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The Primitive Derivation and Discrete Integrals

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Abstract. The modules of logarithmic derivations for the (extended) Catalan and Shi arrangements associated with root systems are known to be free. However, except for a few cases, explicit bases for such modules are not known. In this paper, we construct explicit bases for type A root systems. Our construction is based on Bandlow–Musiker's integral formula for a basis of the space of quasiinvariants. The integral formula can be considered as an expression for the inverse of the primitive derivation introduced by K. Saito. We prove that the discrete analogues of the integral formulas provide bases for Catalan and Shi arrangements.

Key words: hyperplane arrangements; freeness; Catalan arrangements; Shi arrangements

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Dedicated to Professor Kyoji Saito
on the occasion of his 77th birthday

1 Introduction

1.1 Background

Let V be an ℓ-dimensional linear space over $\mathbb{C}$. Let $S = S(V^*)$ be the set of polynomial functions on V. Choose $x_1, \dots, x_\ell$ as a basis of V^* and identify S with the polynomial ring $\mathbb{C}[x_1, \dots, x_\ell]$. Let $\mathrm{Der}_S = \bigoplus_{i=1}^{\ell} S\partial_i$ be the module of polynomial vector fields ($\mathbb{C}$-linear derivations of S).

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes. For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $H = \mathrm{Ker}(\alpha_H)$. For each nonnegative integer m, let

$$D(\mathcal{A}, m) := \big\{\delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^m),\ \text{for any } H \in \mathcal{A}\big\}.$$

If $m = 1$, $D(\mathcal{A}, 1)$ is simply denoted by $D(\mathcal{A})$, whose elements are called logarithmic derivations. The module $D(\mathcal{A})$ was introduced in [13] for the purpose of computing Gauss–Manin connections, and $D(\mathcal{A}, m)$ was introduced by Ziegler [26] for studying restrictions of free arrangements. The algebraic structures of these modules are thought to reflect the combinatorial nature of $\mathcal{A}$ (see [11, 25]).

Now we assume that $\mathcal{A}$ is a Coxeter arrangement, that is, the set of reflecting hyperplanes of a finite irreducible real reflection group $W \subset \mathrm{GL}(V)$. The definition of $D(\mathcal{A}, m)$ naturally gives rise to a filtration:

$$\mathrm{Der}_S = D(\mathcal{A}, 0) \supset D(\mathcal{A}, 1) \supset D(\mathcal{A}, 2) \supset \cdots. \tag{1.1}$$

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The filtration (1.1) is closely related to several important structures. First, taking W-invariant parts, we have

$$D(\mathcal{A},0)^W = D(\mathcal{A},1)^W \supset D(\mathcal{A},2)^W = D(\mathcal{A},3)^W \supset \cdots.$$

This filtration is known to be equal to the semi-infinite Hodge filtration studied by K. Saito [15, 16, 21, 23], which is a crucial structure in his theory of primitive forms [14]. In particular, the inverse operator of the so-called *primitive derivation* ∇_D describes the filtration as

$$D(\mathcal{A},2m+1)^W = \left(\nabla_D^{-1}\right)^m D(\mathcal{A})^W.$$

As indicated by Misha Feigin (see forthcoming paper [1] for details), these spaces are also isomorphic to the isotypic component of the spaces of m-quasiinvariants, which were introduced in the study of the Calogero–Moser system [4, 5, 8].

Around 2000, Terao proved that the module $D(\mathcal{A},m)$ is an S-free module using these structures [20]. Terao's results on the freeness of $D(\mathcal{A},m)$ opened new perspectives between the primitive derivation and enumerative combinatorics of Catalan/Shi arrangements.

Catalan and Shi arrangements are classes of finite truncations of affine Weyl arrangements for root systems. The terminology "Catalan arrangement" is explained by the fact that the number of chambers in the fundamental region of a type A root system is equal to the Catalan number [12]. The Shi arrangement was introduced by J.-Y. Shi in [17] in the study of affine Weyl groups. In 1996, Edelman–Reiner [6] posed a conjecture concerning the freeness of cones of Catalan and Shi arrangements for root systems. This conjecture was proved for type A root system by Edelman–Reiner [6] and Athanasiadis [3], and later for all root systems by [24]. The freeness of $D(\mathcal{A},m)$ played crucial role in the proof of [24] because $D(\mathcal{A},m)$ can be regarded as the "leading terms" of the logarithmic vector fields for Catalan and Shi arrangements. In the present paper we focus on the construction of explicit bases for these modules.

