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Nonsymmetric Macdonald Superpolynomials

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Abstract. There are representations of the type-A Hecke algebra on spaces of polynomials in anti-commuting variables. Luque and the author [*Sém. Lothar. Combin.* **66** (2012), Art. B66b, 68 pages, arXiv:1106.0875] constructed nonsymmetric Macdonald polynomials taking values in arbitrary modules of the Hecke algebra. In this paper the two ideas are combined to define and study nonsymmetric Macdonald polynomials taking values in the aforementioned anti-commuting polynomials, in other words, superpolynomials. The modules, their orthogonal bases and their properties are first derived. In terms of the standard Young tableau approach to representations these modules correspond to hook tableaux. The details of the Dunkl–Luque theory and the particular application are presented. There is an inner product on the polynomials for which the Macdonald polynomials are mutually orthogonal. The squared norms for this product are determined. By using techniques of Baker and Forrester [*Ann. Comb.* **3** (1999), 159–170, arXiv:q-alg/9707001] symmetric Macdonald polynomials are built up from the nonsymmetric theory. Here "symmetric" means in the Hecke algebra sense, not in the classical group sense. There is a concise formula for the squared norm of the minimal symmetric polynomial, and some formulas for anti-symmetric polynomials. For both symmetric and anti-symmetric polynomials there is a factorization when the polynomials are evaluated at special points.

Key words: superpolynomials; Hecke algebra; symmetrization; norms

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1 Introduction

Nonsymmetric Macdonald [13] polynomials are simultaneous eigenfunctions of a set of mutually commuting operators derived from an action of the type-A Hecke algebra on the space of polynomials in N variables. They are significantly different from the symmetric Macdonald polynomials in the technique of their respective definitions and yet Baker and Forrester [1] established a strong relation between them. In the analogous theory of nonsymmetric Jack polynomials Griffeth [11] constructed such polynomials which take values in modules of the underlying groups, specifically the complex reflection groups in the infinite family $G(\ell,p,N)$. These polynomials constitute a standard module of the rational Cherednik algebra. Luque and the author [9] extended the theory of nonsymmetric Macdonald polynomials in the direction suggested by Griffeth's work by studying polynomials taking values in modules of the Hecke algebra. The development relies on exploiting standard Young tableaux and the Yang–Baxter graph technique of Lascoux [12].

The superpolynomials considered here are generated by N anti-commuting and N commuting variables. By defining representations of the Hecke algebra on anti-commuting variables the theory of vector-valued nonsymmetric Macdonald polynomials is applied to define and analyze superpolynomials. There is a theory of symmetric Macdonald superpolynomials initiated by Blondeau-Fournier, Desrosiers, Lapointe, and Mathieu [3] with further developments on norm and special point values by González and Lapointe [10]. Their approach and definitions are based on differential operators and linear combinations of the classical nonsymmetric Macdonald polynomials, whose coefficients involve anti-commuting variables. The theory developed in the present paper is different due to the method of using anti-commuting variables to form Hecke algebra modules.

Nonsymmetric Macdonald polynomials associated with general root systems were intensively studied by Cherednik [5]. By specializing to root systems of type A it becomes possible to develop more detailed relations, formulas and structure. In particular, the papers of Noumi and Mimachi [14], Baker and Forrester [1] provide important background for the present paper. Note that some authors use different axioms for the quadratic relations of the Hecke algebra, such as $\big(T-t^{1/2}\big)\big(T+t^{-1/2}\big)=0$, rather than $(T-t)(T+1)=0$.

The theory of Hecke algebras of type A and their representations is briefly described in Section 2 and then applied to modules of polynomials in anti-commuting variables. In general the irreducible representations are constructed as spans of standard Young tableaux whose shape corresponds to a fixed partition of N. In the present situation it is the hook tableaux which arise. The basis vectors are constructed and the important transformation formulas are stated. There is an inner product in which the generators of the Hecke algebra are self-adjoint which leads to evaluation of the squared norms of the basis elements.

In Section 3 the theory of vector-valued nonsymmetric Macdonald polynomials developed in [9] is applied to produce superpolynomials, considered as polynomials taking values in modules of anti-commuting variables. The main results are stated without proofs but some important details are carefully worked out. In [8] the author constructed an inner product in which the nonsymmetric Macdonald polynomials are mutually orthogonal, in the general vector-valued situation. This structure is worked out for the superpolynomials in Section 3.3 and the squared norms are computed. In Section 4 the techniques of Baker and Forrester [1] are used to produce supersymmetric Macdonald polynomials, and the squared norms. From results of [9] the labels of these polynomials correspond to the superpartitions of Desrosiers, Lapointe, and Mathieu [6]. It has to be emphasized that in this paper the meaning of symmetric is with respect to the Hecke algebra, not the symmetric group. Also the squared norm of the lowest degree supersymmetric polynomial is determined – the formula is more elegant than the general formula; its calculation is able to use telescoping arguments for simplifications. There is a derivation of formulas for antisymmetric Macdonald polynomials in Section 4.5. In the conclusion some further topics of investigation, such as evaluation at special points, are discussed.

2 The Hecke algebra of type A

2.1 Definitions and Jucys–Murphy elements

The Hecke algebra $\mathcal{H}_N(t)$ of type A_{N-1} with parameter t is the associative algebra over an extension field of $\mathbb{Q}$, generated by $\{T_1,\dots,T_{N-1}\}$ subject to the braid relations

$$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\qquad 1\le i<N-1, \tag{2.1a}$$
$$T_iT_j=T_jT_i,\qquad |i-j|\ge 2, \tag{2.1b}$$

and the quadratic relations

$$(T_i-t)(T_i+1)=0,\qquad 1\le i<N, \tag{2.2}$$

where t is a generic parameter (this means $t^n\neq 1$ for $2\le n\le N$). The quadratic relation implies $T_i^{-1}=\frac{1}{t}(T_i+1-t)$. There is a commutative set in $\mathcal{H}_N(t)$ of *Jucys–Murphy elements* defined by $\omega_N = 1$, $\omega_i = t^{-1}T_i\omega_{i+1}T_i$ for $1 \le i < N$, that is,

$$\omega_i = t^{i-N}T_iT_{i+1}\cdots T_{N-1}T_{N-1}T_{N-2}\cdots T_i.$$

Simultaneous eigenvectors of $\{\omega_i\}$ form bases of irreducible representations of the algebra. The symmetric group $\mathcal{S}_N$ is the group of permutations of $\{1, 2, \dots, N\}$ and is generated by the simple reflections (adjacent transpositions) $\{s_i \colon 1 \le i < N\}$, where s_i interchanges i, $i+1$ and fixes the other points (the s_i satisfy the braid relations and $s_i^2 = 1$). There is a linear isomorphism $\mathbb{Z}\mathcal{S}_N \to \mathcal{H}_N(t)$ given by $\sum_{u\in\mathcal{S}_N} a_u u \to \sum_{u\in\mathcal{S}_N} a_u T(u)$, where $T(u) = T_{i_1}\cdots T_{i_\ell}$ with $u = s_{i_1}\cdots s_{i_\ell}$ being a shortest expression for u (in fact $\ell = \#\{(i,j) \colon i < j,\ u(i) > u(j)\}$); $T(u)$ is well-defined because of the braid relations (see [7]).

2.2 Modules of anti-commuting variables

Consider polynomials in N anti-commuting (fermionic) variables $\theta_1, \theta_2, \dots, \theta_N$. They satisfy $\theta_i^2 = 0$ and $\theta_i\theta_j + \theta_j\theta_i = 0$ for $i \ne j$. The basis for these polynomials consists of monomials labeled by subsets of $\{1, 2, \dots, N\}$:

$$\phi_E := \theta_{i_1}\cdots\theta_{i_m}, \qquad E = \{i_1, i_2, \dots, i_m\}, \qquad 1 \le i_1 < i_2 < \cdots < i_m \le N.$$

The polynomials have coefficients in an extension field of $\mathbb{Q}(q,t)$ with transcendental q, t, or generic q, t satisfying $q, t \ne 0$, $q^a \ne 1$, $q^a t^n \ne 1$ for $a \in \mathbb{Z}$ and $n \ne 2, 3, \dots, N$.

Definition 2.1. $\mathcal{P} := \operatorname{span}\big\{\phi_E \colon E \subset \{1, \dots, N\}\big\}$ and $\mathcal{P}_m := \operatorname{span}\big\{\phi_E \colon \#E = m\big\}$ for $0 \le m \le N$. The fermionic degree of ϕ_E is $\#E$.

Some utility formulas are used for working with $\{\phi_E\}$.

Definition 2.2. For a subset $E \subset \{1, 2, \dots, N\}$ and $1 \le i < N$ let

$$\begin{aligned}
&E^C :=, 2, \dots, N\}\backslash E,\\
&\operatorname{inv}(E) := \#\big\{(i,j) \in E \times E^C \colon i < j\big\},\\
&s_i\phi_E = \phi_{s_iE} = \phi_{(E\backslash\{i\})\cup\{i+1\}}, \qquad (i, i+1) \in E \times E^C,\\
&s_i\phi_E = \phi_{s_iE} = \phi_{(E\backslash\{i+1\})\cup\{i\}}, \qquad (i, i+1) \in E^C \times E.
\end{aligned}$$

(When $\{i, i+1\} \subset E$ or $\subset E^C$ then $s_iE = E$ and $s_i\phi_E = \phi_E$.) Introduce a representation of $\mathcal{H}_N(t)$ on $\mathcal{P}$.

Definition 2.3. For $1 \le i < N$

$$T_i\phi_E = \begin{cases} -\phi_E, & \{i, i+1\} \subset E,\\ t\phi_E, & \{i, i+1\} \subset E^C,\\ s_i\phi_E, & (i, i+1) \in E \times E^C,\\ (t-1)\phi_E + ts_i\phi_E, & (i, i+1) \in E^C \times E. \end{cases}$$

Proposition 2.4. *The operators $\{T_i\}$ satisfy the braid and quadratic relations (2.1a) and (2.2).*

Proof. It suffices to verify that $T_1T_2T_1 = T_2T_1T_2$ and $(T_1 - t)(T_1 + 1) = 0$ on the spaces $\operatorname{span}\{\theta_1, \theta_2, \theta_3\}$ and $\operatorname{span}\{\theta_2\theta_3, \theta_1\theta_3, \theta_1\theta_2\}$. The relations are trivially satisfied on $\operatorname{span}\{1\}$ and $\operatorname{span}\{\theta_1\theta_2\theta_3\}$. ■

Remark 2.5. For symbolic computation and to verify the previous proposition use

$$T_i f(\theta_1,\dots,\theta_N) = tf + (t\theta_i - \theta_{i+1})\left(\frac{\partial}{\partial\theta_{i+1}} - \frac{\partial}{\partial\theta_i}\right)f - \left(t\theta_i^2 + \theta_{i+1}^2\right)\frac{\partial^2}{\partial\theta_i\partial\theta_{i+1}}f,$$
$$T_i^{-1} f(\theta_1,\dots,\theta_N) = \frac{1}{t}f + \left(\theta_i - \frac{1}{t}\theta_{i+1}\right)\left(\frac{\partial}{\partial\theta_{i+1}} - \frac{\partial}{\partial\theta_i}\right)f - \left(\theta_i^2 + \frac{1}{t}\theta_{i+1}^2\right)\frac{\partial^2}{\partial\theta_i\partial\theta_{i+1}}f,$$

with the partial derivatives being formal (the order of variables is ignored).

There is a symmetric bilinear form on $\mathcal{P}$ which is positive-definite for $t > 0$ and in which T_i is self-adjoint for $1 \le i < N$. The purpose of the form is to make the simultaneous eigenvectors of $\{\omega_i\}$ mutually perpendicular.

Definition 2.6. For $E, F \subset \{1, 2, \dots, N\}$ define $\langle\phi_E, \phi_F\rangle = \delta_{E,F} t^{-\operatorname{inv}(E)}$ and extend the form to $\mathcal{P}$ by linearity.

Proposition 2.7. *Suppose $f, g \in \mathcal{P}$ and $1 \le i < N$ then $\langle T_i f, g\rangle = \langle f, T_i g\rangle$.*

Proof. It suffices to consider T_1. Let $F \subset \{3, 4, \dots, N\}$. Then $T_1\phi_F = t\phi_F$ and $T_1\phi_{\{1,2\}\cup F} = -\phi_{\{1,2\}\cup F}$. Let $F_i = F \cup \{i\}$ for $i = 1, 2$ and $T_1\phi_{F_1} = \phi_{F_2}$ and $T_1\phi_{F_2} = (t-1)\phi_{F_2} + t\phi_{F_1}$ so that

$$\langle T_1\phi_{F_1}, \phi_{F_2}\rangle = \langle\phi_{F_2}, \phi_{F_2}\rangle = t^{-\operatorname{inv}(F_2)},$$
$$\langle\phi_{F_1}, T_1\phi_{F_2}\rangle = \langle\phi_{F_1}, (t-1)\phi_{F_2} + t\phi_{F_1}\rangle = t\,\langle\phi_{F_1}, \phi_{F_1}\rangle = t^{1-\operatorname{inv}(F_1)},$$

and $\operatorname{inv}(F_1) = \operatorname{inv}(F_2) + 1$ by counting the pair $(1, 2) \in F_1 \times F_1^C$. ■

Corollary 2.8. *If $f, g \in \mathcal{P}$ and $1 \le i \le N$ then $\langle\omega_i f, g\rangle = \langle f, \omega_i g\rangle$.*

Proof. This follows from $\omega_N = 1$ and $\omega_i = t^{-1}T_i\omega_{i+1}T_i$ for $i < N$. ■

There are two degree-changing linear maps which commute with the Hecke algebra action.

Definition 2.9. For $n \in \mathbb{Z}$ set $\sigma(n) := (-1)^n$ and for $E \subset \{1, 2, \dots, N\}$, $1 \le i \le N$ set $s(i, E) := \#\{j \in E \colon j < i\}$. Define the operators ∂_i and $\widehat{\theta}_i$ by $\partial_i\theta_i\phi_E = \phi_E$, $\partial_i\phi_E = 0$ and $\widehat{\theta}_i\phi_E = \theta_i\phi_E = \sigma(s(i, E))\phi_{E\cup\{i\}}$ for $i \notin E$, while $\widehat{\theta}_i\phi_E = 0$ for $i \in E$ $\big($also $i \in E$ implies $\phi_E = \sigma(s(i, E))\theta_i\phi_{E\setminus\{i\}}$ and $\partial_i\phi_E = \sigma(s(i, E))\phi_{E\setminus\{i\}}\big)$. Define $M := \sum_{i=1}^N \widehat{\theta}_i$ and $D := \sum_{i=1}^N t^{i-1}\partial_i$.

By direct computation one can show that $\widehat{\theta}_i\partial_j = -\partial_j\widehat{\theta}_i$ for $i \ne j$.

Proposition 2.10. *M and D commute with T_i for $1 \le i < N$.*

Proof. It follows from the definitions that ∂_i and $\widehat{\theta}_j$ commute with T_i when $j < i$ or $j > i + 1$. It suffices to show $\partial_1 + t\partial_2$ and $\widehat{\theta}_1 + \widehat{\theta}_2$ commute with T_1 applied to $p_1 := \phi_F$, $p_2 := (\theta_1 + \theta_2)\phi_F$, $p_3 := (t\theta_1 - \theta_2)\phi_F$, $p_4 := \theta_1\theta_2\phi_F$ with $1, 2 \notin F$. Then $T_1 p_i = tp_i$ for $i = 1, 2$, $T_1 p_i = -p_i$ for $i = 3, 4$ and

$$(\partial_1 + t\partial_2)[p_1, p_2, p_3, p_4] = [0, (t+1)p_1, 0, -p_3],$$
$$(\widehat{\theta}_1 + \widehat{\theta}_2)[p_1, p_2, p_3, p_4] = [p_2, 0, -(t+1)p_4, 0].$$

This concludes the proof. ■

It is clear that $D^2 = 0 = M^2$. For $n = 0, 1, 2, \dots$ let $[n]_t := \frac{1-t^n}{1-t}$ and $[n]_t! := [1]_t[2]_t \cdots [n]_t$.

