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# On a Yang–Mills Type Functional

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**Abstract.** We study a functional that derives from the classical Yang–Mills functional and Born–Infeld theory. We establish its first variation formula and prove the existence of critical points. We also obtain the second variation formula.

*Key words:* curvature; vector bundle; Yang–Mills connections; variations

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## 1 Motivations

Let $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then the graph of $u$

$$G_u = \left\{(x,z) \in \mathbb{R}^{n+1} \,|\, z = u(x),\, x \in \Omega\right\},$$

is a minimal hypersurface if and only if satisfies the following differential equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{1.1}$$

In 1970 Calabi, in a paper in which he studied examples of Bernstein problems, noticed that if $n = 2$, $u$ is an $F$-harmonic map for the function $F(t) = \sqrt{1+2t} - 1$. We recall that $u$ is an $F$-harmonic map if it is a critical point of the following functional

$$E_F(u) = \int_{\mathbb{R}^2} F\left(\frac{\|du\|^2}{2}\right) \vartheta_g,$$

with respect to any compactly supported variation, $\|du\|^2$ being the Hilbert–Schmidt norm.

Following Calabi's ideas, Yang and then Sibner showed that for $n = 3$, the equation (1.1) is equivalent, over a simply connected domain, to the vector equation

$$\nabla \times \left(\frac{\nabla \times A}{\sqrt{1+|\nabla \times A|^2}}\right) = 0,$$

which arises in the nonlinear electromagnetic theory of Born and Infeld. Here $A$ is a vector field in $\mathbb{R}^3$ and $\nabla \times (\,\cdot\,)$ is the curl of $(\,\cdot\,)$. Born–Infeld theory is of contemporary interest due to its relevance in string theory.

This observation lead Yang to give a generalized treatment of equation (1.1), expressed in terms of differential forms, as follows:

$$\delta\left(\frac{d\omega}{\sqrt{1+\|d\omega\|^2}}\right) = 0, \tag{1.2}$$

for any $\omega \in A^p(\mathbb{R}^4)$. It is not very difficult to verify that the solution of equation (1.2) is a critical point of the following integral

$$\int_{\mathbb{R}^4} \left(\sqrt{1+\|d\omega\|^2}-1\right)\vartheta_g.$$

These facts give us the motivation to study a similar functional, namely the Yang–Mills–Born–Infeld functional

$$\mathrm{YM}_{\mathrm{BI}}(D) = \int_M \left(\sqrt{1+\left\|R^D\right\|^2}-1\right)\vartheta_g,$$

defined more generally on Riemannian manifolds. The definition of the above functional is similar to the definition of the well-known Yang–Mills functional (see also [3]).

The paper is organized as follows. In Section 2 we give some preliminaries and define the functional. In Section 3 we derive its Euler–Lagrange equations and we obtain the main result of the paper (Theorem 3.2). In dimension $\geq 5$, we give criteria for which a metric is conformal to a metric with respect to which a $G$-connection is critical for $\mathrm{YM}_{\mathrm{BI}}$. Section 4 is devoted to a conservation law of the functional. Finally in Section 5 we derive the second variation formula.

## 2 The functional

Let $E$ be a smooth real vector bundle over a compact $n$-dimensional Riemannian manifold $(M^n, g)$, such that its structure group $G$ is a compact Lie subgroup of the orthogonal group $O(n)$.

For any vector bundle $F$ over $M$ we denote by $\Gamma(F)$ the space of smooth cross sections of $F$ and for each $p \geq 0$ we denote by $\Omega^p(F) = \Gamma(\Lambda^p T^*M \otimes F)$ the space of all smooth $p$-forms on $M$ with values in $F$. Note that $\Omega^0(F) = \Gamma(F)$.

A connection $D$ on the vector bundle $E$ is defined by specifying a covariant derivative, that is a linear map

$$D\colon\ \Omega^0(E) \to \Omega^1(E),$$

such that $D(fs) = df \otimes s + f Ds$, for any section $s \in \Omega^0(E)$ and any smooth function $f \in C^\infty(M)$.

A connection $D$ is called a $G$-connection if the natural extension of $D$ to tensor bundles of $E$ annihilates the tensors which define the $G$-structure. We denote by $\mathcal{C}(E)$ the space of all smooth $G$-connections $D$ on $E$.