1.2 Constructions of explicit bases

In this section, we introduce Catalan and Shi arrangements (of type $A_{\ell-1}$). We define m-Catalan arrangement $\mathrm{Cat}_\ell(m)$ as

$$\prod_{\substack{1\le i<j\le \ell\\ -m\le k\le m}} (x_i - x_j - k) = 0,$$

and m-Shi arrangement $\mathrm{Shi}_\ell(m)$ by

$$\prod_{1-m\le k\le m}\ \prod_{1\le i<j\le \ell} (x_i - x_j - k) = 0.$$

We also denote $\mathrm{Cat}_\ell(0)$ by $\mathcal{B}_\ell$. The arrangement $\mathcal{B}_\ell$ is defined as the polynomial $\prod_{1\le i<j\le\ell}(x_i-x_j)$ and called the braid arrangement.

The cones $c\,\mathrm{Cat}_\ell(m)$ and $c\,\mathrm{Shi}_\ell(m)$ are defined by the homogeneous polynomials

$$z\prod_{\substack{1\le i<j\le \ell\\ -m\le k\le m}} (x_i - x_j - kz) = 0,$$

and

$$z\prod_{1-m\le k\le m}\ \prod_{1\le i<j\le \ell} (x_i - x_j - kz) = 0,$$

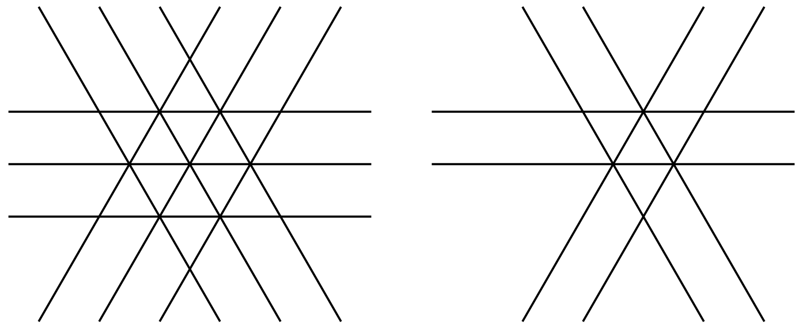

Figure 1. $\mathrm{Cat}_3(1)$ and $\mathrm{Shi}_3(1)$ (the intersections with the plane $x_1+x_2+x_3=0$ are drawn).

respectively. As we have already noted, the modules $D(c\,\mathrm{Cat}_\ell(m))$ and $D(c\,\mathrm{Shi}_\ell(m))$ are free. The exponents, that is, the degrees of the homogeneous bases of the modules, are as follows

$$\exp(c\,\mathrm{Cat}_\ell(m)) = \{0, 1, m\ell+1, m\ell+2, \dots, m\ell+\ell-1\},$$
$$\exp(c\,\mathrm{Shi}_\ell(m)) = \{0, 1, \underbrace{m\ell, m\ell, \dots, m\ell}_{\ell-1}\}.$$

We note that explicit bases were not constructed in the known proofs. Indeed, the proofs by Edelman–Reiner [6] and Athanasiadis [3] used Terao's addition–deletion theorem of freeness [11, Theorem 4.51], and that in [24] used cohomological arguments to guarantee the existence of global sections of certain coherent sheaves associated with the graded module $D(\mathcal{A})$. Since then, a number of efforts have been made to construct explicit bases for $D(c\,\mathrm{Cat}_\ell(m))$ and $D(c\,\mathrm{Shi}_\ell(m))$. First, in [19], a basis for $D(c\,\mathrm{Shi}_\ell(1))$ was constructed using the Bernoulli polynomial. Subsequently, in [10] and [18], similar bases were constructed for root systems of type B, C, and D. Note that these works are for Shi arrangements with $m=1$. Catalan arrangements and Shi arrangements with $m>1$ have not been covered. For larger m, the type A_2 was the only known case. Namely, explicit bases were constructed for $c\,\mathrm{Cat}_3(m)$ and $c\,\mathrm{Shi}_3(m)$, $m\geq 1$, in [2].