Proposition 2.11. *$MD + DM = [N]_t$.*

Proof. Fix ϕ_E, $\#E = m$; $\partial_j\phi_E = \sigma(s(j, E))\phi_{E\setminus\{j\}}$, then $\widehat{\theta}_j\partial_j\phi_E = \sigma(s(j, E))\theta_j\phi_{E\setminus\{j\}} = \phi_E$ thus the coefficient of ϕ_E in MD is $\sum_{j\in E} t^{j-1}$. Also $M\phi_E = \sum_{i\notin E}\theta_i\phi_E$ and the coefficient of ϕ_E in $DM\phi_E$ is $\sum_{i\notin E} t^{i-1}$ so that the coefficient of ϕ_E in $MD + DM$ is $\sum_{j=1}^N t^{j-1} = [N]_t$. Suppose $i \in E$, $j \notin E$ then $\widehat{\theta}_j\phi_{E\setminus\{i\}}$ appears in MD with coefficient $t^{i-1}\sigma(s(i, E))$ while $t^{i-1}\partial_i\theta_j\phi_E = \sigma(s(i, E))t^{i-1}\partial_i\theta_j\theta_i\phi_{E\setminus\{i\}} = -\sigma(s(i, E))t^{i-1}\widehat{\theta}_j\phi_{E\setminus\{i\}}$, and this term is canceled out in $MD + DM$. ■

2.3 Representations of $\mathcal{H}_N(t)$

These representations correspond to partitions of N, namely $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{N}_0^N$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ and $\sum_{i=1}^N \lambda_i = N$. The length of λ is $\ell(\lambda) = \max\{i \colon \lambda_i \ge 1\}$. There is a graphical device to picture λ, called the *Ferrers diagram*, which has boxes at $\{(i,j) \colon 1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i\}$ (integer points). A reverse standard tableau (RSYT) is a filling of the Ferrers diagram with the numbers $\{1, 2, \dots, N\}$ such that the entries decrease in each row and in each column. The relevant representation of $\mathcal{H}_N(t)$ is defined on the span of the RSYT's of shape λ in such a way that $\omega_i Y = t^{c(i,Y)} Y$ for $1 \le i \le N$, where $Y[a,b] = i$, $c(i,Y) = b - a$ ($b - a$ is called the *content* of $[a,b]$), and Y is a RSYT of shape λ. In the present work only hook tableaux will occur, namely partitions of the form $\lambda = (N-n, 1^n)$ (the part 1 is repeated n times), so that $\ell(\lambda) = n+1$.

We will show that $\mathcal{P}_m$ is a direct sum of the $\mathcal{H}_N(t)$-modules corresponding to $(N-m, 1^m)$ and $(N+1-m, 1^{m-1})$. Here is a structure for labeling the ϕ_E of interest.

Definition 2.12. Let $\mathcal{Y}_0 := \{E \colon \#E = m+1,\ N \in E\}$ and $\mathcal{Y}_1 := \{E \colon \#E = m-1,\ N \notin E\}$.

These sets are associated to RSYT's of shape $\big(N-m, 1^m\big)$ and $\big(N-m+1, 1^{m-1}\big)$ respectively, and this correspondence will be used to define content vectors for E.

Definition 2.13. Suppose $E \in \mathcal{Y}_0$ and $E = \{i_1, \dots, i_m, i_{m+1}\}$, $E^C = \{j_1, \dots, j_{N-m-1}\}$ with $i_1 < i_2 < \cdots < i_{m+1} = N$ and $j_1 < j_2 < \cdots$ then Y_E is the RSYT of shape $(N-m, 1^m)$ given by $Y_E[k,1] = i_{m+2-k}$ for $1 \le k \le m+1$, and $Y_E[1,k] = j_{N-m+1-k}$ for $2 \le k \le N-m$. Suppose $E \in \mathcal{Y}_1$ and $E = \{i_1, \dots, i_{m-1}\}$, $E^C = \{j_1, \dots, j_{N-m+1}\}$ with $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots < j_{N-m+1} = N$ then Y_E is the RSYT of shape $(N-m+1, 1^{m-1})$ given by $Y_E[k,1] = i_{m+1-k}$ for $2 \le k \le m$, $Y_E[1,k] = j_{N-m+2-k}$ for $1 \le k \le N-m+1$. In both cases define the content vector $c(i,E) = c\big(i, Y_E\big)$ for $1 \le i \le N$.

For space-saving convenience the RSYT's are displayed in two rows, with the second row consisting of the entries $Y_E[2,1]$, $Y_E[3,1], \dots$. Recall the *content* of cell $[i,j]$ is $j - i$.

As example let $N = 8$, $m = 3$, $E = \{2, 5, 7, 8\}$ then

$$Y_E = \begin{bmatrix} 8 & 6 & 4 & 3 & 1 \\ \cdot & 7 & 5 & 2 & \end{bmatrix}$$

and $[c(i,E)]_{i=1}^8 = [4, -3, 3, 2, -2, 1, -1, 0]$.

We will construct for each $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ a polynomial $\tau_E \in \mathcal{P}_m$ such that $\omega_i \tau_E = t^{c(i,E)} \tau_E$ for $1 \le i \le N$. To start let $E_0 := \{N-m, N-m+1, \dots, N\} \in \mathcal{Y}_0$. Then

$$Y_{E_0} = \begin{bmatrix} N & N-m-1 & N-m-2 & \cdots & \cdots & 1 \\ \cdot & N-1 & N-2 & \cdots & N-m & \end{bmatrix},$$

$$[c(i,E_0)]_{i=1}^N = [N-m-1, N-m-2, \dots, 1, -m, 1-m, \dots, -1, 0].$$

Theorem 2.14. *Let $\psi_0 = D\phi_{E_0} \in \ker D \cap \mathcal{P}_m$, then $\omega_i \psi_0 = t^{c(i,E_0)} \psi_0$ for $1 \le i \le N$.*

Proof. If $N - m \le i < N$ then $T_i \phi_{E_0} = -\phi_{E_0}$ and so $T_i \psi_0 = -\psi_0$, because $T_i D = D T_i$ and $\psi_0 = D\phi_{E_0}$, thus $\omega_i \psi_0 = t^{i-N} \psi_0$. It is clear that $T_i \psi_0 = t\psi_0$ for $1 \le i < N-m-1$ and so it remains to prove $\omega_{N-m-1} \psi_0 = t\psi_0$. (The remaining part of the argument is straightforward, and is at the end of this proof; for example $\omega_{N-m-2}\psi_0 = t^{-1} T_{N-m-2} \omega_{N-m-1} T_{N-m-2} \psi_0 = T_{N-m-2} \omega_{N-m-1} \psi_0 = t^2 \psi_0$.) Let $F := \{N-m-1, N-m, \dots, N\}$ and $F_j := F \backslash \{j\}$, $p_j = \phi_{F_j}$, (so that $p_{N-m-1} = \phi_{E_0}$) then $T_i p_j = -p_j$ if $i > j$ or $N-m-1 \le i < j-1$, $T_j p_j = (1-t) p_j + t p_{j+1}$

and $T_j p_{j+1} = p_j$. To set up an induction argument let $U_{N-m-1} = T_{N-m-1}$ and $U_{i+1} = T_{i+1}U_i$ for $i < N-1$. We claim

$$U_i \phi_{E_0} = t^{i-N+m+2} p_{i+1} + (t-1) \sum_{j=N-m-1}^{i} (-1)^{i-j} t^{j-N+m+1} p_j.$$

At the start of the induction $T_{N-m-1} p_{N-m-1} = t p_{N-m} + (t-1) p_{N-m-1}$. Suppose the formula holds for i then

$$\begin{aligned} U_{i+1}\phi_{E_0} &= t^{i-N+m+2} T_{i+1} p_{i+1} + (t-1) \sum_{j=N-m-1}^{i} (-1)^{i-j} t^{j-N+m+1} T_{i+1} p_j \\ &= t^{i-N+m+2}(t p_{i+2} + (t-1) p_{i+1}) - (t-1) \sum_{j=N-m-1}^{i} (-1)^{i-j} t^{j-N+m+1} p_j \\ &= t^{i-N+m+3} p_{i+2} + (t-1) \sum_{j=N-m}^{i+1} (-1)^{j-i+1} t^{j-N+m+1} p_j, \end{aligned}$$

as is to be shown. We also need

$$D\phi_F = \sum_{j=N-m-1}^{N-1} (-1)^{j-N+m+1} t^{j-1} p_j + (-1)^{m+1} t^{N-1} p_N.$$

Thus

$$\begin{aligned} U_{N-1}\phi_{E_0} &= t^{m+1} p_N + (t-1) \sum_{j=N-m-1}^{N-1} (-1)^{N-1-j} t^{j-N+m+1} p_j \\ &= t^{m+1} p_N + (t-1)(-1)^m t^{-N+m+2} \left\{ D\phi_F - (-1)^{m+1} t^{N-1} p_N \right\} \\ &= t^{m+2} p_N + (t-1)(-1)^m t^{-N+m+2} D\phi_F. \end{aligned}$$

Then $U_{N-1}\psi_0 = DU_{N-1}\phi_{E_0} = t^{m+2} Dp_N$ and $T_{N-m-1}T_{N-m}\cdots T_{N-1}U_{N-1}\psi_0 = t^{m+2}Dp_{N-m-1}$ $= t^{m+2}\psi_0$ since $T_j p_{j+1} = p_j$ for $N-m-1 \le j < N$. Hence $\omega_{N-m-1}\psi_0 = t^{N-m-1-N} t^{m+2}\psi_0$ $= t\psi_0$. It follows that

$$\begin{aligned} \omega_i \psi_0 &= t^{i-N+m+1} T_i \cdots T_{N-m-2} \omega_{N-m-1} T_{N-m-2} \cdots T_i \psi_0 \\ &= t^{i-N+m+1} t^{1+2(N-m-1-i)} \psi_0 = t^{N-m-i} \psi_0, \end{aligned}$$

for $1 \le i \le N-m-1$. ■

Turning to the isotype $\left(N-m+1, 1^{m-1}\right)$, let $E_1 := \{1, 2, \ldots, m-1\} \in \mathcal{Y}_1$ so that

$$\begin{aligned} Y_{E_1} &= \begin{bmatrix} N & N-1 & N-2 & \cdots & \cdots & m \\ \cdot & m-1 & m-2 & \cdots & 1 & \end{bmatrix}, \\ [c(i,E_1)]_{i=1}^N &= [1-m, 2-m, \ldots, -1, N-m, N-m-1, \ldots, 1, 0]. \end{aligned}$$

Theorem 2.15. *Let $\eta_{E_1} = M\phi_{E_1} \in \ker M \cap \mathcal{P}_m$. Then $\omega_i \eta_{E_1} = t^{c(i,E_1)} \eta_{E_1}$ for $1 \le i \le N$.*

Proof. Since $T_i \phi_{E_1} = t\phi_{E_1}$ for $m \le i < N$ it follows that $\omega_i M\phi_{E_1} = t^{N-i} M\phi_{E_1}$. Also $T_i \phi_{E_1} = -\phi_{E_1}$ and $T_i M\phi_{E_1} = -M\phi_{E_1}$ for $1 \le i < m-1$ and so it remains to show $\omega_{m-1}\eta_{E_1} = t^{-1}\eta_{E_1}$. Let $F = \{1, 2, \ldots, m-2\}$ and for $m-1 \le j \le N$ let $F_j = F \cup \{j\}$ and $p_j = \phi_{F_j}$ $(\phi_{E_1} = p_{m-1})$.

Then $T_jp_j = p_{j+1}$ so that $T_{N-1}T_{N-2}\cdots T_{m-1}p_{m-1} = p_N$. Also $T_jp_{j+1} = tp_j + (t-1)p_{j+1}$ and $T_jp_i = tp_i$ for $i > j+1$. By induction we prove that

$$T_iT_{i+1}\cdots T_{N-1}p_N = t^{N-i}p_i + (t-1)t^{N-i-1}\sum_{j=i+1}^{N} p_j.$$

The formula is valid for $i = N-1$ and assuming it is true for i apply T_{i-1} to both sides, then the first term becomes $t^{N-i}\left(p_{i-1} + (t-1)p_i\right)$ and the second term is multiplied by t. Substitute $M\phi_F = \sum_{j=m-1}^{N} p_j$ in the formula with $i = m-1$ to obtain

$$\begin{aligned} T_{m-1}\cdots T_{N-1}p_N &= t^{N-m+1}p_{m-1} + (t-1)t^{N-m}\left\{M\phi_F - p_{m-1}\right\} \\ &= t^{N-m}p_{m-1} + (t-1)t^{N-m}M\phi_F. \end{aligned}$$

Thus $\omega_{m-1}\eta_{E_1} = t^{m-1-N}t^{N-m}Mp_{m-1} = t^{-1}\eta_{E_1}$ (since $M^2 = 0$). From $T_i\eta_{E_1} = -\eta_{E_1}$ for $1 \le i < m-1$ it follows that $\omega_i\eta_{E_1} = t^{i-m}\eta_{E_1}$. Thus $\omega_i\eta_{E_1} = t^{c(i,E_1)}\eta_{E_1}$ for $1 \le i \le N$. ∎

2.4 Steps

Having found two polynomials which are $\{\omega_i\}$ simultaneous eigenfunctions we describe the method for constructing for each $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ a polynomial $\tau_E \in \mathcal{P}_m$ such that $\omega_i\tau_E = t^{c(i,E)}\tau_E$ for $1 \le i \le N$. Recall the standard properties $T_i\omega_j = \omega_jT_i$ for $i < j-1$ (obvious) and for $i > j$ (suppose $i = j+1$) then

$$\begin{aligned} T_{j+1}\omega_j &= t^{-2}T_{j+1}T_jT_{j+1}\omega_{j+2}T_{j+1}T_j = t^{-2}T_jT_{j+1}T_j\omega_{j+2}T_{j+1}T_j = t^{-2}T_jT_{j+1}\omega_{j+2}T_jT_{j+1}T_j \\ &= t^{-2}T_jT_{j+1}\omega_{j+2}T_{j=1}T_jT_{j+1} = \omega_jT_{j+1}, \end{aligned}$$

by the braid relations; and

$$\begin{aligned} &T_j\omega_j = t^{-1}T_j^2\omega_{j+1}T_j = t^{-1}\{(t-1)T_j + t\}\omega_{j+1}T_j = (t-1)\omega_j + \omega_{j+1}T_j, \\ &\omega_jT_j = T_j\omega_{j+1} + (t-1)\omega_j. \end{aligned} \tag{2.3}$$

Proposition 2.16. *Suppose $\omega_jf = \lambda_jf$ for $1 \le j \le N$ ($f \ne 0$), $\lambda_i \ne \lambda_{i+1}$ and*

$$g := T_if + \frac{(t-1)\lambda_i}{\lambda_{i+1} - \lambda_i}f$$

then $\omega_jg = \lambda_jg$ for all $j \ne i, i+1$ and $\omega_ig = \lambda_{i+1}g$, $\omega_{i+1}g = \lambda_ig$. If $\lambda_{i+1} \ne t^{\pm 1}\lambda_i$ then $g \ne 0$.