Given a connection on $E$, the map $D\colon \Omega^0(E) \to \Omega^1(E)$ can be extended to a generalized de Rham sequence

$$\Omega^0(E) \xrightarrow{d^D = D} \Omega^1(E) \xrightarrow{d^D} \Omega^2(E) \xrightarrow{d^D} \cdots .$$

For each $G$-connection $D$ of the vector bundle $E$, the curvature tensor of $D$, denoted by $R^D$, is determined by $\left(d^D\right)^2\colon \Omega^0(E) \to \Omega^2(E)$. If we suppose that $E$ carries an inner product compatible with $G$, it is easy to see that $R^D \in \Omega^2(g_E)$, where $g_E \subset \mathrm{End}(E)$ is the subbundle of skew-symmetric endomorphisms of $E$.

Given metrics on $M$ and $E$, there are naturally induced metrics on all associated bundles, such as $\Lambda^p T^*M \otimes \mathrm{End}(E)$:

$$\langle \varphi, \psi \rangle_x = \sum_{1<i_1<\cdots<i_p<n} \left\langle \varphi^t(e_{i_1}, \dots, e_{i_p}), \psi(e_{i_1}, \dots, e_{i_p}) \right\rangle,$$

where, for any point $x \in M$, $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_xM$ with respect to the metric $g$. The pointwise inner product gives an $L^2$-norm on $\Omega^p(E)$ by setting

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \vartheta_g.$$

With respect to this norm, the formal adjoint of $d^D$ it is denoted by $\delta^D$ (the coderivative) and satisfies

$$\left(d^D\varphi, \psi\right) = \left(\varphi, \delta^D\psi\right).$$

In particular, for any $G$-connection $D$, the norm of the curvature $R^D$ is defined by

$$\left\|R^D\right\|_x^2 = \sum_{i<j} \left\|R^D_{e_i,e_j}\right\|_x^2,$$

for any point $x \in M$ and any orthonormal basis $\{e_i\}_{i=\overline{1,n}}$ on $T_xM$. The norm of $R^D_{e_i,e_j}$ is the usual one on $\mathrm{End}(E)$, namely $\langle A, B\rangle = \frac{1}{2}\operatorname{tr}\left(A^t \circ B\right)$.

We are able to define the Yang–Mills–Born–Infeld functional $\mathrm{YM}_{\mathrm{BI}}\colon \mathcal{C}(E) \to \mathbb{R}$ (see also [3]) by

$$\mathrm{YM}_{\mathrm{BI}}(D) = \int_M \left(\sqrt{1 + \left\|R^D\right\|^2} - 1\right)\vartheta_g.$$

## 3 The first variation formula. Existence result

In the following we shall derive the Euler–Lagrange equations of the functional $\mathrm{YM}_{\mathrm{BI}}$. These equations were also obtained in [3] for the $F$-Yang–Mills functional.

**Theorem 3.1.** *The first variation formula of the functional* $\mathrm{YM}_{\mathrm{BI}}$ *is given by*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{YM}_{\mathrm{BI}}\left(D^t\right) = \int_M \left\langle B, \delta^D\left(\frac{1}{\sqrt{1+\left\|R^D\right\|^2}}R^D\right)\right\rangle \vartheta_g,$$

*where*

$$B = \left.\frac{d}{dt}\right|_{t=0} D^t.$$

*Consequently, $D$ is a critical point of* $\mathrm{YM}_{\mathrm{BI}}$ *if and only if*

$$\delta^D\left(\frac{1}{\sqrt{1+\left\|R^D\right\|^2}}R^D\right) = 0,$$

*which are the Euler–Lagrange equations of* $\mathrm{YM}_{\mathrm{BI}}$.

**Proof.** Let $D$ a $G$-connection $D \in \mathcal{C}(E)$ and consider a smooth curve $D^t = D + \alpha^t$ on $\mathcal{C}(E)$, $t \in (-\epsilon, \epsilon)$, such that $\alpha^0 = 0$, where $\alpha^t \in \Omega^1(g_E)$. The corresponding curvature is given by

$$R^{D^t} = R^D + d^D\alpha^t + \tfrac{1}{2}\left[\alpha^t \wedge \alpha^t\right],$$

where we define the bracket of $g_E$-valued 1 forms $\varphi$ and $\psi$ by the formula $[\varphi \wedge \psi](X, Y) = [\varphi(X), \psi(Y)] - [\varphi(Y), \psi(X)]$ for any vector fields $X, Y \in \Gamma(TM)$. Indeed for any vector fields $X, Y \in \Gamma(TM)$ and $u \in \Gamma(E)$ we have