Our purpose in the present paper is to construct an explicit basis for $D(c\,\mathrm{Cat}_\ell(m))$ and $D(c\,\mathrm{Shi}_\ell(m))$, for all $\ell\geq 2$ and $m\geq 1$. The paper is organized as follows. The starting point of our study is Bandlow–Musiker's integral expression [4] (which goes back to Felder–Veselov's integral expression [9]) for a basis of the space of quasiinvariants introduced in [5, 8]. Misha Feigin [7] communicated to us that Bandlow–Musiker's formula provides a basis for the multiarrangement $D(\mathcal{B}_\ell, 2m+1)$. More precisely, for appropriate choices of k, the integral expressions

$$\sum_{i,j=1}^{\ell}\left(\int_{x_i}^{x_j} t^k\left(\prod_{p=1}^{\ell}(t-x_p)\right)^m \mathrm{d}t\right)\partial_i \tag{1.2}$$

provide a basis for $D(\mathcal{B}_\ell, 2m+1)$. We discuss these facts in addition to a basis for $D(\mathcal{B}_\ell, 2m)$ in Section 2.

After introducing the notion of "discrete integrals" in Section 3, we present the main results in Section 4, that is, we prove the following: a basis for $D(c\,\mathrm{Cat}_\ell(m))$ is obtained from (1.2) by simply replacing the integration "$\int_a^b \mathrm{d}t$" with the discrete integration "$\sum_a^b \Delta t$" as follows

$$\sum_{i,j=1}^{\ell}\left(\sum\nolimits_{x_i}^{x_j} t^k\left(\prod_{p=1}^{\ell}(t-x_p)\right)^{\underline{m}} \Delta t\right)\partial_i.$$

(To be precise, we need to homogenize the above polynomial vector field, see Sections 3 and 4 for notations and details.) We also provide a basis for $D(c\,\mathrm{Shi}_\ell(m))$.

2 Bandlow–Musiker's expression

Recall that $\mathcal{B}_\ell$ denotes the braid arrangement defined by $\prod_{1\le i<j\le\ell}(x_i-x_j)$. The symmetric group $W=\mathfrak{S}_\ell$ naturally acts on $\mathcal{B}_\ell$ by the permutation of coordinates. In this section, we construct a basis for $D(\mathcal{B}_\ell,m)$, $m\geq 1$. Note that because the vector field

$$\theta_0 := \partial_1+\partial_2+\cdots+\partial_\ell$$

annihilates the linear form x_i-x_j, we have $\theta_0\in D(\mathcal{B}_\ell,m)$ for any m.

We also set $g(t):=(t-x_1)(t-x_2)\cdots(t-x_\ell)\in\mathbb{C}[t,x_1,\dots,x_\ell]$. Following Bandlow–Musiker [4] and Feigin [7], for $m,k\geq 0$, we introduce the following vector field

$$\eta_k^m=\sum_{i=1}^{\ell}\left(\sum_{j=1}^{\ell}\int_{x_i}^{x_j}t^k g(t)^m\mathrm{d}t\right)\partial_i. \tag{2.1}$$

Proposition 2.1 ([4, 7])**.** *The vector fields $\eta_0^m,\eta_1^m,\dots,\eta_{\ell-2}^m,\theta_0$ form a basis for $D(\mathcal{B}_\ell,2m+1)^W$ as an S^W-module.*

Recall that Terao [20] proved that there exists a W-invariant basis for $D(\mathcal{B}_\ell,2m+1)$. Terao's invariant basis generates a submodule of $D(\mathcal{B}_\ell,2m+1)^W$ over S^W. However, these must coincide since the degrees of Terao's basis are equal to those of the above basis. Thus we have the following.

Corollary 2.2. *The vector fields $\eta_0^m,\eta_1^m,\dots,\eta_{\ell-2}^m,\theta_0$ form a basis of $D(\mathcal{B}_\ell,2m+1)$ as an S-module.*

Here we give a direct proof of Corollary 2.2 in order to see the relationship between integral expression (2.1) and the primitive derivation. First, we prove $\eta_k^m\in D(\mathcal{B}_\ell,2m+1)$. Since η_k^m is W-symmetric, it is sufficient to show that

$$\eta_k^m(x_1-x_2)=\ell\int_{x_1}^{x_2}t^k g(t)^m\mathrm{d}t$$

is divisible by $(x_1-x_2)^{2m+1}$. This can be checked by the change of variables $t'=t-x_1$. Indeed, since the integrand is divisible by $(t-x_1)^m(t-x_2)^m$, after the change of variable, it is divisible by $(t')^m(t'-(x_2-x_1))^m$. Therefore, the definite integral $\int_0^{x_2-x_1}\mathrm{d}t'$ is divisible by $(x_2-x_1)^{2m+1}$.