Proof. If $j > i+1$ or $j < i$ then $\omega_jT_i = T_i\omega_j$ and thus $\omega_jg = \lambda_jg$. By (2.3)

$$\begin{aligned} \omega_ig &= \omega_iT_if + \lambda_i\frac{(t-1)\lambda_i}{\lambda_{i+1} - \lambda_i}f = T_i\omega_{i+1}f + \left\{t - 1 + \frac{(t-1)\lambda_i}{\lambda_{i+1} - \lambda_i}\right\}\lambda_if \\ &= \lambda_{i+1}T_if + \frac{(t-1)\lambda_{i+1}\lambda_i}{\lambda_{i+1} - \lambda_i}f = \lambda_{i+1}g. \end{aligned}$$

A similar calculation using $\omega_{i+1}T_i = T_i\omega_i - (t-1)\omega_i$ shows that $\omega_{i+1}g = \lambda_ig$. Since $T_i^2 = (t-1)T_i + t$

$$\left(T_i + \frac{(t-1)\lambda_{i+1}}{\lambda_i - \lambda_{i+1}}\right)\left(T_i + \frac{(t-1)\lambda_i}{\lambda_{i+1} - \lambda_i}\right) = \frac{(\lambda_it - \lambda_{i+1})(\lambda_i - t\lambda_{i+1})}{(\lambda_i - \lambda_{i+1})^2},$$

thus $\lambda_{i+1} \ne t^{\pm 1}\lambda_i$ implies $g \ne 0$. ∎

Given the hypotheses of the proposition and the self-adjointness of ω_i (Corollary 2.8) it follows that $\langle f,g\rangle=0$ ($\lambda_i\langle f,g\rangle=\langle\omega_i f,g\rangle=\langle f,\omega_i g\rangle=\lambda_{i+1}\langle f,g\rangle$).

Lemma 2.17. *Suppose $g=(T_i+b)f$ and $\langle f,g\rangle=0$ then $\|g\|^2=(1-b)(t+b)\|f\|^2$.*

Proof. It follows from T_i being self-adjoint that $\langle T_if,T_if\rangle=\langle T_i^2f,f\rangle=(t-1)\langle T_if,f\rangle+t\|f\|^2$ and $\langle f,g\rangle=0$ implies $\langle T_if,f\rangle+b\|f\|^2=0$. Thus

$$\begin{aligned}\|g\|^2&=\|T_if\|^2+2b\langle T_if,f\rangle+b^2\|f\|^2=(t-1+2b)\langle T_if,f\rangle+(t+b^2)\|f\|^2\\&=\{-b(t-1+2b)+t+b^2\}\|f\|^2=\big(-b^2-b(t-1)+t\big)\|f\|^2=(1-b)(t+b)\|f\|^2.\end{aligned}$$ ■

Corollary 2.18. *Suppose $g=(T_i+b)f$, $\langle g,f\rangle=0$ and $b=1$ or $b=-t$ then $g=0$.*

2.4.1 Isotype $(N-m,1^m)$

This concerns the polynomials in $\mathcal{P}_{m,0}=\ker D\cap\mathcal{P}_m=D\mathcal{P}_{m+1}$. Recall $\mathcal{Y}_0=\{E\colon \#E=m+1, N\in E\}$.

Definition 2.19. For $\#E=m+1$ define $\psi_E=D\phi_E$.

The set $\{\psi_E\colon E\in\mathcal{Y}_0\}$ spans $\mathcal{P}_{m,0}$; for suppose $N\notin E$ then $\theta_ND\phi_E$ is a linear combination of ϕ_F with $F\in\mathcal{Y}_0$ and $t^{1-N}D\theta_ND\phi_E=D\phi_E=\psi_E$. The map $p(\theta_1,\dots,\theta_N)\to p(\theta_1,\dots,\theta_{N-1},0)$ takes ψ_E to $t^{N-1}\phi_{E\backslash\{N\}}$; thus $\dim\mathcal{P}_{m,0}=\binom{N-1}{m}$. The function $\operatorname{inv}(E)$ provides a partial order on $\mathcal{Y}_0$.

Definition 2.20. For $0\le n\le m(N-1-m)$ let

$$\mathcal{P}_{m,0}^{(n)}:=\operatorname{span}\{\psi_E\colon E\in\mathcal{Y}_0,\ \operatorname{inv}(E)\le n\}.$$

The extreme cases are $\operatorname{inv}(\{N-m,\dots,N\})=0$ and $\operatorname{inv}(\{1,2,\dots,m,N\})=m(N-1-m)$.

Lemma 2.21. *Suppose $E\in\mathcal{Y}_0$ and $\operatorname{inv}(E)=n$. If $\{i,i+1\}\subset E$ then $T_i\psi_E=-\psi_E$, or if $\{i,i+1\}\cap E=\varnothing$ then $T_i\psi_E=t\psi_E$. If $(i,i+1)\in E\times E^C$ then $\operatorname{inv}(s_iE)=n-1$ and $T_i\psi_E=\psi_{s_iE}$. If $i(i,i+1)\in E^C\times E$ then $\operatorname{inv}(s_iE)=n+1$ and $T_i\psi_E=(t-1)\psi_E+t\psi_{s_iE}\in\mathcal{P}_{m,0}^{(n+1)}$. That is, $T_i\mathcal{P}_{m,0}^{(n)}\subset\mathcal{P}_{m,0}^{(n+1)}$.*

Proof. The transformation rules follow from Definition 2.3 and $DT_i=T_iD$ (see Proposition 2.10). ■

Theorem 2.22. *Suppose for some n and for each $E\in\mathcal{Y}_0$ with $\operatorname{inv}(E)=n$ there is a polynomial $\tau_E=t^n\psi_E+p_E$ with $p_E\in\mathcal{P}_{m,0}^{(n-1)}$ such that $\omega_i\tau_E=t^{c(i,E)}\tau_E$ for all i then this property holds for $n+1$.*

Proof. Suppose $E\in\mathcal{Y}_0$ with $\operatorname{inv}(E)=n+1$ then for some i it holds that $(i,i+1)\in E\times E^C$ (otherwise $E=E_0=\{N-m,N-m+1,\dots,N\}$ and $\operatorname{inv}(E_0)=0$), then let $F=s_iE$ so that $\operatorname{inv}(F)=n$. Then $c(j,E)=c(j,F)$ for $j\ne i,i+1$ and $c(i,E)=c(i+1,F)\le-1$, $c(i+1,E)=c(i,F)\ge1$. By Proposition 2.16 let

$$\tau_E=\left(T_i+\frac{(t-1)t^{c(i,F)}}{t^{c(i+1,F)}-t^{c(i,F)}}\right)\tau_F.$$

Then $\omega_j\tau_E=t^{c(j,E)}\tau_E$ for all j; and $\tau_E=(T_i+b)\big(t^n\psi_F+p_F\big)=t^{n+1}\psi_E+t^n(t-1+b)\psi_F+(T_i+b)p_E$ (for a constant b). By the lemma $t^n(t-1+b)\psi_F+(T_i+b)p_E\in\mathcal{P}_{m,0}^{(n)}$. ■

Corollary 2.23. *For each $E \in \mathcal{Y}_0$ and $\operatorname{inv}(E) = n$ there is a unique $\tau_E \in \mathcal{P}_{m,0}$ with $\tau_E = t^n \psi_E + p_E$ and $p_E \in \mathcal{P}_{m,0}^{(n-1)}$ such that $\omega_i \tau_E = t^{c(i,E)} \tau_E$ for all i.*

Proof. The existence follows from induction starting with $\tau_{E_0} = \psi_{E_0}$ and Theorem 2.14. Uniqueness follows from the leading term. The $\{\omega_i\}$-eigenvalues of τ_E determine E uniquely. ■

Corollary 2.24. *Suppose $E \in \mathcal{Y}_0$, if $\{i, i+1\} \subset E$ then $T_i \tau_E = -\tau_E$ and if $\{i, i+1\} \cap E = \varnothing$ or $i = N-1 \notin E$ then $T_i \tau_E = t\tau_E$.*

Proof. Let $b = \frac{(t-1)t^{c(i,E)}}{t^{c(i+1,E)} - t^{c(i,E)}} = \frac{t-1}{t^{c(i+1,E)-c(i,E)} - 1}$. If $\{i, i+1\} \subset E$ then $c(i+1, E) = 1 + c(i, E)$ and $b = 1$; thus $\langle (T_i + 1)\tau_E, \tau_E \rangle = 0$ (by the comment after Proposition 2.16) and $\|(T_i + 1)\tau_E\|^2 = 0$ by Corollary 2.18. If $\{i, i+1\} \cap E = \varnothing$ (or $i = N-1 \notin E$) then $c(i+1, E) = c(i, E) - 1$ and $b = \frac{t-1}{t^{-1}-1} = -t$; thus $\|(T_i - t)\tau_E\|^2 = 0$. ■

Definition 2.25. Let $u(z) := \frac{(t-z)(1-tz)}{(1-z)^2}$. Suppose $v \in \mathbb{Z}^N$ and $v_j \neq 0$ for $j < N$, $v_N = 0$, then

$$\mathcal{C}(v) := \prod_{1 \le i < j < N} \left\{ u\left(t^{v_i - v_j}\right) : v_i < 0 < v_j \right\}. \tag{2.4}$$

Proposition 2.26. *Suppose $E \in \mathcal{Y}_0$ then*

$$\|\tau_E\|^2 = t^{2(N-m-1)} [m+1]_t \mathcal{C}\left([c(i, E)]_{i=1}^N\right).$$

Proof. By definition $\tau_{E_0} = \sum_{j=N-m}^N t^{j-1}(-1)^{N-m-j} \phi_{E_0 \backslash \{j\}}$ and $\|\tau_{E_0}\|^2 = \sum_{j=N-m}^N t^{2j-2} t^{-i_j}$, where $i_j = \operatorname{inv}\left(E_0 \backslash \{j\}\right) = j - N + m$; thus $\|\tau_{E_0}\|^2 = t^{2(N-m-1)} \sum_{j=0}^m t^j$. Suppose the formula is valid for all E with $\operatorname{inv}(E) \le n$ and $\operatorname{inv}(E) = n+1$. Then $E = s_i F$ for some $i \in E$ with $i+1 \notin E$ and $\operatorname{inv}(F) = n$ (so $i+1 \in F$, $i \notin F$). By Lemma 2.17 $\|\tau_E\|^2 = (1-b)(t+b)\|\tau_F\|^2$, where $b = \frac{t-1}{t^{c(i+1,F)-c(i,F)} - 1}$. Write $z = t^{c(i+1,F)-c(i,F)} = t^{c(i,E)-c(i+1,E)}$ then

$$\|\tau_E\|^2 = \frac{(t-z)(1-tz)}{(1-z)^2} \|\tau_F\|^2 = u(z) \|\tau_F\|^2.$$

In the product for $\|\tau_F\|^2$ the factors for pairs (i) (k, ℓ) with $\{k, \ell\} \cap \{i, i+1\} = \varnothing$, (ii) (k, i), $k \in F$, $k < i$, (iii) $(i+1, \ell)$, $\ell \notin F, \ell > i+1$ are the same in the product for $\|\tau_E\|^2$ for the pairs (i) (k, ℓ), (ii) $(k, i+1)$, (iii) (i, ℓ), respectively. The extra factor in the product for $\|\tau_E\|^2$ has the desired value. ■

2.4.2 Isotype $(N-m+1, 1^{m-1})$

This concerns the polynomials in $\mathcal{P}_{m,1} = \ker M \cap \mathcal{P}_m = M\mathcal{P}_{m-1}$.

Definition 2.27. For $\#E = m-1$ define $\eta_E := M\phi_E$. The set $\mathcal{Y}_1 = \left\{ E : \#E = m-1, N \notin E \right\}$.

The set $\left\{ \eta_E : E \in \mathcal{Y}_1 \right\}$ spans $\mathcal{P}_{m,1}$; for suppose $N \in E$ then $\partial_N M\phi_E$ is a linear combination of ϕ_F with $F \in \mathcal{Y}_1$ and $M\left(\partial_N M \phi_E\right) = M\phi_E = \eta_E$. Furthermore $\partial_N \eta_E = \phi_E$ and thus $\dim \mathcal{P}_{m,1} = \binom{N-1}{m-1}$. The function $\operatorname{inv}(E)$ provides a partial order on $\mathcal{Y}_1$. The extreme cases are $\operatorname{inv}(\{N-m+1, \dots, N-1\}) = m-1$ and $\operatorname{inv}(\{1, 2, \dots, m-1\}) = (m-1)(N-m+1)$.

Definition 2.28. For $0 \le n \le (m-1)(N-m+1)$ let

$$\mathcal{P}_{m,1}^{(n)} := \operatorname{span}\left\{ \eta_E : E \in \mathcal{Y}_1, \operatorname{inv}(E) \ge n \right\}.$$

Lemma 2.29. *Suppose $E \in \mathcal{Y}_1$ and $\operatorname{inv}(E) = n$. If $\{i, i+1\} \subset E$ then $T_i \eta_E = -\eta_E$, or if $\{i, i+1\} \cap E = \varnothing$ then $T_i \eta_E = t\eta_E$. If $i \notin E$, $i+1 \in E$ then $\operatorname{inv}(s_i E) = n+1$ and $T_i \eta_E = (t-1)\eta_E + t\eta_{s_i E} \in \mathcal{P}_{m,1}^{(n)}$. If $i \in E$, $i+1 \notin E$ then $\operatorname{inv}(s_i E) = n-1$ and $T_i \eta_E = \eta_{s_i E} \in \mathcal{P}_{m,1}^{(n-1)}$.*

Proof. The transformation rules follow from Definition 2.3 and $MT_i = T_iM$. ∎

Theorem 2.30. *Suppose for some n and for each $E \in \mathcal{Y}_1$ with $\operatorname{inv}(E) = n$ there is a polynomial $\tau_E = \eta_E + p_E$ with $p_E \in \mathcal{P}_{m,1}^{(n+1)}$ such that $\omega_i \tau_E = t^{c(i,E)}\tau_E$ for all i then this property holds for $n-1$.*

Proof. Suppose $E \in \mathcal{Y}_1$ with $\operatorname{inv}(E) = n-1$ then for some i it holds that $i \notin E, i+1 \in E$ (otherwise $E = E_1 = \{1,\dots,m-1\}$ and $\operatorname{inv}(E_1) = (m-1)(N-m+1)$), then let $F = s_iE$ so that $\operatorname{inv}(F) = n$. Then $c(j,E) = c(j,F)$ for $j \neq i, i+1$ and $c(i,E) = c(i+1,F) \geq 1$, $c(i+1,E) = c(i,F) \leq -1$. By Proposition 2.16 let

$$\tau_E = \left(T_i + \frac{(t-1)t^{c(i,F)}}{t^{c(i+1,F)} - t^{c(i,F)}}\right)\tau_F.$$

Then $\omega_j\tau_E = t^{c(j,E)}\tau_E$ for all j; and $\tau_E = (T_i + b)\big(\eta_F + p_F\big) = \eta_E + b\eta_F + (T_i+b)p_F$ (for a constant b). By the previous lemma $b\eta_F + (T_i + b)p_E \in \mathcal{P}_{m,1}^{(n)}$. ∎

Corollary 2.31. *For each $E \in \mathcal{Y}_1$ and $\operatorname{inv}(E) = n \leq (m-1)(N-m+1)$ there is a unique $\tau_E \in \mathcal{P}_{m,1}$ with $\tau_E = \eta_E + p_E$ and $p_E \in \mathcal{P}_{m,1}^{(n+1)}$ such that $\omega_i\tau_E = t^{c(i,E)}\tau_E$ for all i.*

