, \dots, b_m\}, \qquad b_k = \pm 1, \tag{4.18}$$

$$D = \operatorname{diag}\{d_1, d_2, \dots, d_m\} > 0, \qquad \widetilde{D} = \operatorname{diag}\{\widetilde{d}_1, \widetilde{d}_2, \dots, \widetilde{d}_m\} > 0$$

was dealt with in [2, 53]. This system is a generalization of the well-known N-wave (nonlinear optics) equation $[D,\xi_t] - [\widetilde{D}, \xi_x] = [[D,\xi],[\widetilde{D},\xi]]$ first studied in [51] (see also [3]). GBDT version of the Bäcklund–Darboux transformation for GNOE was constructed in [36]. When the initial system in GBDT for GNOE [36, Theorem 4] is trivial (i.e., $\xi_0 \equiv 0$), Theorem 4 from [36] takes the form:

Proposition 4.5. *Let an $n \times m$ matrix function Π and an $n \times n$ matrix function S satisfy equations*

$$\Pi_x = \Pi_y D, \qquad \Pi_t = \Pi_y \widetilde{D}, \tag{4.19}$$

$$S_y = -\Pi B \Pi^*, \qquad S_x = -\Pi B D \Pi^*, \qquad S_t = -\Pi B \widetilde{D} \Pi^*. \tag{4.20}$$

Then the matrix function

$$\xi = \Pi^* S^{-1} \Pi B$$

satisfies (in the points of invertibility of S) GNOE (4.17) and reduction condition (4.18).

In order to construct pseudo-exponential-type solutions ξ, we will consider matrix functions Π and S of the form (2.1) and (2.5), respectively, where E_A will depend on three variables and $N = ml$, $l \in \mathbb{N}$. Namely, we set

$$\Pi(x,t,y) = CE_A(x,t,y)\widehat{C}, \qquad E_A(x,t,y) = \exp\{xA_1 + tA_2 + yA_3\}, \tag{4.21}$$

$$A_1 = D \otimes A, \qquad A_2 = \widetilde{D} \otimes A, \qquad A_3 = I_m \otimes A, \tag{4.22}$$

$$\widehat{C} = \sum_{k=1}^{m} (e_k e_k^*) \otimes (\widehat{c} e_k), \qquad e_k = \{\delta_{ik}\}_{i=1}^m \in \mathbb{C}^m, \tag{4.23}$$

where C is an $n \times N$ matrix, A is an $l \times l$ matrix, $N = ml$, $\otimes$ is Kronecker product, $\widehat{c}$ is an $l \times m$ matrix, e_k is a column vector and δ_{ik} is Kronecker's delta. It is immediate that the matrices A_k ($k = 1, 2, 3$) commute. Hence, we see that matrices A, C and $\widehat{c}$ determine (via (4.21)–(4.23)) matrix function Π satisfying (4.19).

Proposition 4.6. *Let relations (4.21)–(4.23) hold and set*

$$S(x,t,y) = S_0 + CE_A(x,t,y)RE_A(x,t,y)^*C^*, \qquad S_0 = S_0^*, \tag{4.24}$$

where the $N \times N$ matrix R ($N = ml$, $R = R^*$) satisfies matrix identities

$$A_1R + RA_1^* = -\widehat{C}BD\widehat{C}^*, \qquad A_2R + RA_2^* = -\widehat{C}B\widetilde{D}\widehat{C}^*, \tag{4.25}$$

$$A_3R + RA_3^* = -\widehat{C}B\widehat{C}^*. \tag{4.26}$$

Then, the matrix function $\xi = \Pi^* S^{-1} \Pi B$ satisfies (in the points of invertibility of S) GNOE (4.17) and reduction condition (4.18).

Proof. We mentioned above that Π given by (4.21)–(4.23) satisfies (4.19). Moreover, relations (4.21) and (4.24)–(4.26) yield (4.20). Thus, the conditions of Proposition 4.5 are fulfilled. ∎

We note that, according to (4.23), the right-hand sides of the equalities in (4.25) and (4.26) are block diagonal matrices with $l \times l$ blocks. Therefore, we will construct block diagonal matrix R, the blocks R_{kk} of which are also $l \times l$ matrices:

$$R = \operatorname{diag}\{R_{11}, R_{22}, \dots, R_{mm}\}. \tag{4.27}$$

Taking into account (4.22), we see that for R of the form (4.27) identities

$$AR_{kk} + R_{kk}A^* = -b_k(\widehat{c}e_k)(\widehat{c}e_k)^*, \qquad 1 \le k \le m, \tag{4.28}$$

imply that identities (4.25) and (4.26) hold.

Corollary 4.7. *Let relations (4.21)–(4.23) and (4.28) hold. Then, the matrix function $\xi = \Pi^* S^{-1} \Pi B$, where S is given by (4.24) and (4.27), satisfies (in the points of invertibility of S) GNOE (4.17) and reduction condition (4.18).*

Remark 4.8. If $\sigma(A) \cap \sigma(-A^*) = \varnothing$, there exist unique solutions R_{kk} satisfying (4.28). For that case we have also $R_{kk} = R_{kk}^*$ (i.e., $R = R^*$). Clearly, R_{kk} is immediately recovered if $\sigma(A) \cap \sigma(-A^*) = \varnothing$ and A is a diagonal matrix.

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