Example 2.3. When $\ell=2$, $m=1$, we obtain the following famous formula

$$\int_{x_1}^{x_2}(t-x_1)(t-x_2)\,\mathrm{d}t=\frac{(x_1-x_2)^3}{6}.$$

Next we describe the action of the primitive derivation on the vector field η_k^m. Let ∇ denote the integrable connection with flat sections $\partial_1,\dots,\partial_\ell$. More explicitly, for polynomial vector fields δ and $\eta=\sum_{i=1}^{\ell}f_i\partial_i$, we define

$$\nabla_\delta\eta=\sum_{i=1}^{\ell}(\delta f_i)\,\partial_i.$$

Let P_i denote the coefficient of $t^{\ell-i}$ in $g(t)$, that is, $g(t)=t^\ell+P_1t^{\ell-1}+\cdots+P_\ell$. Then $P_1,\dots,P_\ell\in\mathbb{Z}[x_1,\dots,x_\ell]$ are elementary symmetric functions (up to the sign), and satisfy $\mathbb{C}[x_1,\dots,x_\ell]^W=\mathbb{C}[P_1,\dots,P_\ell]$. Therefore, we can regard $P_1,\dots,P_\ell$ as a system of coordinates on the quotient space $\mathbb{C}^\ell/W$. The vector field $D = \frac{\partial}{\partial P_\ell}$ is called the primitive derivation [15, 16]. The lift of D by the natural projection $\pi\colon \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell/W$ is expressed as

$$D = \frac{1}{Q}\det\begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_2} & \frac{\partial}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix},$$

where $Q = \det\left(\frac{\partial P_i}{\partial x_j}\right)$ is the Jacobian of the projection π. For simplicity, we also denote by D the lift of the primitive derivation $\frac{\partial}{\partial P_\ell}$. Note that other vector fields $\frac{\partial}{\partial P_1}, \frac{\partial}{\partial P_2}, \dots, \frac{\partial}{\partial P_{\ell-1}}$ also have similar expressions. The definition (2.1) can be written as

$$\eta_k^m = \sum_{i=1}^{\ell}\left(\sum_{j=1}^{\ell}\int_{x_i}^{x_j} t^k\bigl(t^\ell + P_1 t^{\ell-1} + \cdots + P_\ell\bigr)^m \mathrm{d}t\right)\partial_i.$$

Then, if $m > 0$, differentiation by $\frac{\partial}{\partial P_j}$ yields (see Remark 2.5 for details)

$$\nabla_{\frac{\partial}{\partial P_j}}\eta_k^m = m\,\eta_{k+\ell-j}^{m-1}. \tag{2.2}$$

When $m = 0$, it is checked by Saito's criterion [11, 13] that the vector fields $\eta_0^0, \eta_1^0, \dots, \eta_{\ell-2}^0, \theta_0$ form a basis of $D(\mathcal{B}_\ell, 1)$. For $m > 0$, we have $\nabla_D^m \eta_k^m = m! \cdot \eta_k^0$. Recall that $D(\mathcal{B}_\ell \cap H_0, 2m+1)^W = \nabla_D^{-m} D(\mathcal{B}_\ell \cap H_0, 1)^W$ [23, Corollary 10]. Therefore, $\nabla_D^{-m}\eta_0^0, \dots, \nabla_D^{-m}\eta_{\ell-1}^0$ form a basis of $D(\mathcal{B}_\ell \cap H_0, 2m+1)$, where H_0 is the hyperplane $x_1 + \cdots + x_\ell = 0$. Adding θ_0, we obtain a basis for $D(\mathcal{B}_\ell, 2m+1)$. This completes the proof of Corollary 2.2.