Proof. The existence follows from induction starting with $\tau_{E_1} = M\phi_{E_1} = \eta_{E_1}$ and Theorem 2.15. Uniqueness follows from the leading term. The eigenvalues of τ_E determine E uniquely. ∎

Corollary 2.32. *Suppose $E \in \mathcal{Y}_1$, if $\{i,i+1\} \in E$ then $T_i\tau_E = -\tau_E$ and if $\{i,i+1\} \cap E = \varnothing$ then $T_i\tau_E = t\tau_E$.*

Proof. This has the same proof as Corollary 2.24. ∎

Proposition 2.33. *Suppose $E \in \mathcal{Y}_1$ then*

$$\|\tau_E\|^2 = t^{-m(N-m)}[N-m+1]_t\, \mathcal{C}\big([-c(i,E)]_{i=1}^N\big).$$

Proof. By definition $\tau_{E_1} = (-1)^{m-1}\sum_{j=m}^N \phi_{E_1\cup\{j\}}$ and $\|\tau_{E_1}\|^2 = \sum_{j=m}^N t^{-i_j}$, where $i_j = \operatorname{inv}\big(E_1 \cup \{j\}\big) = m(N+1-m) - j$; thus $\|\tau_{E_1}\|^2 = t^{-m(N-m)}\sum_{j=0}^{N-m} t^j$. Suppose the formula is valid for all E with $\operatorname{inv}(E) \geq n$ and $\operatorname{inv}(E) = n-1$. Then $E = s_iF$ for some $i \notin E$ with $i+1 \in E$ and $\operatorname{inv}(F) = n$ (so $(i,i+1) \in F \times F^C$). By Lemma 2.17 $\|\tau_E\|^2 = (1-b)(t+b)\|\tau_F\|^2$, where $b = \frac{t-1}{t^{c(i+1,F)-c(i,F)}-1}$. Write $z = t^{c(i+1,F)-c(i,F)} = t^{c(i,E)-c(i+1,E)}$ then

$$\|\tau_E\|^2 = \frac{(t-z)(1-tz)}{(1-z)^2}\|\tau_F\|^2 = u(z)\|\tau_F\|^2.$$

In the product for $|\tau_F|^2$ the factors for pairs (i) (k,ℓ) with $\{k,\ell\} \cap \{i,i+1\} = \varnothing$, (ii) (k,i), $k \notin F$, $k < i$, (iii) $(i+1,\ell)$, $\ell \in F$, $\ell > i+1$ are the same in the product for $|\tau_E|^2$ for the pairs (i) (k,ℓ), (ii) $(k,i+1)$, (iii) (i,ℓ), respectively. The extra factor in the product for $|\tau_E|^2$ has the desired value. ∎

Proposition 2.34. *Let $F_0 = \{1,\dots,m,N\}$, $F_1 = \{1,\dots,m\}$ then $\tau_{F_0} \in \mathcal{P}_{m,0}$ and $\tau_{F_1} \in \mathcal{P}_{m+1,1}$ have the same $\{\omega_i\}$-eigenvalues and*

$$\|\tau_{F_0}\|^2 = t^{(m+2)(N-m-1)}\frac{[N]_t}{[N-m]_t}, \qquad \|\tau_{F_1}\|^2 = t^{-(m+1)(N-m-1)}[N-m]_t,$$
$$\|\tau_{F_0}\|^2 = t^{(2m+3)(N-m-1)}[N]_t[N-m]_t^{-2}\|\tau_{F_1}\|^2,$$
$$D\tau_{F_1} = t^{-(m+1)(N-m-1)}[N-m]_t\tau_{F_0}, \qquad \|D\tau_{F_1}\|^2 = t^{N-m-1}[N]_t\|\tau_{F_1}\|^2.$$

Proof. The content vector for F_0 is $[-m, 1-m, \ldots, -1, N-m-1, \ldots, 1, 0]$ (the same as F_1) so that

$$\begin{aligned}
\mathcal{C}\big([c(i,F_0)]_{i=1}^N\big) &= \prod_{i=1}^{m}\prod_{j=1}^{N-m-1}\frac{(t-t^{-i-j})(1-t^{1-i-j})}{(1-t^{-i-j})^2}\\
&= t^{m(N-m-1)}\prod_{i=1}^{m}\prod_{j=1}^{N-m-1}\left(\frac{t^{i+j+1}-1}{t^{i+j}-1}\right)\left(\frac{t^{i+j-1}-1}{t^{i+j}-1}\right)\\
&= t^{m(N-m-1)}\prod_{j=1}^{N-m-1}\left(\frac{t^{m+1+j}-1}{t^{1+j}-1}\right)\left(\frac{t^{j}-1}{t^{m+j}-1}\right)\\
&= t^{m(N-m-1)}\left(\frac{t-1}{t^{N-m}-1}\right)\left(\frac{t^N-1}{t^{m+1}-1}\right) = \frac{t^{m(N-m-1)}[N]_t}{[N-m]_t[m+1]_t}
\end{aligned}$$

by use of telescoping arguments. Thus $\|\tau_{F_0}\|^2 = t^{(m+2)(N-m-1)}\frac{[N]_t}{[N-m]_t}$ by Proposition 2.26. The value of $\|\tau_{F_1}\|^2$ is from Proposition 2.33. By definition $\tau_{F_1} = M\phi_{F_1} = (\theta_{m+1}+\cdots+\theta_N)\times\theta_1\theta_2\cdots\theta_m$ and the coefficient of ϕ_{F_1} in $D\tau_{F_1}$ is $\sum_{i=m+1}^N t^{i-1} = t^m[N-m]_t$. The coefficient of ϕ_{F_1} in $\tau_{F_0} = D\phi_{F_1}$ is $(-1)^m t^{m(N-1-m)+N-1}$(see Theorem 2.22). From $D\tau_{F_1}$ and τ_{F_0} having the same $\{\omega_i\}$-eigenvalues it follows that $D\tau_{F_1} = a\tau_{F_0}$ for some constant. ■

2.5 Isomorphisms

This section concerns the action of the maps M, D on the irreducible $\mathcal{H}_N(t)$-modules. The following is a version of Schur's lemma for irreducible representations.

Lemma 2.35. *Suppose μ is a linear isomorphism $V_1 \to V_2$ of irreducible $\mathcal{H}_N(t)$-modules such that $T_i\mu = \mu T_i$ for $1 \le i < N$ and V_1, V_2 are equipped with inner products in which each T_i is self-adjoint, then $\|\mu f\|^2/\|f\|^2$ is constant for $f \in V_1$.*

Proof. The argument is based on orthogonal bases defined in the previous sections. By hypothesis V_1 has an orthogonal basis consisting of $\{\omega_i\}$-eigenfunctions. The image of this basis under μ has the same property. For a typical basis element $f \in V_1$ suppose $\omega_j f = \lambda_j f$ for all j and $\lambda_{i+1} \ne t^{\pm1}\lambda_i$ then $g = (T_i + b)f$ satisfies $\omega_i g = \lambda_{i+1}g, \omega_{i+1}g = \lambda_i g$ for $b = \frac{(t-1)\lambda_i}{\lambda_{i+1}-\lambda_i}$ and $\|g\|^2 = (1-b)(t+b)\|f\|^2$ (this equation follows from T_i being self-adjoint and $\langle f, g\rangle = 0$). By hypothesis $\omega_j\mu f = \lambda_j\mu f$ for all j and $\mu g = (T_i+b)\mu f$ satisfies $\omega_i\mu g = \lambda_{i+1}\mu g, \omega_{i+1}\mu g = \lambda_i\mu g$. By Lemma 2.17 $\|\mu g\|^2 = (1-b)(t+b)\|\mu f\|^2$ and so $\gamma := \|\mu g\|^2/\|g\|^2 = \|\mu f\|^2/\|f\|^2$. By the step constructions $\|\mu f\|^2/\|f\|^2 = \gamma$ holds for every basis vector of V_1. ■

The relation $MD + DM = [N]_t$ (Proposition 2.11) implies that $\mathcal{P}_m$ is a direct sum of $\mathcal{P}_{m,0} = \mathcal{P}_m \cap \ker D$ and $\mathcal{P}_{m,1} = \mathcal{P}_m \cap \ker M$.

Theorem 2.36. *The maps M, D are linear isomorphisms $\mathcal{P}_{m,0} \to \mathcal{P}_{m+1,1}$, $\mathcal{P}_{m+1,1} \to \mathcal{P}_{m,0}$ respectively, of $\mathcal{H}_N(t)$-modules, $\|Mf\|^2 = t^{m+1-N}[N]_t\|f\|^2$ for $f \in \mathcal{P}_{m,0}$ and $\|Dg\|^2 = t^{N-m-1}\times[N]_t\|g\|^2$ for $g \in \mathcal{P}_{m+1,1}$.*

Proof. M and D commute with each T_i and hence with each ω_i. Furthermore if $f \in \mathcal{P}_{m,0}$ then $(MD+DM)f = DMf = [N]_t f$ (by Proposition 2.11) and thus M is one-to-one on $\mathcal{P}_{m,0}$. Similarly if $g \in \mathcal{P}_{m+1,1}$ then $[N]_t g = (MD+DM)g = MDg$ and D is one-to-one. By the lemma there are constants γ_1, γ_2 such that $\|Mf\|^2 = \gamma_1\|f\|^2$ and $\|Dg\|^2 = \gamma_2\|g\|^2$. From $(MD+DM)f = DMf$ it follows that $\|DMf\|^2 = [N]_t^2\|f\|^2 = \gamma_2\|Mf\|^2$ and $\gamma_1 = [N]_t^2/\gamma_2$. By Proposition 2.34 $\gamma_2 = t^{N-m-1}[N]_t$ and thus $\gamma_1 = t^{m+1-N}[N]_t$. ■

3 Nonsymmetric Macdonald polynomials

3.1 Operators on polynomials

The following presents the key concepts for our constructions: the definition of the action of $\mathcal{H}_N(t)$ on superpolynomials and the ingredients necessary to define the Cherednik operators whose simultaneous eigenvectors are the nonsymmetric Macdonald superpolynomials. Here we extend the polynomials in $\{\theta_i\}$ by adjoining N commuting variables $x_1,\dots,x_N$ (that is $[x_i,x_j]=0$, $[x_i,\theta_j]=0$, $\theta_i\theta_j=-\theta_j\theta_i$ for all i, j). Each polynomial is a sum of monomials $x^\alpha\phi_E$, where $E\subset\{1,2,\dots,N\}$ and $\alpha\in\mathbb{N}_0^N$, $x^\alpha:=\prod_{i=1}^N x_i^{\alpha_i}$. The partitions in $\mathbb{N}_0^N$ are denoted by $\mathbb{N}_0^{N,+}$ ($\lambda\in\mathbb{N}_0^{N,+}$ if and only if $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N$). The *fermionic* degree of this monomial is $\#E$ and the *bosonic* degree is $|\alpha|:=\sum_{i=1}^N\alpha_i$. Let $s\mathcal{P}_m:=\operatorname{span}\{x^\alpha\phi_E\colon\alpha\in\mathbb{N}_0^N,\ \#E=m\}$. Then using the decomposition $\mathcal{P}_m=\mathcal{P}_{m,0}\oplus\mathcal{P}_{m,1}$ let

$$
\begin{aligned}
&s\mathcal{P}_{m,0}=\operatorname{span}\{x^\alpha\psi_E\colon\alpha\in\mathbb{N}_0^N,\ E\in\mathcal{Y}_0\},\\
&s\mathcal{P}_{m,1}=\operatorname{span}\{x^\alpha\eta_E\colon\alpha\in\mathbb{N}_0^N,\ E\in\mathcal{Y}_1\}.
\end{aligned}
$$

The Hecke algebra $\mathcal{H}_N(t)$ is represented on $s\mathcal{P}_m$. This allows us to apply the theory of nonsymmetric Macdonald polynomials taking values in $\mathcal{H}_N(t)$-modules (see [8, 9]).

Definition 3.1. Suppose $p\in s\mathcal{P}_m$ and $1\leq i<N$ then set

$$
\boldsymbol{T}_i p(x;\theta):=(1-t)x_{i+1}\frac{p(x;\theta)-p(xs_i;\theta)}{x_i-x_{i+1}}+T_i p(xs_i;\theta).
$$

Note that T_i acts on the θ variables according to Definition 2.3.

Theorem 3.2 ([9, Proposition 3.5])**.** *Suppose $1\leq i<N-1$ then $\boldsymbol{T}_i\boldsymbol{T}_{i+1}\boldsymbol{T}_i=\boldsymbol{T}_{i+1}\boldsymbol{T}_i\boldsymbol{T}_{i+1}$, if $1\leq i<N$ then $(\boldsymbol{T}_i+1)(\boldsymbol{T}_i-t)=0$ and if $1\leq i<j-1\leq N-2$ then $\boldsymbol{T}_i\boldsymbol{T}_j=\boldsymbol{T}_j\boldsymbol{T}_i$.*

We also use

$$
\boldsymbol{T}_i^{-1}p(x;\theta)=\left(\frac{1-t}{t}\right)x_i\frac{p(x;\theta)-p(xs_i;\theta)}{x_i-x_{i+1}}+T_i^{-1}p(xs_i;\theta).
$$

Definition 3.3. Let $T^{(N)}=T_{N-1}T_{N-2}\cdots T_1$ and for $p\in s\mathcal{P}_m$ and $1\leq i\leq N$

$$
\begin{aligned}
&\boldsymbol{w}p(x;\theta):=T^{(N)}p(qx_N,x_1,x_2,\dots,x_{N-1};\theta),\\
&\xi_i p(x;\theta):=t^{i-N}\boldsymbol{T}_i\boldsymbol{T}_{i+1}\cdots\boldsymbol{T}_{N-1}\boldsymbol{w}\boldsymbol{T}_1^{-1}\boldsymbol{T}_2^{-1}\cdots\boldsymbol{T}_{i-1}^{-1}p(x;\theta).
\end{aligned}
$$

The operators ξ_i are Cherednik operators, defined by Baker and Forrester [1] (see Braverman et al. [4] for the significance of these operators in double affine Hecke algebras). They mutually commute (the proof in the vector-valued situation is in [9, Theorem 3.8]). Observe $\xi_{i-1}=t^{-1}\boldsymbol{T}_{i-1}\xi_i\boldsymbol{T}_{i-1}$. Their key properties are

$$
\begin{aligned}
&T^{(N)}T_{i+1}=T_{N-1}\cdots T_{i+1}T_iT_{i+1}T_{i-1}\cdots T_1=T_{N-1}\cdots T_{i+2}T_iT_{i+1}T_i\cdots T_1=T_iT^{(N)},\\
&\big(T^{(N)}\big)^2=T^{(N)}\big(T_{N-1}\cdots T_2\big)T_1=\big(T_{N-2}\cdots T_1\big)T^{(N)}T_1=T_{N-1}^{-1}\big(T^{(N)}\big)^2T_1,\\
&\big(T^{(N)}\big)^2T_1=T_{N-1}\big(T^{(N)}\big)^2,\\
&\boldsymbol{w}\boldsymbol{T}_{i+1}=\boldsymbol{T}_i\boldsymbol{w},\qquad \boldsymbol{w}^2\boldsymbol{T}_1=\boldsymbol{T}_{N-1}\boldsymbol{w}^2,\\
&\boldsymbol{w}^{-1}p(x;\theta)=T_1^{-1}T_2^{-1}\cdots T_{N-1}^{-1}p(x_2,x_3,\dots,x_N,q^{-1}x_1;\theta).
\end{aligned}\tag{3.1}
$$

There is a basis of $s\mathcal{P}_m$ consisting of simultaneous eigenvectors of $\{\xi_i\}$ and these are the nonsymmetric Macdonald superpolynomials (henceforth abbreviated to "NSMP").