$$\begin{aligned} R^{D^t}(X,Y)(u) &= D^t_X\left(D^t_Y u\right) - D^t_Y\left(D^t_X u\right) - D^t_{[X,Y]}u \\ &= D^t_X\left(D_Y u + \alpha^t(Y)(u)\right) - D^t_Y\left(D_X u + \alpha^t(X)(u)\right) \\ &\quad - D^t_X\left(D_{[X,Y]}u + \alpha^t([X,Y])(u)\right) \end{aligned}$$

$$
\begin{aligned}
&= D_X\big(D_Y u + \alpha^t(Y)(u)\big) + \alpha^t(X)\big(D_Y u + \alpha^t(Y)(u)\big)\\
&\quad - D_Y\big(D_X u + \alpha^t(X)(u)\big) - \alpha^t(Y)\big(D_X u + \alpha^t(X)(u)\big)\\
&\quad - D_{[X,Y]}u - \alpha([X,Y])(u)\\
&= R^D(X,Y)(u) + D_X\big(\alpha^t(Y)(u)\big) - \alpha^t(Y)(D_X u)\\
&\quad - \big(D_Y(\alpha^t(X)(u)) - \alpha^t(X)(D_Y u)\big) - \alpha^t([X,Y])(u)\\
&\quad + \alpha^t(X)\big(\alpha^t(Y)(u)\big) - \alpha^t(Y)\big(\alpha^t(X)(u)\big)\\
&= R^D(X,Y)(u) + \big(D_X\big(\alpha^t(Y)\big)(u)\big) - \big(D_Y\big(\alpha^t(X)\big)(u)\big)\\
&\quad - \alpha^t([X,Y])(u) + \tfrac{1}{2}\big[\alpha^t \wedge \alpha^t\big](X,Y)(u)\\
&= R^D(X,Y)(u) + \big(d^D\alpha^t\big)(X,Y)(u) + \tfrac{1}{2}\big[\alpha^t \wedge \alpha^t\big](X,Y)(u).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\Big(\sqrt{1+\|R^{D^t}\|^2}-1\Big) &= \frac{1}{\sqrt{1+\|R^D\|^2}}\frac{d}{dt}\bigg|_{t=0}\frac{1}{2}\|R^{D^t}\|^2\\
&= \frac{1}{\sqrt{1+\|R^D\|^2}}\left\langle \frac{d}{dt}R^{D^t}, R^D\right\rangle\bigg|_{t=0}\\
&= \frac{1}{\sqrt{1+\|R^D\|^2}}\langle d^D B, R^D\rangle,
\end{aligned}
$$

where $B = \frac{d}{dt}\big|_{t=0} D^t \in \Omega^1(g_E)$.

Thus we obtain

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\mathrm{YM}_{\mathrm{BI}}\big(D^t\big) &= \int_M \frac{1}{\sqrt{1+\|R^D\|^2}}\langle d^D B, R^D\rangle \vartheta_g\\
&= \int_M \left\langle B, \delta^D\left(\frac{1}{\sqrt{1+\|R^D\|^2}}R^D\right)\right\rangle \vartheta_g. \qquad \blacksquare
\end{aligned}
$$

After deriving the Euler–Lagrange equations, we look for their solutions. We next prove an existence result for the critical points of the functional $\mathrm{YM}_{\mathrm{BI}}$.

**Theorem 3.2.** *Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold with $n \geq 5$, $G$ a compact Lie group, and $E$ a smooth $G$-vector bundle over $M$. Then there exists a Riemannian metric $\tilde{g}$ on $M$ in the conformal class of $g$, and a $G$-connection $D$ on $E$ such that $D$ is a critical point of the functional* $\mathrm{YM}_{\mathrm{BI}}$.

**Proof.** We prove the theorem in two steps.

**Step 1.** Consider the functional $F_p\colon \mathcal{C}(E) \to \mathbb{R}$, defined by

$$
F_p(D) = \frac{1}{2}\int_M \big(1 + \|R^D\|_g^2\big)^{(p-2)/2}\vartheta_g.
$$

By [4] this functional satisfies the Palais–Smale conditions and attains the minimum if $2p > n$. The Euler–Lagrange equation associated to $F_p(D)$ is

$$
\delta_g^D\big(\big(1 + \|R^D\|_g^2\big)^{(p-2)/2}R^D\big) = 0.
$$

This equation has a solution $D$ for $2p > n$. Define the function $f\colon M \to \mathbb{R}$ by $f = \big(1 + \|R^D\|_g^2\big)^{(p-2)/n-4}$ and the metric $\overline{g} = fg$, conformally equivalent to $g$. As $\delta_g^D\left(f^{(n-4)/2}R^D\right) = 0$, it is easy to see that $\delta_{\overline{g}}^D\big(R^D\big)=0$. Hence there exists a Riemannian metric $\overline{g}$ on $M$, conformaly equivalent to $g$, and a $G$-connection $D$ on $E$ such that $D$ is a Yang–Mills connection with respect to $\overline{g}$.