Now we consider $D(\mathcal{B}_\ell, 2m)$. Let $g_k(t) := \frac{g(t)}{(t-x_k)} = (t-x_1)\cdots\widehat{(t-x_k)}\cdots(t-x_\ell)$ for $1 \le k \le \ell$. For $m > 0$ and $k = 1, \dots, \ell$, we define the vector field σ_k^m as

$$\sigma_k^m := \sum_{i=1}^{\ell}\left(\sum_{j=1}^{\ell}\int_{x_i}^{x_j} g(t)^{m-1} g_k(t)\,\mathrm{d}t\right)\partial_i. \tag{2.3}$$

Proposition 2.4. *Vector fields $\theta_0, \sigma_1^m - \sigma_2^m, \sigma_2^m - \sigma_3^m, \dots, \sigma_{\ell-1}^m - \sigma_\ell^m$ form a basis of $D(\mathcal{B}_\ell, 2m)$.*

Proof. We first note that $\frac{\partial g(t)}{\partial x_k} = -g_k(t)$. Hence we have

$$\nabla_{\partial_k}\eta_0^m = -m\sigma_k^m, \tag{2.4}$$

and

$$\nabla_{(\partial_i - \partial_{i+1})}\eta_0^m = -m\bigl(\sigma_i^m - \sigma_{i+1}^m\bigr).$$

Since $\partial_1 - \partial_2, \dots, \partial_{\ell-1} - \partial_\ell$ form a basis of H_0, by [23, Theorem 7], $\theta_0, \nabla_{(\partial_1-\partial_2)}\eta_0^m, \dots, \nabla_{(\partial_{\ell-1}-\partial_\ell)}\eta_0^m$ form a basis of $D(\mathcal{B}_\ell, 2m)$. ∎

Remark 2.5. Recall that for certain functions $a(x)$, $b(x)$, $f(x,t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t = \frac{\mathrm{d}b(x)}{\mathrm{d}x} f(x, b(x)) - \frac{\mathrm{d}a(x)}{\mathrm{d}x} f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\mathrm{d}t.$$

In our setting, since the integrand $t^k g(t)^m$ vanishes at x_i, the first two terms do not contribute. Thus we have the equations (2.2) and (2.4).

3 Discrete integrals

In this section we only consider polynomial functions. For a function $f(t)$, we define the difference operator Δ as $\Delta f(t) = f(t+1)-f(t)$. When $\Delta F(t) = f(t)$, $F(t)$ is called an indefinite summation (or antidifference) of $f(t)$, and denoted by

$$F(t) = \sum f(t)\Delta t.$$

Let $F(t)$ be an indefinite summation of $f(t)$. Then we define the definite summation as

$$\sum_a^b f(t)\Delta t = F(b) - F(a).$$

Obviously we have the following.

$$\sum_b^a f(t)\Delta t = -\sum_a^b f(t)\Delta t,$$
$$\sum_a^c f(t)\Delta t = \sum_a^b f(t)\Delta t + \sum_b^c f(t)\Delta t.$$

Note that if $b-a=n$ is a positive integer, the definite summation is nothing but the finite sum

$$\sum_a^b f(t)\Delta t = f(a) + f(a+1) + \cdots + f(b-1). \tag{3.1}$$

Example 3.1. Recall that the Bernoulli polynomial $B_n(t)$ is a monic polynomial with rational coefficients defined by

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1}, \tag{3.2}$$

$\big($e.g., $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \dots\big)$. By applying the difference operator with respect to the variable x to the equation (3.2), we have

$$\sum_{n=0}^{\infty} \frac{\Delta B_n(x)}{n!} t^n = t\mathrm{e}^{xt} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} t^n.$$

Thus we have $\Delta B_n(t) = nt^{n-1}$. Therefore, the monomial x^n has an indefinite summation $\frac{B_{n+1}(x)}{n+1}$. Furthermore, an arbitrary polynomial $f(x) = \sum_i a_i x^i$ has an indefinite summation $\sum_i a_i \frac{B_{i+1}(x)}{i+1}$.

The leading part of a definite summation is equal to a definite integral. More precisely, we have the following.