Suppose $p(\theta)$ is independent of x then $\boldsymbol{T}_i p = T_i p$ and

$$\begin{aligned}\xi_i p(\theta) &= t^{i-N} T_i T_{i+1} \cdots T_{N-1} \big(T_{N-1} \cdots T_2 T_1\big) T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} p(\theta) \\ &= t^{i-N} T_i T_{i+1} \cdots T_{N-1} T_{N-1} \cdots T_i p(\theta) = \omega_i p(\theta),\end{aligned}$$

that is ξ_i agrees with ω_i on polynomials of bosonic degree 0. Also $\boldsymbol{w}\boldsymbol{T}_{i+1} = \boldsymbol{T}_i \boldsymbol{w}$. Suppose $j > i+1$ then

$$\begin{aligned}\xi_i \boldsymbol{T}_j &= t^{i-N} \boldsymbol{T}_i \boldsymbol{T}_{i+1} \cdots \boldsymbol{T}_{N-1} w \boldsymbol{T}_j \boldsymbol{T}_1^{-1} \boldsymbol{T}_2^{-1} \cdots \boldsymbol{T}_{i-1}^{-1} \\ &= t^{i-N} \boldsymbol{T}_i \boldsymbol{T}_{i+1} \cdots \boldsymbol{T}_{N-1} \boldsymbol{T}_{j-1} w \boldsymbol{T}_1^{-1} \boldsymbol{T}_2^{-1} \cdots \boldsymbol{T}_{i-1}^{-1} \\ &= t^{i-N} \boldsymbol{T}_i \cdots (\boldsymbol{T}_{j-1} \boldsymbol{T}_j \boldsymbol{T}_{j-1}) \cdots \boldsymbol{T}_{N-1} w \boldsymbol{T}_1^{-1} \boldsymbol{T}_2^{-1} \cdots \boldsymbol{T}_{i-1}^{-1} \\ &= t^{i-N} \boldsymbol{T}_i \cdots (\boldsymbol{T}_j \boldsymbol{T}_{j-1} \boldsymbol{T}_j) \cdots \boldsymbol{T}_{N-1} w \boldsymbol{T}_1^{-1} \boldsymbol{T}_2^{-1} \cdots \boldsymbol{T}_{i-1}^{-1} = \boldsymbol{T}_j \xi_i.\end{aligned}$$

A similar argument shows $\xi_i \boldsymbol{T}_j = \boldsymbol{T}_j \xi_i$ when $j < i-1$, by using $\boldsymbol{T}_j^{-1} \boldsymbol{T}_{j+1}^{-1} \boldsymbol{T}_j = \boldsymbol{T}_{j+1} \boldsymbol{T}_j^{-1} \boldsymbol{T}_{j+1}^{-1}$.

3.2 Properties of nonsymmetric Macdonald polynomials

They have a triangularity property with respect to the partial order $\rhd$ on the compositions $\mathbb{N}_0^N$, which is derived from the dominance order:

$$\begin{aligned}&\alpha \prec \beta \iff \sum_{j=1}^{i} \alpha_j \le \sum_{j=1}^{i} \beta_j, \qquad 1 \le i \le N, \qquad \alpha \ne \beta, \\ &\alpha \lhd \beta \iff (|\alpha| = |\beta|) \wedge \big[\big(\alpha^+ \prec \beta^+\big) \vee \big(\alpha^+ = \beta^+ \wedge \alpha \prec \beta\big)\big].\end{aligned}$$

The rank function on compositions is involved in the formula for an NSMP.

Definition 3.4. For $\alpha \in \mathbb{N}_0^N$, $1 \le i \le N$

$$r_\alpha(i) := \#\{j\colon \alpha_j > \alpha_i\} + \#\{j\colon 1 \le j \le i,\ \alpha_j = \alpha_i\},$$

then $r_\alpha \in \mathcal{S}_N$. There is a shortest expression $r_\alpha = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $R_\alpha := (T_{i_1} T_{i_2} \cdots T_{i_k})^{-1} \in \mathcal{H}_N(t)$ (that is, $R_\alpha = T(r_\alpha)^{-1}$).

A consequence is that $r_\alpha \alpha = \alpha^+$, the *nonincreasing rearrangement* of α, for any $\alpha \in \mathbb{N}_0^N$. For example if $\alpha = (1,2,0,5,4,5)$ then $r_\alpha = [5,4,6,1,3,2]$ and $r_\alpha \alpha = \alpha^+ = (5,5,4,2,1,0)$ (recall $(u\alpha)_i = \alpha_{u^{-1}(i)}$). Also $r_\alpha = I$ if and only if $\alpha \in \mathbb{N}_0^{N,+}$.

Theorem 3.5 ([9, Theorem 4.12])**.** *Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_k$, $k = 0,1$ then there exists a (ξ_i)-simultaneous eigenfunction*

$$M_{\alpha,E}(x;\theta) = t^{e(\alpha^+)} q^{b(\alpha)} x^\alpha R_\alpha(\tau_E(\theta)) + \sum_{\beta \lhd \alpha} x^\beta v_{\alpha,\beta,E}(\theta; q, t), \tag{3.2}$$

where $v_{\alpha,\beta,E}(\theta;q,t) \in \mathcal{P}_{m,k}$ and its coefficients are rational functions of q, t. Also $\xi_i M_{\alpha,E}(x;\theta) = \zeta_{\alpha,E}(i) M_{\alpha,E}(x;\theta)$, where $\zeta_{\alpha,E}(i) = q^{\alpha_i} t^{c(r_\alpha(i),E)}$ for $1 \le i \le N$. The exponent $b(\alpha) := \sum_{i=1}^N \binom{\alpha_i}{2}$ and $e(\alpha^+) := \sum_{i=1}^N \alpha_i^+ (N - i + c(i,E))$.

The *spectral vector is* $[\zeta_{\alpha,E}(i)]_{i=1}^N$. Note that the leading term involves the element $R_\alpha(\tau_E(\theta))$ of $\mathcal{H}_N(t)$ acting on fermionic variables. The explanation for the exponents $e(\alpha^+)$ and $b(\alpha)$ is in Proposition 3.10 below. The relations (2.3) hold when ω_i, T_i is replaced by ξ_i, $\boldsymbol{T}_i$ respectively

$$\begin{aligned}&\boldsymbol{T}_j \xi_j = (t-1)\xi_j + \xi_{j+1} \boldsymbol{T}_j, \\ &\xi_j \boldsymbol{T}_j = \boldsymbol{T}_j \xi_{j+1} + (t-1)\xi_j,\end{aligned}$$

and this leads to the following, which has the same proof as Proposition 2.16:

Proposition 3.6. *Suppose $\xi_j f = \lambda_j f$ for $1 \le j \le N$ ($f \neq 0$ and $f \in s\mathcal{P}_m$) and $g := \boldsymbol{T}_i f + \frac{t-1}{\lambda_{i+1}/\lambda_i - 1} f$ then $\xi_j g = \lambda_j g$ for all $j \neq i, i+1$ and $\xi_i g = \lambda_{i+1} g$, $\xi_{i+1} g = \lambda_i g$. If $\lambda_{i+1} \neq t^{\pm 1}\lambda_i$ then $g \neq 0$.*

This together with a degree-raising operation provides the method for constructing the Macdonald polynomials.

Suppose $\alpha \in \mathbb{N}_0^N$, $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ and $\alpha_i \neq \alpha_{i+1}$ then let $z = \zeta_{\alpha,E}(i+1)/\zeta_{\alpha,E}(i)$ and

$$M_{s_i\alpha,E} = c_{\alpha,E}\left(\boldsymbol{T}_i + \frac{t-1}{z-1}\right) M_{\alpha,E}, \tag{3.3}$$

where $c_{\alpha,E} = 1$ if $\alpha_i < \alpha_{i+1}$ and $c_{\alpha,E} = u(z)$ if $\alpha_i > \alpha_{i+1}$. The spectral vector $\zeta_{s_i\alpha,E} = s_i\zeta_{\alpha,E}$.

Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ and $\alpha_i = \alpha_{i+1}$, then let $j = r_\alpha(i)$. If $\{j, j+1\} \in E$ or $j = N-1 \in E$ then $\boldsymbol{T}_i M_{\alpha,E} = -M_{\alpha,E}$. If $\{j, j+1\} \cap E = \varnothing$ or $j = N-1 \notin E$ then $\boldsymbol{T}_i M_{\alpha,E} = t M_{\alpha,E}$. If $j < N-2$ and $(j, j+1) \in E \times E^C$ or $(j, j+1) \in E^C \times E$ then $z = \zeta_{\alpha,E}(i+1)/\zeta_{\alpha,E}(i) = t^{c(j+1,E)-c(j,E)}$ and

$$M_{\alpha,s_jE} = c_{\alpha,E}\left(\boldsymbol{T}_i + \frac{t-1}{z-1}\right) M_{\alpha,E},$$

where $c_{\alpha,E} = 1$ if (1) $E \in \mathcal{Y}_0$ and $(j, j+1) \in E^C \times E$ or if (2) $E \in \mathcal{Y}_1$ and $(j, j+1) \in E \times E^C$, or $c_{\alpha,E} = u(z)^{-1}$ if (1) $E \in \mathcal{Y}_0$ and $(j, j+1) \in E \times E^C$ or if (2) $E \in \mathcal{Y}_1$ and $(j, j+1) \in E^C \times E$. In all cases $\zeta_{\alpha,s_jE} = s_i\zeta_{\alpha,E}$.

The above equations are implicit formulas for $\boldsymbol{T}_i M_{\alpha,E}$. Formula (3.3) is the same as that for the scalar case, as in [1, 14].

The *affine* step is defined as follows: for $\alpha \in \mathbb{N}_0^N$, $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$

$$\begin{aligned} &\Phi\alpha = (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1), \\ &\zeta_{\Phi\alpha,E} = \left[\zeta_{\alpha,E}(2), \zeta_{\alpha,E}(3), \ldots, \zeta_{\alpha,E}(N), q\zeta_{\alpha,E}(1)\right], \\ &M_{\Phi\alpha,E} = x_N \boldsymbol{w} M_{\alpha,E}. \end{aligned}$$

This is based on the relations $\xi_N x_N \boldsymbol{w} = q x_N \boldsymbol{w} \xi_1$ and $\xi_i x_N \boldsymbol{w} = x_N \boldsymbol{w} \xi_{i+1}$ for $1 \le i < N$.

Denote $\mathbf{0} = (0, \ldots, 0) \in \mathbb{N}_0^N$ and recall $E_0 = \{N-m, \ldots, N\}$, $E_1 = \{1, 2, \ldots, m-1\}$. In [9, Section 4.1] a Yang–Baxter directed graph method is used to inductively construct the $M_{\alpha,E}$ (this technique is due to Lascoux [12]). Label the nodes (α, E); for $\mathcal{Y}_0$ the root is $(\mathbf{0}, E_0)$ with $M_{\mathbf{0},E_0} = \tau_{E_0} = D\phi_{E_0}$; and for $\mathcal{Y}_1$ the root is $(\mathbf{0}, E_1)$ with $M_{\mathbf{0},E_1} = \tau_{E_1} = M\phi_{E_1}$. The equations have been set up so that $M_{\mathbf{0},E} = \tau_E$. The arrows in the graph point from (α, E) to $(\Phi\alpha, E)$, and from (α, E) to $(s_i\alpha, E)$ ($\alpha_i < \alpha_{i+1}$) or to (α, s_jE) when $j = r_\alpha(E)$ and $c_{\alpha,E} = 1$ in the cases described above.

Here is a brief discussion of the effect of $\boldsymbol{T}_i$ on $x^\alpha R_\alpha \tau_E$ for the $c_{\alpha,E} = 1$ cases. For $\alpha \in \mathbb{N}_0^N$ let

$$\operatorname{inv}(\alpha) := \#\{(i,j)\colon i < j,\ \alpha_i < \alpha_j\}$$

then $r_\alpha\alpha = \alpha^+$ and $r_\alpha = s_{i_1} \cdots s_{i_\ell}$, where $\ell = \operatorname{inv}(\alpha)$. Recall $R_\alpha = T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}$ and the value of R_α is independent of the expressions for r_α of length ℓ. Suppose $\alpha_i < \alpha_{i+1}$ then $\operatorname{inv}(\alpha) = \operatorname{inv}(s_i\alpha) + 1$; write $s_i\alpha = r_{s_i\alpha}^{-1}\alpha_+$ and $r_{s_i\alpha} = s_{i_1} \cdots s_{i_\ell}$ with $\ell = \operatorname{inv}(s_i\alpha)$. Thus $r_\alpha^{-1} = s_i s_{i_1} \cdots s_{i_\ell}$ and $R_\alpha = T_i^{-1} R_{s_i\alpha}$ and so $\boldsymbol{T}_i x^\alpha R_\alpha \tau_E = x^{s_i\alpha} R_{s_i\alpha}\tau_E + p(x;\theta)$, where p is a sum of $x^\beta p'(\theta)$ with $s_i\alpha \rhd \beta$.

Proposition 3.7. *Suppose $\alpha_i = \alpha_{i+1}$ with $j = r_\alpha(i)$ then $\boldsymbol{T}_i x^\alpha R_\alpha \tau_E = x^\alpha T_i R_\alpha \tau_E$ and $T_i R_\alpha = R_\alpha T_j$.*

Proof. Let $r_\alpha = s_{i_1} \cdots s_{i_\ell}$ with $\ell = \operatorname{inv}(\alpha)$. Let $\beta = \alpha$ except $\beta_{i+1} = \alpha_i + \frac{1}{2}$ then $\operatorname{inv}(\beta) = \ell + 1$. From $r_\alpha \alpha = \alpha^+$ it follows that $r_\alpha s_i \beta = \beta^+$. By definition of j we obtain $\beta_j^+ = \alpha_i + \frac{1}{2}, \beta_{j+1}^+ = \alpha_i$ and $r_\alpha^{-1} s_j \beta^+ = \beta$. Now $r_\alpha s_i \beta = \beta^+$ and $s_j r_\alpha \beta = \beta^+$ and $r_\alpha s_i, s_j r_\alpha$ have (at least) $\ell + 1$ factors s_{i_n} and $\operatorname{inv}(\beta) = \ell + 1$. This implies $r_\alpha s_i = s_j r_\alpha$ and $T_{i_1} T_{i_2} \cdots T_{i_\ell} T_i = T_j T_{i_1} T_{i_2} \cdots T_{i_\ell}$, that is, $R_\alpha^{-1} T_i = T_j R_\alpha^{-1}$. ■

Thus in the case $\alpha_i = \alpha_{i+1}$ the transformation laws (Theorem 2.22 and Corollary 2.24) apply to $\left(T_j + \frac{t-1}{z-1}\right)\tau_E$ with $z = t^{c(j+1,E)-c(j,E)}$ (with the possible results $-\tau_E, t\tau_E, \tau_{s_j E}, \dots$).

Hence only the affine step changes the power of t in the coefficient of x^α. It remains to consider $x^N \boldsymbol{w} x^\alpha R_\alpha \tau_E$.