**Step 2.** Now we look for a "good" function $\sigma$ such that $\tilde{g}=\sigma^{-1}g$. Taking into account the first step, we can start with an Yang–Mills connection $D$ with respect to the metric $g$. It is clear that

$$\delta_g^D R^D=0 \qquad \text{if and only if} \quad \delta_{\tilde{g}}^D\big(\sigma^{\frac{n-4}{2}}R^D\big)=0,$$

for any $G$-connection.

The function $\sigma$ is good if it satisfies the following functional equation

$$\sigma^{\frac{n-4}{2}}=\frac{1}{\sqrt{1+\sigma^2\big\|R^D\big\|_g^2}}\left(=\frac{1}{\sqrt{1+\big\|R^D\big\|_{\tilde{g}}^2}}\right).$$

So, what we have to do next is to solve the above functional equation.

Let $h\colon[0,\infty)\to[0,\infty)$ given by $h(t)=\sqrt{1+2t}-1$. It is clear that its derivative is a strictly decreasing function and let $H\colon(0,1]\to[0,\infty)$ be its smooth inverse. Consider the smooth function $F\colon(0,1]\to[0,\infty)$ given by

$$F(y)=\frac{H\big(y^{(n-4)/2}\big)}{y^2}.$$

It is not difficult to prove that $F$ is invertible. Denote by $\Phi\colon[0,\infty)\to(0,1]$ the smooth inverse of $F$. We define the positive smooth function $\sigma$ by

$$\sigma=\Phi\big(\tfrac{1}{2}\big\|R^D\big\|_g^2\big).$$

We then have

$$\begin{aligned}0&=\delta_{\tilde{g}}^D\big(\sigma^{(n-4)/2}R^D\big)=\delta_{\tilde{g}}^D\Big(\big(\Phi\big(\tfrac{1}{2}\big\|R^D\big\|_g^2\big)\big)^{(n-4)/2}R^D\Big)\\&=\delta_{\tilde{g}}^D\left(\frac{1}{\sqrt{1+\sigma^2\big\|R^D\big\|_g^2}}R^D\right)=\delta_{\tilde{g}}^D\left(\frac{1}{\sqrt{1+\big\|R^D\big\|_{\tilde{g}}^2}}R^D\right),\end{aligned}$$

which proves that the Yang–Mills connection $D$ is also a critical point of the functional $\mathrm{YM}_{\mathrm{BI}}$ with respect to the metric $\tilde{g}$. ■

**Remark 3.3.** The condition $n\geq 5$ is crucial in the previous proof because the Euler–Lagrange equations are conformally invariant in dimension $n=4$.

## 4 The stress-energy tensor. Conservation law

Motivated by Feynman's ideas on stationary electromagnetic field, in 1982 Baird and Eells introduced the stress-energy tensor associated to any smooth map $f\colon(M,g)\to(N,h)$ between two Riemannian manifolds, The stress-energy tensor is $S_f:=e(f)g-f^*h$, where $e(f)$ is the energy density of $f$. In the same spirit, to any $G$-connection $D$ one associates an analouguous 2-tensor (related to the Yang–Mills–Born–Infeld functional) by (see also [3])

$$S_D=\Big(\sqrt{1+\big\|R^D\big\|^2}-1\Big)g-\frac{1}{\sqrt{1+\big\|R^D\big\|^2}}R^D\odot R^D,$$

where $R^D\odot R^D$ is the symmetric product defined by $R^D\odot R^D=\big\langle i_XR^D,i_YR^D\big\rangle$.