Proposition 3.2. *Let $f(x_0, x_1, \dots, x_n) \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d. Let*

$$F(y_1, y_2, x_1, \dots, x_n) := \sum_{y_1}^{y_2} f(x_0, x_1, \dots, x_n)\Delta x_0.$$

Then $F(y_1, y_2, x_1, \dots, x_n)$ is a (not necessarily homogeneous) polynomial of degree $d+1$ in $y_1, y_2, x_1, \dots, x_n$ whose highest degree part is

$$\lim_{z\to 0} z^{d+1} F\left(\frac{y_1}{z}, \frac{y_2}{z}, \frac{x_1}{z}, \dots, \frac{x_n}{z}\right) = \int_{y_1}^{y_2} f(x_0, x_1, \dots, x_n)\,\mathrm{d}x_0.$$

Proof. This is straightforward from the fact that the leading term of $\frac{B_{n+1}(t)}{n+1}$ is $\frac{t^{n+1}}{n+1}$, which is an indefinite integral of t^n. ■

The following is a discrete analogue of the power. Let $n > 0$ be a positive integer. We define the falling power $f(t)^{\underline{n}}$ as

$$f(t)^{\underline{n}} = f(t)f(t-1)\cdots f(t-n+1).$$

4 Main results

4.1 A basis for the Catalan arrangement

Let $\delta = \sum_{i=1}^{\ell} f_i(x_1, \dots, x_\ell)\, \partial_i$ be a polynomial vector field ($f_1, \dots, f_\ell$ are not necessarily homogeneous). Let $d := \max\{\deg f_1, \dots, \deg f_\ell\}$. We define the homogenization of δ by

$$\widetilde{\delta} = \sum_{i=1}^{\ell} z^d\, f_i\left(\frac{x_1}{z}, \dots, \frac{x_\ell}{z}\right)\partial_i.$$

Using the notion of discrete integrals, we define ζ_k^m as follows.

$$\zeta_k^m = \sum_{i,j=1}^{\ell} \left(\sum\nolimits_{x_i}^{x_j} t^k g(t)^{\underline{m}} \Delta t\right) \partial_i.$$

Note that the definition of ζ_k^m is a discrete analogue of (2.1). The next result shows that the homogenizations of these vector fields form a basis for the Catalan arrangement.

Theorem 4.1. *$\widetilde{\zeta}_0^m, \widetilde{\zeta}_1^m, \dots, \widetilde{\zeta}_{\ell-2}^m, \theta_0$ and the Euler vector field $\theta_E := z\partial_z + x_1\partial_1 + \dots + x_\ell\partial_\ell$ form a basis of $D(c\,\mathrm{Cat}_\ell(m))$.*

Proof. Note that the restriction of $c\,\mathrm{Cat}_\ell(m)$ to the hyperplane $z = 0$ is equal to the braid arrangement $\mathcal{B}_\ell$ (with multiplicity $2m+1$). In view of Ziegler's characterization of freeness [26] (see also [25, Corollary 1.35]), it is sufficient to prove the following:

(a) $\widetilde{\zeta}_0^m, \widetilde{\zeta}_1^m, \dots, \widetilde{\zeta}_{\ell-2}^m \in D(c\,\mathrm{Cat}_\ell(m))$.

(b) The restrictions $\widetilde{\zeta}_0^m|_{z=0}, \widetilde{\zeta}_1^m|_{z=0}, \dots, \widetilde{\zeta}_{\ell-2}^m|_{z=0}, \theta_0|_{z=0}$ form a basis of $D(\mathcal{B}_\ell, 2m+1)$.

First we prove (b). By Proposition 3.2, we have

$$\widetilde{\zeta}_0^m|_{z=0} = \eta_k^m.$$

Now (b) is obtained from Proposition 2.1. To prove (a), since ζ_k^m is symmetric, it is sufficient to show that $\zeta_k^m(x_1 - x_2)$ is divisible by $(x_1 - x_2 - p)$ for $-m \le p \le m$. If $p = 0$, it is clear that

$$\zeta_k^m(x_1 - x_2) = -\ell \sum\nolimits_{x_2}^{x_1} t^k g(t)^{\underline{m}} \Delta t$$

is divisible by $x_1 - x_2$. Now we assume $p > 0$. To prove divisibility by $(x_i - x_j - p)$, we need to show that

$$\sum\nolimits_{x_2}^{x_2+p} t^k g(t)^{\underline{m}} \Delta t = 0. \tag{4.1}$$

Actually, using (3.1), the left-hand side is equal to

$$\sum_{t=x_2}^{x_2+p-1} t^k g(t) g(t-1) \cdots g(t-m+1).$$

For each $t = x_2 + i$ $(0 \le i \le p-1)$, clearly we have $g(t-i) = 0$. Thus (4.1) holds. In the case $-m \le p < 0$, we need to consider $\sum_{x_1}^{x_2} \Delta t = \sum_{x_1}^{x_1-p} \Delta t$ instead of (4.1). The remaining arguments are similar. ■