Proposition 3.8. *Suppose $\alpha \in \mathbb{N}_0^N$ and $r_\alpha(1) = j$ then $T^{(N)} R_\alpha = t^{N-j} R_{\Phi\alpha} \omega_j$.*

Proof. Let $\ell = \#\{(i,j) \colon 1 < i < j \le N,\, \alpha_i < \alpha_j\}$. Then there is a product $u = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ such that $u\alpha = \left(\alpha_1, \alpha_1^+, \dots, \alpha_{j-1}^+, \alpha_{j+1}^+, \dots, \alpha_N^+\right)$, and each $i_k > 1$. By definition $i > j$ implies $\alpha_1^+ \le \alpha_i < \alpha_1 + 1$, and $i < j$ implies $\alpha_i^+ \ge \alpha_1 + 1$. Then $s_{j-1} s_{j-2} \cdots s_1 u \alpha = \alpha^+$ and $R_\alpha^{-1} = \left(T_{j-1} \cdots T_1\right) U$, where $U = T_{i_1} \cdots T_{i_\ell}$. Let $u' = s_{i_1 - 1} \cdots s_{i_\ell - 1}$ then

$$u'(\Phi\alpha) = \left(\alpha_1^+, \dots, \alpha_{j-1}^+, \alpha_{j+1}^+, \dots, \alpha_N^+, \alpha_1 + 1\right)$$

and $s_j s_{j+1} \cdots s_{N-1} u'(\Phi\alpha) = (\Phi\alpha)^+$. Thus $R_{\Phi\alpha}^{-1} = \left(T_j \cdots T_{N-1}\right) U'$ and $U' = T_{i_1 - 1} \cdots T_{i_\ell - 1}$. By (3.1) $T^{(N)} U = U' T^{(N)}$ and

$$\begin{aligned} R_{\Phi\alpha}^{-1} T^{(N)} &= \left(T_j \cdots T_{N-1}\right) U' T^{(N)} = \left(T_j \cdots T_{N-1}\right) T^{(N)} U \\ &= \left(T_j \cdots T_{N-1} T_{N-1} \cdots T_j\right) T_{j-1} \cdots T_1 U = t^{N-j} \omega_j R_\alpha^{-1}. \end{aligned}$$ ■

As a consequence $x_N \boldsymbol{w} x^\alpha R_\alpha \tau_E = q^{\alpha_1} t^{N-j+c(j,E)} x^{\Phi\alpha} R_{\Phi\alpha} \tau_E$ with $j = r_\alpha(1)$. Denote $(\alpha_1 + 1, \alpha_2 + 1, \cdots, \alpha_N + 1)$ by $\alpha + \mathbf{1}$.

Corollary 3.9. $(x_N \boldsymbol{w})^N M_{\alpha,E} = M_{\alpha+\mathbf{1},E} = q^{|\alpha|} t^v (x_1 x_2 \cdots x_N) M_{\alpha,E}$, *where* $v = \frac{N(N-1)}{2} + \sum_{i=1}^N c(i,E)$.

Proof. Suppose the coefficient of $x^\alpha R_\alpha \left(\tau_E(\theta)\right)$ in $M_{\alpha,E}$ is $q^a t^b$ then

$$x_N \boldsymbol{w} M_{\alpha,E} = M_{\Phi\alpha,E} = q^{a+\alpha_1} t^{N - r_\alpha(1) + c(r_\alpha(1),E)} x^{\Phi\alpha} R_{\Phi\alpha}(\tau_E) + \cdots.$$

Then $(x_N \boldsymbol{w})^2 M_{\alpha,E}$ involves $r_{\Phi\alpha}(1) = r_\alpha(2)$. Repeating this process yields a factor of $q^{|\alpha|}$ and the t-exponent $\sum_{i=1}^N (N - r_\alpha(i) + c(r_\alpha(i),E)) = \frac{N(N-1)}{2} + \sum_{i=1}^N c(i,E)$. Furthermore $\Phi^N \alpha = \alpha + \mathbf{1}$ and $R_{\alpha+\mathbf{1}} = R_\alpha$. ■

If $E \in \mathcal{Y}_0, \mathcal{Y}_1$ then $\sum_{i=1}^N c(i,E) = \frac{N(N-1)}{2} - Nm$, $\frac{N(N-1)}{2} - N(m-1)$ respectively (so $v = N(N-m-1)$ or $N(N-m)$).

Proposition 3.10. *The exponent on q in (3.2) is $b(\alpha) = \sum_{i=1}^N \binom{\alpha_i}{2}$. The exponent on t in (3.2) is $\sum_{i=1}^N \lambda_i (N - i + c(i,E))$.*

Proof. For the value of $b(\alpha)$ we use induction on $|\alpha|$. The statement $b(\mathbf{0}) = 0$ is true since $M_{\mathbf{0},E} = \tau_E$. The steps at $\alpha_i < \alpha_{i+1}$ taking x^α to $x^{s_i \alpha}$ do not involve q (which does affect the other terms of the polynomial), and indeed $b(\alpha)$ is invariant under s_i. The affine step takes x^α to $x^{\Phi\alpha}$ and multiplies by q^{α_1}, that is, $b(\Phi\alpha) = b(\alpha) + \alpha_1$, and $\binom{\alpha_1}{2} + \alpha_1 = \binom{\alpha_1+1}{2}$. Induct on $\ell(\lambda)$ for the t-exponent. Note that only the affine step affects the exponent so it depends only on $\lambda = \alpha^+$. Suppose $\lambda_k \ge 1$ and $\lambda_i = 0$ for $i > k$. The affine step from $\lambda^{(k)}$ with $\lambda_i^{(k)} = \lambda_i$ except $\lambda_k^{(k)} = \lambda_k - 1$ proceeds by $\lambda^{(k)} \to (\lambda_k - 1, \lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0) \to (\lambda_1, \dots, \lambda_{k=1}, 0, \dots, 0, \lambda_k) \to \lambda$ and multiplies by $t^{N-k+c(k,E)}$. Thus passing from $(\lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0)$ to λ contributes a factor of $t^{\lambda_k(N-k+c(k,E))}$. ■

Example 3.11. Explicit formulas for $M_{\alpha,E}$ tend to be complicated; here is a fairly simple one. Let $N=5$, $m=2$, $E=\{3,4,5\}$ and $\alpha=(0,0,1,0,0)$:

$$\begin{aligned} M_{\alpha,E} &= t^6 x_3(t^3\theta_2\theta_4 - t^2\theta_2\theta_4+\theta_3\theta_4) \\ &\quad + \frac{(t-1)t^9 q}{qt^3-1}\bigl\{x_4\bigl(t^3\theta_2\theta_3 - t\theta_2\theta_5+\theta_3\theta_5\bigr) - x_5\bigl(t^2\theta_2\theta_3 - t\theta_2\theta_4+\theta_3\theta_4\bigr)\bigr\}. \end{aligned}$$

The spectral vector is $\bigl[t,t^{-2},qt^2,t^{-1},1\bigr]$ and $\boldsymbol{T}_4 M_{\alpha,E}=-M_{\alpha,E}$.

There is a pairwise commuting set of (bosonic) degree lowering operations, namely the Dunkl operators defined by Baker and Forrester [1].

Definition 3.12. Suppose $f\in s\mathcal{P}_m$ then $\boldsymbol{D}_N f := \frac{1}{x_N}(f-\xi_N f)$ and if $i<N$ then $\boldsymbol{D}_i f = \frac{1}{t}\boldsymbol{T}_i\boldsymbol{D}_{i+1}\boldsymbol{T}_i f$.

Assuming the existence of the nonsymmetric Macdonald polynomials $M_{\alpha,E}$ the argument for showing that $\boldsymbol{D}_i f$ is a polynomial is the following:

Proposition 3.13. *Suppose $\alpha\in\mathbb{N}_0^N$ and $E\in\mathcal{Y}_0\cup\mathcal{Y}_1$; if $\alpha_N=0$ then $\boldsymbol{D}_N M_{\alpha,E}=0$ or if $\alpha_N\ge 1$ then $\boldsymbol{D}_N M_{\alpha,E}=(1-\zeta_{\alpha,E}(N))\boldsymbol{w}M_{\beta,E}$, where $\alpha=\Phi\beta$. Also $\boldsymbol{D}_N M_{\Phi\alpha,E}=(1-q\zeta_\alpha(1))\boldsymbol{w}M_{\alpha,E}$.*

Proof. If $\alpha_N=0$ then $r_\alpha(N)=N, c(N,E)=0$ and $\xi_N M_{\alpha,E}=M_{\alpha,E}$ and $(1-\xi_N)M_{\alpha,E}=0$. If $\alpha_N\ge 1$ then $\alpha=\Phi\beta$ with $|\beta|=|\alpha|-1$ and $(1-\xi_N)M_{\alpha,E}=(1-\zeta_{\alpha,E}(N))M_{\alpha,E}=(1-\zeta_{\alpha,E}(N))x_N\boldsymbol{w}M_{\beta,E}$, and thus $\boldsymbol{D}_N M_{\alpha,E}=(1-\zeta_{\alpha,E}(N))\boldsymbol{w}M_{\beta,E}$. For the other statement note that $\zeta_{\Phi\alpha,E}(N)=q\zeta_{\alpha,E}(1)$. ∎

Remark 3.14. We conjecture that there are evaluation formulas for the special case $E=\{1,2,\dots,m,N\}$, $\alpha_i=0$ for $i>m$ and $x_0=\bigl(t^{N-1},t^{N-2},\dots,t,1\bigr)$. Here are some small examples in isotype $(2,1,1)$

$$\begin{aligned} J_{(1,1,0,0),E}(x_0) &= \frac{t^5\bigl(q-t^4\bigr)}{q-t^2}g_{12}(\theta), \\ J_{(1,2,0,0),E}(x_0) &= \frac{t^7\bigl(q-t^4\bigr)\bigl(q^2-t^4\bigr)}{\bigl(q-t^2\bigr)\bigl(q^2-t^3\bigr)}g_{12}(\theta), \\ J_{(2,1,0,0),E}(x_0) &= \frac{t^7\bigl(q-t^2\bigr)\bigl(q-t^4\bigr)\bigl(q^2-t^4\bigr)}{(q-t)\bigl(q-t^2\bigr)\bigl(q^2-t^3\bigr)}g_{12}(\theta), \\ g_{12}(\theta) &= \theta_1\theta_2 - \frac{1}{t^2(1+t)}(t\theta_1-\theta_2)(\theta_3+\theta_4). \end{aligned}$$

Replace $g_{12}(\theta)$ by $(\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4)$ for the $\mathcal{P}_{3,1}$ version (by applying M).

3.3 Symmetric bilinear form

In this section we define an inner product (symmetric bilinear form) on $s\mathcal{P}_m$ in which $\boldsymbol{T}_i$, ξ_i are self-adjoint, the Macdonald polynomials are pairwise orthogonal and it is positive-definite for $t,q>0$, $q\ne 1$ and $\min\bigl(q^{1/N},q^{-1/N}\bigr)<t<\max\bigl(q^{1/N},q^{-1/N}\bigr)$. The background and proofs for this section are in [8]. The hypotheses $\langle\boldsymbol{T}_i f,g\rangle=\langle f,\boldsymbol{T}_i g\rangle$ for $1\le i<N$ and $\langle\xi_N f,g\rangle=\langle f,\xi_N g\rangle$ already imply that $\langle\xi_i f,g\rangle=\langle f,\xi_i g\rangle$ for all i since $\xi_i=t^{-1}\boldsymbol{T}_i\xi_{i+1}\boldsymbol{T}_i$ and thus $\langle M_{\alpha,E},M_{\beta,F}\rangle=0$ if $(\alpha,E)\ne(\beta,F)$ (at least one different $\{\xi_i\}$-eigenvalue). Denote $\langle f,f\rangle=\|f\|^2$, even if possibly nonpositive. The aim is to determine a formula for $\|M_{\alpha,E}\|^2$ which, other than leading coefficients q^*t^*, involves only linear factors of the form $(1-q^qt^b)$ with $a\in\mathbb{N}_0$, $b\in\mathbb{Z}$, $|b|\le N$. Recall $u(z):=\frac{(t-z)(1-tz)}{(1-z)^2}$ from Definition 2.25.

Proposition 3.15. *Suppose there is a symmetric bilinear form on $s\mathcal{P}$ in which each $\boldsymbol{T}_i$ and ξ_i is self-adjoint and suppose $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, $\alpha \in \mathbb{N}_0^N$ and $\alpha_i < \alpha_{i+1}$ for some i then*

$$\|M_{s_i\alpha,E}\|^2 = u\big(q^{\alpha_{i+1}-\alpha_i} t^{c(r_\alpha(i+1),E)-c(r_\alpha(i),E)}\big)\|M_{\alpha,E}\|^2.$$

Proof. This is the same argument used in Lemma 2.17. ■

We introduce a product for expressing $\|M_{\alpha,E}\|^2$ in terms of $\|M_{\alpha^+,E}\|^2$.

Definition 3.16. For $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, $\alpha \in \mathbb{N}_0^N$ let

$$\mathcal{R}(\alpha, E) := \prod_{1\le i<j\le N,\, \alpha_i<\alpha_j} u\big(q^{\alpha_j-\alpha_i} t^{c(r_\alpha(j),E)-c(r_\alpha(i),E)}\big).$$

There are $\operatorname{inv}(\alpha)$ terms in the product. The next proposition assumes the same hypotheses on the bilinear form.

Proposition 3.17. *Suppose $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, $\alpha \in \mathbb{N}_0^N$ then*

$$\|M_{\alpha^+,E}\|^2 = \mathcal{R}(\alpha, E)\|M_{\alpha,E}\|^2.$$

Proof. With the same argument as in Proposition 2.26 one shows $\alpha_i < \alpha_{i+1}$ implies

$$\frac{\mathcal{R}(\alpha, E)}{\mathcal{R}(s_i\alpha, E)} = u\big(q^{\alpha_{i+1}-\alpha_i} t^{c(r_\alpha(i+1),E)-c(r_\alpha(i),E)}\big). \qquad ■$$

Another hypothesis is required to define the inner product for all polynomials starting with bosonic degree 0 ($M_{\mathbf{0},E} = \tau_E$). The approach of making $\boldsymbol{D}_i$ the adjoint of multiplication by x_i, or making an isometry out of the latter (torus norm) as is done in the Jack polynomial situation, does not work here without a modification.

Theorem 3.18 ([8, Section 3.3])**.** *There is a unique symmetric bilinear form on $s\mathcal{P}_m$ which extends the form in Definition 2.6 and satisfies (for $f, g \in s\mathcal{P}_m$ and $1 \le i < N$)*

$$\langle \boldsymbol{T}_i f, g\rangle = \langle f, \boldsymbol{T}_i g\rangle, \tag{3.4}$$
$$\langle \xi_N f, g\rangle = \langle f, \xi_N g\rangle, \tag{3.5}$$
$$\langle \boldsymbol{w}^{-1}\boldsymbol{D}_N f, g\rangle = (1-q)\langle f, x_N \boldsymbol{w} g\rangle. \tag{3.6}$$

It follows from $\xi_i = t^{-1}\boldsymbol{T}_i\xi_{i+1}\boldsymbol{T}_i$ that $\langle \xi_i f, g\rangle = \langle f, \xi_i g\rangle$ for all i. The reason for the factor $(1-q)$ is to allow the limit as $t \to 1$ when $q = t^{1/\kappa}$ to obtain nonsymmetric Jack polynomials.