It is natural to look for the geometric interpretation of this tensor. There exists a variational interpretation which we shall explain in the following. Consider the following functional

$$\mathcal{E}_D(g) = \int_M \left(\sqrt{1 + \|R^D\|^2} - 1\right)\vartheta_g.$$

The difference between this functional and $\mathrm{YM_{BI}}$ is that $\mathcal{E}_D$ is defined on the space of smooth Riemannian metrics on the base manifold $M$ and the connection $D$ is fixed. In order to compute the rate of change of $\mathcal{E}_D(g)$ when the metric on the base manifold is changed, we consider a smooth family of metrics $g_s$ with $s \in (-\varepsilon, +\varepsilon)$, such that $g_0 = g$. The "tangent" vector at $g$ to the curve of metrics $g_s$ is denoted by $\delta g = \frac{dg_s}{ds}\big|_{s=0}$ and can be viewed as a smooth 2-covariant symmetric tensor field on $M$. Using the formulae obtained by Baird (see [1])

$$\frac{d\|R^D\|_{g_s}}{ds}\bigg|_{s=0} = -\langle R^D \odot R^D, \delta g\rangle,$$

and

$$\frac{d}{ds}\vartheta_{g_s}\bigg|_{s=0} = \frac{1}{2}\langle g, \delta g\rangle\vartheta_g$$

we obtain

$$\begin{aligned}
\frac{d\mathcal{E}_D(g_s)}{ds}\bigg|_{s=0} &= \int_M \frac{1}{\sqrt{1+\|R^D\|^2}}\frac{d}{ds}\left(\frac{1}{2}\|R^D\|^2\right)\bigg|_{s=0}\vartheta_g \\
&\quad + \int_M \left(\sqrt{1+\|R^D\|^2} - 1\right)\frac{d}{ds}\vartheta_{g_s}\bigg|_{s=0} \\
&= \frac{1}{2}\int_M \left\langle \left(\sqrt{1+\|R^D\|^2} - 1\right)g - \frac{1}{\sqrt{1+\|R^D\|^2}}R^D \odot R^D, \delta g\right\rangle \vartheta_g \\
&= \frac{1}{2}\int_M \langle S_D, g\rangle \vartheta_g.
\end{aligned}$$

Recall now

**Definition 4.1.** A $G$-connection $D$ is said to satisfy a conservation law if $S_D$ is divergence free.

Concerning this notion we obtain the following result (see [3] for the general case of $F$-Yang–Mills fields).

**Proposition 4.2.** *Any critical point of the functional* $\mathrm{YM_{BI}}$ *is conservative.*

**Proof.** The following formula for the divergence of the stress-energy tensor is true (see [3])

$$\begin{aligned}
\operatorname{div} S_D(X) &= \left\langle \frac{1}{\sqrt{1+\|R^D\|^2}}\delta^D R^D - i_{\operatorname{grad}\left(\frac{1}{\sqrt{1+\|R^D\|^2}}\right)} R^D, i_X R^D \right\rangle \\
&\quad + \frac{1}{\sqrt{1+\|R^D\|^2}}\langle i_X d^D R^D, R^D\rangle,
\end{aligned}$$

for any vector field $X$ on $M$. Using the Bianchi identity and the Euler–Lagrange equation of the functional $\mathrm{YM_{BI}}$, we derive $\operatorname{div} S_D = 0$. ∎

## 5 The second variation formula

In this section we obtain the second variation formula of the functional $\mathrm{YM}_{\mathrm{BI}}$. Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold, $G$ a compact Lie group and $E$ a $G$-vector bundle over $M$. Let $D$ be a critical point of the functional $\mathrm{YM}_{\mathrm{BI}}$ and $D^t$ a smooth curve on $\mathcal{C}(E)$ such that $D^t = D+\alpha^t$, where $\alpha^t \in \Omega^1(g_E)$ for all $t\in(-\varepsilon,\varepsilon)$, and $\alpha^0=0$. The infinitesimal variation of the connection associated to $D^t$ at $t=0$ is

$$B := \left.\frac{d\alpha^t}{dt}\right|_{t=0} \in \Omega(g_E).$$

According to [2], we define the endomorphism $\mathcal{R}^D$ of $\Omega^1(g_E)$ by

$$\mathcal{R}^D(\varphi)(X) := \sum_{i=1}^n \left[R^D(e_i,X),\varphi(e_i)\right],$$

for $\varphi\in\Omega(g_E)$ and $X\in\Gamma(TM)$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on $(M,g)$. With these notations we have

**Theorem 5.1.** *Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold, $G$ a compact Lie group and $E$ a $G$-vector bundle over $M$. Let $D$ be a critical point of* $\mathrm{YM}_{\mathrm{BI}}$. *The second variation of the functional* $\mathrm{YM}_{\mathrm{BI}}$ *is given by*