4.2 A basis for the Shi arrangement

To construct a basis for the Shi arrangement, we need the following. For $0 \leq k \leq \ell$ and $m \geq 0$, let

$$g_k^{(m)}(t) := g(t)^{\underline{m-1}} \prod_{1 \leq i < k} (t - x_i + 1) \prod_{k < i \leq \ell} (t - x_i - m + 1).$$

We define the following vector field which is a discrete analogue of (2.3)

$$\tau_k^m = \sum_{i,j=1}^{\ell} \left(\sum\nolimits_{x_i}^{x_j} g_k^{(m)}(t) \Delta t \right) \partial_i.$$

Using these vector fields, we can construct an explicit basis for the Shi arrangement.

Theorem 4.2. *$\widetilde{\tau}_1^m - \widetilde{\tau}_2^m, \widetilde{\tau}_2^m - \widetilde{\tau}_3^m, \dots, \widetilde{\tau}_{\ell-1}^m - \widetilde{\tau}_\ell^m, \theta_0$ and the Euler vector field $\theta_E := z\partial_z + x_1\partial_1 + \dots + x_\ell\partial_\ell$ form a basis for $D(c\,\mathrm{Shi}_\ell(m))$.*

Proof. The strategy is similar to the proof of Theorem 4.1. It is sufficient to prove the following:

(a) $\widetilde{\tau}_1^m - \widetilde{\tau}_2^m, \widetilde{\tau}_2^m - \widetilde{\tau}_3^m, \dots, \widetilde{\tau}_{\ell-1}^m - \widetilde{\tau}_\ell^m \in D(c\,\mathrm{Shi}_\ell(m))$.

(b) The restrictions $(\widetilde{\tau}_1^m - \widetilde{\tau}_2^m)|_{z=0}, (\widetilde{\tau}_2^m - \widetilde{\tau}_3^m)|_{z=0}, \dots, (\widetilde{\tau}_{\ell-1}^m - \widetilde{\tau}_\ell^m)|_{z=0}$ form a basis of $D(\mathcal{B}_\ell, 2m)$.

As in the proof of Theorem 4.1, (b) is obtained from Propositions 3.2 and 2.4.

To prove (a), we need to show that for $1 \leq u < v \leq \ell$,

$$\tau_k^m(x_u - x_v) = \ell \sum\nolimits_{x_u}^{x_v} g_k^{(m)}(t) \Delta t$$

is divisible by $(x_u - x_v - p)$ for any $1 - m \leq p \leq m$. The case $p = 0$ is obvious. We assume $0 < p$. In this case, we need to show that

$$\sum\nolimits_{x_v}^{x_v+p} g_k^{(m)}(t) \Delta t = 0.$$

By the formula (3.1), the left-hand side is equal to

$$\sum_{s=0}^{p-1} g_k^{(m)}(x_v + s).$$

Suppose $0 < p < m$. In this case, we have $m > 1$. Then $g(t)^{\underline{(m-1)}}$ vanishes when $t = x_v + s$, $0 \leq s \leq m - 2$. The remaining case is $p = m$. Then $x_u = x_v + m$ and $t = x_v + m - 1$. In this case, the product

$$\prod_{1 \leq i < k} (t - x_i + 1) \prod_{k < i \leq \ell} (t - x_i - m + 1)$$

is equal to zero. Indeed, if $v > k$, the factor $(t - x_v - m + 1)$ vanishes. If $v \leq k$, then $u < k$. Then the factor $(t - x_u + 1)$ vanishes.

The case $1 - m \leq p < 0$ is similar (simpler). ∎

Remark 4.3. In [22], generalizations of integral formulas of Bandlow–Musiker are discussed. It would be interesting to consider discrete versions of such generalizations.

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