In [8] hypothesis (3.6) is stated in the equivalent form

$$\langle \boldsymbol{D}_N f, g\rangle = (1-q)\langle f, x_N \boldsymbol{w}\boldsymbol{w}^* g\rangle,$$

where

$$\boldsymbol{w}^* = \boldsymbol{T}_1\boldsymbol{T}_2\cdots\boldsymbol{T}_{N-1}\boldsymbol{w}\boldsymbol{T}_1^{-1}\boldsymbol{T}_2^{-1}\cdots\boldsymbol{T}_{N-1}^{-1},$$
$$\langle \boldsymbol{w} f, g\rangle = \langle f, \boldsymbol{w}^* g\rangle,$$

and this expression follows from $\boldsymbol{w} = t^{N-1}\boldsymbol{T}_{N-1}^{-1}\cdots\boldsymbol{T}_2^{-1}\boldsymbol{T}_1^{-1}\xi_1$. Next we use hypothesis (3.6) to relate norms for polynomials of different bosonic degrees.

Proposition 3.19. *Suppose $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, $\alpha \in \mathbb{N}_0^N$ then*

$$\|M_{\Phi\alpha,E}\|^2 = \frac{1 - q^{\alpha_1+1}t^{c(r_\alpha(1),E)}}{1-q}\|M_{\alpha,E}\|^2.$$

Proof. In (3.6) set $g = M_{\alpha,E}$ and $f = M_{\Phi\alpha,E}$ then $(1-q)\langle f, x_N \boldsymbol{w} g\rangle = (1-q)\|M_{\Phi\alpha,E}\|^2$. On the other hand

$$\begin{aligned} &\boldsymbol{D}_N f = \frac{1}{x_N}(1-\xi_N)f = \frac{1}{x_N}(1-\zeta_{\Phi\alpha,E}(N))M_{\Phi\alpha,E} = (1-\zeta_{\Phi\alpha,E}(N))\boldsymbol{w}M_{\alpha,E},\\ &\langle \boldsymbol{w}^{-1}\boldsymbol{D}_N f, g\rangle = (1-\zeta_{\Phi\alpha,E}(N))\langle M_{\alpha,E}, M_{\alpha,E}\rangle, \end{aligned}$$

thus $\|M_{\Phi\alpha,E}\|^2 = \frac{1-\zeta_{\Phi\alpha,E}(N)}{1-q}\|M_{\alpha,E}\|^2$ and $\zeta_{\Phi\alpha,E}(N) = q\zeta_{\alpha,E}(1) = q^{\alpha_1+1}t^{c(r_\alpha(1),E)}$. ■

With this formula and Proposition 3.17 we can use induction to find $\|M_{\alpha,E}\|^2$ for any α. The first step uses $\alpha = \mathbf{0}$ and any $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, where $M_{\mathbf{0},E} = \tau_E$ and the spectral vector $[\zeta_{\mathbf{0},E}(i)]_{i=1}^N = \left[t^{c(i,E)}\right]_{i=1}^N$. Then $\Phi\alpha = (0,\dots,0,1)$ and $M_{\Phi\alpha,E} = x_N T^{(N)}\tau_E$, $\|M_{\Phi\alpha,E}\|^2 = \frac{1-qt^{c(1,E)}}{1-q}\|\tau_E\|^2$ (see Propositions 2.26 and 2.33 for this value).

The argument for establishing the formula for $\|M_{\lambda,E}\|^2$, where $\lambda \in \mathbb{N}_0^{N,+}$ uses the following steps, starting with the assumption that $\lambda_k \geq 1$ and $\lambda_j = 0$ for $k < j \leq N$. Throughout E is fixed. Let

$$\begin{aligned} &\mu = (\lambda_1,\dots,\lambda_{k-1},\lambda_k-1,0..,0) = \alpha^+,\\ &\alpha = (\lambda_k-1,\lambda_1,\dots,\lambda_{k-1},0,\dots,0),\\ &\beta = (\lambda_1,\dots,\lambda_{k-1},0,\dots,0,\lambda_k) = \Phi\alpha \end{aligned}$$

so that $\|M_{\alpha,E}\|^2 = \mathcal{R}(\alpha,E)^{-1}\|M_{\mu,E}\|^2$, $\|M_{\beta,E}\|^2 = \frac{1-q^{\lambda_k}t^{c(k,E)}}{1-q}\|M_{\alpha,E}\|^2$ (since $r_\alpha(1) = k$) and $\|M_{\lambda,E}\|^2 = \mathcal{R}(\beta,E)\|M_{\beta,E}\|^2$. In the resulting formula we use a slightly different expression for $u(z) = \frac{(t-z)(1-tz)}{(z-1)^2} = t\frac{(1-z/t)(1-tz)}{(z-1)^2}$ because $u(z)/t$ is invariant under $t \to \frac{1}{t}$, $z \to \frac{1}{z}$, but this causes a power of t to appear in the form $k(\lambda) := \sum_{i=1}^N (N-2i+1)\lambda_i$ for $\lambda \in \mathbb{N}_0^{N,+}$. The shifted q-factorial $(z;q)_0 = 1$, $(z;q)_{n+1} = (z;q)_n(1-zq^n)$, $n = 0,1,2,\dots$ is used.

Theorem 3.20 ([8]). *Suppose $\lambda \in \mathbb{N}_0^{N,+}$, $\alpha,\beta \in \mathbb{N}_0^N$ and $E,F \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ then the following satisfy the hypotheses* (3.4), (3.5), *and* (3.6):

$$\begin{aligned} &\langle M_{\alpha,E}, M_{\beta,F}\rangle = 0, \qquad (\alpha,E) \neq (\beta,F),\\ &\|M_{\alpha,E}\|^2 = \mathcal{R}(\alpha,E)^{-1}\|M_{\alpha^+,E}\|^2,\\ &\|M_{\lambda,E}\|^2 = t^{k(\lambda)}\|\tau_E\|^2(1-q)^{-|\lambda|}\prod_{i=1}^N \left(qt^{c(i,E)};q\right)_{\lambda_i}\\ &\qquad\times \prod_{1\leq i<j\leq N} \frac{\left(qt^{c(i,E)-c(j,E)-1};q\right)_{\lambda_i-\lambda_j}\left(qt^{c(i,E)-c(j,E)+1};q\right)_{\lambda_i-\lambda_j}}{\left(qt^{c(i,E)-c(j,E)};q\right)^2_{\lambda_i-\lambda_j}}. \end{aligned}$$

As Griffeth [11] pointed out there is not much cancellation between successive terms in general; there is a certain amount for the extreme cases $E_0 = \{N-m, N-m+1,\dots,N\}$ and $E_1 = \{1,2,\dots,m-1\}$. By [8, Proposition 11] $\|M_{\alpha,E}\|^2 > 0$ if $q > 0$ and $\min\left(q^{-1/N}, q^{1/N}\right) < t < \max\left(q^{-1/N}, q^{1/N}\right)$.

4 Symmetric Macdonald superpolynomials

4.1 From nonsymmetric to symmetric

This concerns polynomials p in $s\mathcal{P}_m$ which satisfy $\boldsymbol{T}_i p = tp$ for $1 \leq i < N$ and which are eigenfunctions of $\sum_{i=1}^N \xi_i$. We call polynomials satisfying $\boldsymbol{T}_i p = tp$ for all i, or $\boldsymbol{T}_i p = -p$

for all i, *symmetric* or *antisymmetric*, respectively. (The meaning of *symmetric* here is not the same as for the symmetric group situation, as will be shown by example.) There are two approaches to producing symmetric polynomials. One way is to identify a set of $M_{\alpha,E}$ which is closed under the steps $f \to (\boldsymbol{T}_i + b)f$ of the type described in Proposition 3.6 and then to apply the symmetry conditions to a linear combination of these polynomials with undetermined coefficients. The other way is to apply a symmetrization operator to one polynomial. The original idea for these approaches comes from Baker and Forrester [2].

Definition 4.1. For $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ let $\lfloor \alpha, E \rfloor$ denote the tableau obtained from Y_E by replacing i by α_i^+ for $1 \le i \le N$. Let $\mathcal{M}(\alpha, E) = \operatorname{span}\{M_{\beta,F} \colon \lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor\}$.

Example: let $N = 9$, $m = 4$, $E = \{2,3,6,8,9\}$, $\alpha = (3,5,6,2,2,1,4,4,6)$, $\alpha^+ = (6,6,5,4,4,3,2,2,1)$ and

$$Y_E = \begin{bmatrix} 9 & 7 & 5 & 4 & 1 \\ \cdot & 8 & 6 & 3 & 2 \end{bmatrix}, \qquad \lfloor \alpha, E \rfloor = \begin{bmatrix} 1 & 2 & 4 & 4 & 6 \\ \cdot & 2 & 3 & 5 & 6 \end{bmatrix}.$$

Theorem 4.2 ([9, Proposition 5.2])**.** *Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, then there is a series of transformations of the form $a(\boldsymbol{T}_i + b)$ mapping $M_{\alpha,E}$ to $M_{\beta,F}$ if and only if $\lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor$.*

It is a consequence of the transformation rules that if $\lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor$ then the spectral vector $\zeta_{\beta,F}$ is a permutation of $\zeta_{\alpha,E}$. Furthermore $\mathcal{M}(\alpha, E)$ is an $\mathcal{H}_N(t)$-module.

Theorem 4.3 ([9, Theorem 5.27])**.** *Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_0 \cup \mathcal{Y}_1$ and $\lfloor \alpha, E \rfloor$ is column-strict* (*the entries in column 1 are strictly decreasing*) *then there is a unique symmetric polynomial* (*up to multiplication by a constant*) *in $\mathcal{M}(\alpha, E)$ otherwise there is no nonzero symmetric polynomial.*

In [6] the authors defined a *superpartition* with N parts and fermionic degree m as an N-tuple $\big(\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_N\big)$ which satisfies $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m$ and $\Lambda_{m+1} \ge \Lambda_{m+2} \ge \cdots \ge \Lambda_N$. Suppose $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{Y}_0$ and $\lfloor \lambda, E \rfloor$ is column strict, then $\Lambda_i = \lfloor \lambda, E \rfloor [m+2-i, 1]$ for $1 \le i \le m$ and $\Lambda_i = \lfloor \lambda, E \rfloor [1, N+1-i]$ for $m+1 \le i \le N$, and also $\Lambda_m > \Lambda_N$. Alternatively suppose $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{Y}_1$ and $\lfloor \lambda, E \rfloor$ is column strict, then $\Lambda_i = \lfloor \lambda, E \rfloor [m+1-i, 1]$ for $1 \le i \le m$ and $\Lambda_i = \lfloor \lambda, E \rfloor [1, N+2-i]$ for $m+1 \le i \le N$, and also $\Lambda_m \le \Lambda_N$ (because $\Lambda_m = \lfloor \lambda, E \rfloor [1,1]$ and $\Lambda_N = \lfloor \lambda, E \rfloor [1,2]$). Thus the inequalities $\Lambda_m > \Lambda_N$ and $\Lambda_m \le \Lambda_N$ distinguish $\mathcal{Y}_0$ from $\mathcal{Y}_1$.

As a standardization for the labels use $\lambda = \alpha^+$ and for E use the *root* E_R or the sink E_S

Definition 4.4. Suppose $E \in \mathcal{Y}_0, \lambda \in \mathbb{N}_0^{N,+}$ then the root E_R and the sink E_S (which implicitly depend on λ) satisfy

$$\begin{aligned} &\operatorname{inv}(E_R) = \min\big\{ \operatorname{inv}(F) \colon \lfloor \alpha, F \rfloor = \lfloor \lambda, E \rfloor \big\}, \\ &\operatorname{inv}(E_S) = \max\big\{ \operatorname{inv}(F) \colon \lfloor \alpha, F \rfloor = \lfloor \lambda, E \rfloor \big\}. \end{aligned}$$

The root and the sink are produced by minimizing the entries of F in row 1, respectively minimizing the entries of F in column 1. For $E \in \mathcal{Y}_1$ the definitions of E_R and E_S are reversed.

So in the above example $E_R = E$ and $E_S = \{1,3,6,7,9\}$ and there are four sets F such that $\lfloor \lambda, F \rfloor = \lfloor \lambda, E_S \rfloor$.

Consider $p = \sum_{\lfloor \beta, F \rfloor = \lfloor \lambda, E_R \rfloor} A(\beta, F) M_{\beta,F}$ then the action of $\boldsymbol{T}_i$ decomposes the sum into pairs and singletons. Suppose $\lfloor \beta, F \rfloor = \lfloor \lambda, E_R \rfloor$ for some β with $\beta_i < \beta_{i+1}$, some i. Let $z = \zeta_{\beta,F}(i+1)/\zeta_{\beta,F}(i)$ then

$$\begin{aligned} &\left(\boldsymbol{T}_i + \frac{t-1}{z-1}\right) M_{\beta,F} = M_{s_i\beta,F}, \qquad \boldsymbol{T}_i M_{\beta,F} = -\frac{t-1}{z-1} M_{\beta,F} + M_{s_i\beta,F}, \\ &\left(\boldsymbol{T}_i + \frac{t-1}{z^{-1}-1}\right) M_{s_i\beta,F} = u(z) M_{\beta,F}, \qquad \boldsymbol{T}_i M_{s_i\beta,F} = u(z) M_{\beta,F} - \frac{t-1}{z^{-1}-1} M_{s_i\beta,F}, \end{aligned}$$

and

$$(\boldsymbol{T}_i - t)\big(A(\beta,F)M_{\beta,F} + A(s_i\beta,F)M_{s_i\beta,F}\big) = 0$$

implies $A(\beta,F) = \frac{t-z}{1-z}A(s_i\beta,F)$.

Definition 4.5. Let $u_0(z) := \frac{t-z}{1-z}$, $u_1(z) := \frac{1-tz}{1-z}$ and for $\beta \in \mathbb{N}_0^N$, $F \in \mathcal{Y}_0 \cup \mathcal{Y}_1$, $k = 1,2$ let

$$\mathcal{R}_k(\beta,F) = \prod_{1\le i<j\le N,\, \beta_i<\beta_j} u_k\big(q^{\beta_j-\beta_i}t^{c(r_\beta(j),F)-c(r_\beta(i),F)}\big).$$

Thus $\mathcal{R}(\beta,F) = \mathcal{R}_0(\beta,F)\mathcal{R}_1(\beta,F)$ (see Definition 3.16).

Lemma 4.6. *If $\boldsymbol{T}_i p = tp$ for $1 \le i < N$ then*

$$A(\beta,F) = \mathcal{R}_0(\beta,F)A\big(\beta^+,F\big).$$

Proof. Suppose $\beta_i > \beta_{i+1}$ then

$$\frac{\mathcal{R}_0(s_i\beta,F)}{\mathcal{R}_0(\beta,F)} = u_0\left(\frac{\zeta_{\beta,F}(i)}{\zeta_{\beta,F}(i+1)}\right).$$

The same argument as in Proposition 2.26 applies. ∎

Definition 4.7. For $k = 0,1$ let $\mathcal{C}_k(E) := \prod\limits_{\substack{1\le i<j<N,\\ c(i,E)<0<c(j,E)}} u_k\big(t^{v(i,E)-c(ij,E)}\big)$.

Thus $\mathcal{C}\big[c(i,E)]_{i=1}^N\big) = \mathcal{C}_0(E)\mathcal{C}_1(E)$ (see (2.4)).

Lemma 4.8. $\dfrac{A(\beta,F)}{\mathcal{C}_0(F)} = \dfrac{A\left(\beta,E_S\right)}{\mathcal{C}_0\left(E_S\right)}$.