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0}\mathrm{YM}_{\mathrm{BI}}(D^t) &= -\int_M \frac{1}{(1+\|R^D\|^2)^{3/2}}\langle d^DB,R^D\rangle^2\vartheta_g \\ &\quad+\int_M \frac{1}{\sqrt{1+\|R^D\|^2}}\left(\langle d^DB,d^DB\rangle+\langle B,\mathcal{R}^D(B)\rangle\right)\vartheta_g \\ &= \int_M \langle B,\mathcal{S}^D(B)\rangle\vartheta_g,\end{aligned}$$

*where $\mathcal{S}^D$ is a differential operator acting on $\Omega(g_E)$ defined by*

$$\begin{aligned}\mathcal{S}^D(B) &= -\delta^D\left(\frac{1}{(1+\|R^D\|^2)^{3/2}}\langle d^DB,R^D\rangle^2\right) \\ &\quad+\delta^D\left(\frac{1}{\sqrt{1+\|R^D\|^2}}d^DB\right)++\frac{1}{\sqrt{1+\|R^D\|^2}}\mathcal{R}^D(B).\end{aligned}$$

**Proof.** As $R^{D^t}=R^D+d^D\alpha^t+\frac{1}{2}\left[\alpha^t\wedge\alpha^t\right]$ and $\alpha^0=0$ we obtain that

$$\left.\frac{d^2}{dt^2}\right|_{t=0}\left(\frac{1}{2}\|R^{D^t}\|^2\right) = \langle d^DC+[B,B],R^D\rangle+\langle d^DB,d^DB\rangle,$$

where $C := \frac{d^2}{dt^2}\big|_{t=0}\alpha^t$. Thus we obtain

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0}\mathrm{YM}_{\mathrm{BI}}(D^t) &= \left.\frac{d}{dt}\right|_{t=0}\int_M \frac{1}{2}\frac{1}{\sqrt{1+\|R^D\|^2}}\frac{d}{dt}\|R^{D^t}\|^2\vartheta_g \\ &= -\frac{1}{4}\int_M \frac{1}{(1+\|R^D\|^2)^{3/2}}\left(\left.\frac{d}{dt}\right|_{t=0}\|R^{D^t}\|^2\right)^2\vartheta_g\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_M \frac{1}{\sqrt{1+\|R^D\|^2}} \frac{d^2}{dt^2}\Big|_{t=0} \|R^{D^t}\|^2 \vartheta_g \\
&= -\int_M \frac{1}{\left(1+\|R^D\|^2\right)^{3/2}} \langle d^D B, R^D\rangle^2 \vartheta_g \\
&\quad + \int_M \frac{1}{\sqrt{1+\|R^D\|^2}} \left(\langle d^D C + [B,B], R^D\rangle + \langle d^D B, d^D B\rangle\right)\vartheta_g.
\end{aligned}
$$

On the other hand, since $D$ is a critical point of the functional $\mathrm{YM}_{\mathrm{BI}}$, we have

$$
\int_M \frac{1}{\sqrt{1+\|R^D\|^2}} \langle d^D C, R^D\rangle \vartheta_g = \int_M \left\langle C, \delta^D\left(\frac{1}{\sqrt{1+\|R^D\|^2}} R^D\right)\right\rangle \vartheta_g = 0.
$$

Finally, one can prove that

$$
\langle [B\wedge B], R^D\rangle = \langle B, \mathcal{R}^D(B)\rangle.
$$

Indeed

$$
\begin{aligned}
\langle [B\wedge B], R^D\rangle &= \sum_{i<j} \langle [B\wedge B](e_i,e_j), R^D(e_i,e_j)\rangle \\
&= \sum_{i<j} \langle [B(e_i),B(e_j)] - [B(e_j),B(e_i)], R^D(e_i,e_j)\rangle \\
&= 2\sum_{i<j} \langle [B(e_i),B(e_j)], R^D(e_i,e_j)\rangle = \sum_{i,j=1}^n \langle B(e_i), [B(e_j), R^D(e_i,e_j)]\rangle \\
&= \sum_{i=1}^n \langle B(e_i), \mathcal{R}^D(e_i)\rangle = \langle B, \mathcal{R}^D(B)\rangle,
\end{aligned}
$$

and thus we obtain the second variation formula. ■

The index, nullity and stability of a critical point of $\mathrm{YM}_{\mathrm{BI}}$ can be defined in the same way as in the case of Yang–Mills connection (see [2]) but is rather difficult to analyse them because the form of $\mathcal{S}^D$ is much more complicated compared with the case of Yang–Mills connections.


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