Proof. Consider the possibilities when $\beta_i = \beta_{i+1}$ and $j = r_\beta(i)$: if $c(j,F) = c(j+1,F)+1$, that is, j and $j+1$ are in adjacent cells of row 1 of Y_F then $\boldsymbol{T}_i M_{\beta,F} = tM_{\beta,F}$, imposing no conditions on $A(\beta,F)$; if $c(j,F) = c(j+1,F)-1$ then $\boldsymbol{T}_i M_{\beta,F} = -M_{\beta,F}$ but this occurs only if there are adjacent equal values (β_i) in column 1 of $\lfloor\lambda,E_R\rfloor$, ruled out by hypothesis; $c(j,F) < 0 < c(j+1,F)$. In this case we relate $M_{\beta,F}$ to M_{β,s_jF}, where $\operatorname{inv}(s_jF) = \operatorname{inv}(F)-1$: the formulas similar to (3.2) with $z = \frac{\zeta_{\beta,F}(i)}{\zeta_{\beta,F}(i+1)} = t^{c(j,F)-c(j+1,F)}$ appear here:

$$\boldsymbol{T}_i M_{\beta,s_jF} = -\frac{t-1}{z-1}M_{\beta,s_jF} + M_{\beta,F},$$

$$\boldsymbol{T}_i M_{\beta,F} = u(z)M_{\beta,s_jF} - \frac{t-1}{z^{-1}-1}M_{\beta,F},$$

then $(\boldsymbol{T}_i - t)\big(A(\beta,F)M_{\beta,F} + A(\beta,s_jF)M_{\beta,s_jF}\big) = 0$ implies $A\big(\beta,s_jF\big) = \frac{t-z}{1-z}A(\beta,F)$. ∎

So $A(\beta,F) = \mathcal{R}_0(\beta,F)A(\beta^+,F) = A(\lambda,E_S)\dfrac{\mathcal{R}_0(\beta,F)\mathcal{C}_0(E_S)}{\mathcal{C}_0(F)}$.

Theorem 4.9. *Suppose $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{Y}_0$, and $\lfloor\lambda,E\rfloor$ is column-strict then*

$$p_{\lambda,E} = \sum_{\lfloor\alpha,F\rfloor=\lfloor\lambda,E\rfloor} \frac{\mathcal{C}_0\left(E_S\right)\mathcal{R}_0\left(\alpha,F\right)}{\mathcal{C}_0(F)}M_{\alpha,F}$$

is the supersymmetric polynomial in $\mathcal{M}(\lambda,E)$, unique when the coefficient of M_{λ,E_S} is 1.

To show that this meaning of *symmetric* is different from the group case consider $N = 4$, $E = \{1,2\}$, $m = 3$, $\lambda = (2,1,0,0)$ and the corresponding symmetric polynomial (too large to display here) begins:

$$p = x_1^2x_2\theta_1\theta_2(\theta_3+\theta_4) - x_1^2x_3\theta_1(t\theta_2\theta_3 + (t-1)\theta_2\theta_4 - \theta_3\theta_4) - tx_1^2x_4\theta_1(\theta_2+\theta_3)\theta_4 + \cdots.$$

4.2 Symmetrization operator and norms

The symmetrization operator is defined analogously to the group case.

Definition 4.10. For $n \geq 1$ let $X_0 = 1$ and $X_n = 1 + \boldsymbol{T}_n X_{n-1}$, and $S^{(n)} = X_1 X_2 \cdots X_n$.

Equivalently $X_n = 1 + \boldsymbol{T}_n + \boldsymbol{T}_n \boldsymbol{T}_{n-1} + \cdots + \boldsymbol{T}_n \cdots \boldsymbol{T}_2 \boldsymbol{T}_1$.

Theorem 4.11. *If $1 \leq j \leq n$ then $\big(\boldsymbol{T}_j - t\big) S^{(n)} = 0$.*

Proof. Consider the same formulas with $\boldsymbol{T}_i$ replaced by s_i and denote $\widetilde{X}_n = 1 + s_n \widetilde{X}_{n-1}$. In the full expansion there are $(n+1)!$ terms and the coefficient of t^k in $[n+1]_t!$ is the number of terms with k factors. Claim that $\widetilde{S}^{(n)} = \widetilde{X}_1 \widetilde{X}_2 \cdots \widetilde{X}_n = \sum_{u \in \mathcal{S}_{n+1}} u \in \mathbb{Z}\mathcal{S}_{n+1}$; proceeding by induction the statement is true for $n = 1$, where $\widetilde{X}_1 = 1 + s_1$ and now suppose it is true for n and consider $\sum_{u \in \mathcal{S}_{n+1}} u (1 + s_{n+1} + s_{n+1} s_n + \cdots + s_{N+1} \cdots s_1)$ acting on $\gamma = (\gamma_1, \dots, \gamma_{n+2})$; then $s_{n+1} \cdots s_j \gamma = (\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_{n+2}, \gamma_i)$. Thus $\sum_{u \in \mathcal{S}_{n+1}} u s_{n+1} \cdots s_j$ is the sum of all $u^{(i)}$ such that $\big(u^{(i)} \gamma\big)_{n+2} = \gamma_i$. This shows $\widetilde{S}^{(n+1)} = \sum_{u \in \mathcal{S}_{n+2}} u$. Since the number of terms with k factors in $\widetilde{S}^{(n)}$ is the same as the number of u of length k each term is of minimum length (the shortest expression of u as a product of $\{s_i\}$). Thus replacing each s_i by $\boldsymbol{T}_i$ shows that $S^{(n)} = \sum_{u \in \mathcal{S}_{n+1}} \boldsymbol{T}(u)$.

Replacing $\boldsymbol{T}_i$ by $\boldsymbol{T}_{n+1-i}$ for $1 \leq i \leq n$ in $S^{(n)}$ does not affect the sum (implicitly the braid relations are used). Given $j \leq n$ apply the map $\boldsymbol{T}_i \to \boldsymbol{T}_{j+1-i}$ in $X_1 X_2 \cdots X_j$ to obtain

$$S^{(n)} = \big(1 + \boldsymbol{T}_j\big)\big(1 + \boldsymbol{T}_{j-1} + \boldsymbol{T}_{j-1} \boldsymbol{T}_j\big) \cdots \big(1 + \boldsymbol{T}_1 + \cdots + \boldsymbol{T}_1 \cdots \boldsymbol{T}_j\big) X_{j+1} \cdots X_n$$

and it is now obvious that $\big(\boldsymbol{T}_j - t\big) S^{(n)} = 0$. ∎

Corollary 4.12. *Suppose $f \in s\mathcal{P}_m$ then $\boldsymbol{T}_j \big(S^{(N-1)} f\big) = t S^{(N-1)} f$ for $1 \leq j < N$.*

Corollary 4.13. *$S^{(n)} S^{(n)} = [n+1]_t! S^{(n)}$.*

Proof. The effect of X_j on an invariant polynomial is to multiply by $1 + t + t^2 + \cdots + t^j$. ∎

Corollary 4.14. *Suppose $f, g \in s\mathcal{P}_m$ then $\big\langle S^{(N-1)} f, g \big\rangle = \big\langle f, S^{(N-1)} g \big\rangle$.*

Proof. Suppose $u \in \mathcal{S}_N$ and $u = s_{i_1} \cdots s_{i_\ell}$ is a shortest expression for u so that $\boldsymbol{T}(u) = \boldsymbol{T}_{i_1} \cdots \times \boldsymbol{T}_{i_\ell}$ then $\langle \boldsymbol{T}(u) f, g \rangle = \langle f, \boldsymbol{T}_{i_\ell} \cdots \boldsymbol{T}_{i_1} g \rangle = \big\langle f, \boldsymbol{T}(u^{-1}) g \big\rangle$. Since $\sum_{u \in \mathcal{S}_N} \boldsymbol{T}(u) = \sum_{u \in \mathcal{S}_N} \boldsymbol{T}(u^{-1})$ this completes the proof. ∎

There is a summation-free formula for $\|p_{\lambda,E}\|^2$, derived as follows:

Suppose $\lfloor \alpha, F \rfloor = \lfloor \lambda, E \rfloor$ then $S^{(N-1)} M_{\alpha,F} = c p_{\lambda,E}$ for some constant c, because of the uniqueness of $p_{\lambda,E}$ in $\mathcal{M}(\lambda, E)$. Then

$$\begin{aligned} \big\langle p_{\lambda,E}, S^{(N-1)} M_{\alpha,F} \big\rangle &= c \langle p_{\lambda,E}, p_{\lambda,E} \rangle = \big\langle S^{(N-1)} p_{\lambda,E}, M_{\alpha,F} \big\rangle \\ &= [N]_t! \langle p_{\lambda,E}, M_{\alpha,F} \rangle = [N]_t! \frac{\mathcal{C}_0\big(E_S\big) \mathcal{R}_0(\alpha, F)}{\mathcal{C}_0(F)} \|M_{\alpha,F}\|^2. \end{aligned} \tag{4.1}$$

The evaluation depends on determining c, which can be done by using M_{λ^-,E_R}, where λ^- is the nondecreasing rearrangement of λ. For each $i \leq \lambda_1$ let $m_i = \#\big\{j \colon \lfloor \lambda, E_S \rfloor [1, j] = i\big\}$ (the multiplicity of i in row 1 of $\lfloor \lambda, E_S \rfloor$). We will show that the coefficient of M_{λ,E_S} in $S^{(N-1)} M_{\lambda_-,E_R}$ is $[m_i]_t!$. (This was shown in [9, Theorem 5.39]; we are outlining a proof here with simplifications due to the simple hook shape $(N - m, 1^m)$, also to accommodate the different notation.) Here is an illustration of the following theorem and the method of proof. Suppose $\lambda=(3,2,2,2,1,0)$ and

$$Y_{E_R}=\begin{bmatrix}6&5&3&2\\ \cdot&4&1&\end{bmatrix},\qquad \lfloor\lambda,E_S\rfloor=\begin{bmatrix}0&1&2&2\\ \cdot&2&3&\end{bmatrix},\qquad Y_{E_S}=\begin{bmatrix}6&5&4&3\\ \cdot&2&1&\end{bmatrix}.$$

We demonstrate the effect on the significant terms by using the spectral vectors:

$$\begin{aligned}
&\left(\lambda^-,E_R\right)\simeq\left(1,qt,q^2t^3,q^2t^2,q^2t^{-1},q^3t^{-2}\right),\\
&X_5\colon\ \left(qt,q^2t^3,q^2t^2,q^2t^{-1},q^3t^{-2},1\right),\\
&X_4\colon\ \left(q^2t^3,q^2t^2,q^2t^{-1},q^3t^{-2},qt,1\right),\\
&X_3\colon\ (1+t)\times\left(q^2t^3,q^2t^{-1},q^3t^{-2},q^2t^2,qt,1\right),\\
&X_2\colon\ (1+t)\times\left(q^2t^{-1},q^3t^{-2},q^2t^3,q^2t^2,qt,1\right),\\
&X_1\colon\ (1+t)\times\left(q^3t^{-2},q^2t^{-1},q^2t^3,q^2t^2,qt,1\right),
\end{aligned}$$

and this is the spectral vector of $\left(\lambda,E_S\right)$.

Theorem 4.15. *For each $M_{\beta,F}\in\mathcal{M}\left(\lambda,E_S\right)$ with $(\beta,F)\neq\left(\lambda,E_S\right)$ there is a constant $c_{\beta,F}$ such that*

$$S^{(N-1)}M_{\lambda^-,E_R}=\prod_{i=1}^{\lambda_1}[m_i]_t!M_{\lambda,E_S}+\sum_{\lfloor\beta,F\rfloor=\lfloor\lambda,E_S\rfloor}\left\{c_{\beta,F}M_{\beta,F}\colon(\beta,F)\neq\left(\lambda,E_S\right)\right\}.$$

Proof. The proof relies on identifying the intermediate steps in transforming $x^{\lambda^-}R_{\lambda_-}(\tau_{E_R})$ to $x^\lambda\tau_{E_S}$. Roughly the action of X_{N-1-i} transforms $M_{\alpha(i),E}$ to $M_{\alpha(i+1),E}$ by means of $\boldsymbol{T}_{N-1-i}\times\cdots\boldsymbol{T}_1$, where $\alpha(0)=\lambda^-$ and for $i\geq1$

$$\alpha(i)=\left(\lambda_{N-i},\lambda_{N-2},\dots,\lambda_1,\lambda_{N-i+1},\dots,\lambda_N\right)$$

however the situation is not this simple because repeated values of λ_j have to taken into account. Note that X_j does not affect the variables x_k for $k>j+1$. It (almost) suffices to consider the coefficient of $x^{\alpha(1)}$ in $X_{N-1}M_{\lambda_-,E_R}$. (Throughout we use Σ to denote a linear combination of terms $M_{\beta,F}$ which can not be transformed into M_{λ,E_S} by the operators $X_1\cdots X_j$.) Suppose that $\lambda_N<\lambda_{N-1}$ (that is, $\lambda_1^-<\lambda_2^-$) then $\boldsymbol{T}_{N-1-i}\cdots\boldsymbol{T}_1M_{\lambda^-,E_R}=M_{\alpha(1),E_R}+\Sigma$, and the process is repeated with $M_{\alpha(1),E_R}$. The other possibility is that $\lambda_{N-k}>\lambda_{N-k+1}=\cdots=\lambda_N$ for some $k\geq2$. This implies $r_{\lambda^-}(i)=N-k+i$ for $1\leq i\leq k$ and $c\left(r_{\lambda^-}(i),E_R\right)=k-i$, because the entries $N-k+1,\dots,N$ are adjacent in row 1 of Y_{E_R} (by hypothesis $\lfloor\lambda,E\rfloor[2,1]>\lambda_N$ and so $\lambda_N=\lambda_{N-1}$ implies $\lfloor\lambda,E\rfloor[1,2]=N-1$). Thus $\boldsymbol{T}_iM_{\lambda^-,E_R}=tM_{\lambda^-,E_R}$ for $1\leq i\leq k-1$ and

$$\begin{aligned}
&\boldsymbol{T}_{N-1}\cdots\boldsymbol{T}_k(1+\boldsymbol{T}_{k-1}+\boldsymbol{T}_{k-1}\boldsymbol{T}_{k-2}+\cdots+\boldsymbol{T}_{k-1}\cdots\boldsymbol{T}_1)M_{\lambda^-,E_R}\\
&\qquad=\left(1+t+t^2+\cdots+t^{k-1}\right)\boldsymbol{T}_{N-1}\cdots\boldsymbol{T}_kM_{\lambda^-,E_R}=[k]_t\left\{M_{\alpha(1),E_R}+\Sigma\right\}.
\end{aligned}$$

Then $\alpha(1)=\left(\lambda_{N-1},\dots,\lambda_{N-k+1},\lambda_{N-k},\dots,\lambda_1,\lambda_N\right)$ and the previous argument applies with $k-1$ replacing k. The result of applying $X_{N-k}\cdots X_{N-1}$ is $[k]_tM_{\alpha(k),E^r}+\Sigma$. Now $\alpha(k)=\left(\lambda_{N-k},\lambda_{N-k+1},\cdots\right)$ and the $\lambda_N<\lambda_{N-1}$ type process applies.

The last case to consider is $\lambda_{N-i-1-k}>\lambda_{N-i-k}=\cdots=\lambda_{N-i}>\lambda_{N-i+1}$, where $Y_{E_R}[\ell,1]=N-i$ and the entries $N-i-1,N-i-2,\dots,N-i-k$ are adjacent in row 1 of Y_{E_R}. Then

$$\alpha(i)=\left(\lambda_{N-i},\dots,\lambda_{N-i-k},\lambda_{N-i-k-1}\right),$$

and similarly to the previous case $\boldsymbol{T}_i M_{\alpha(i),E'} = tM_{\alpha(i),E'}$ for $1 \le i \le k-1$. The set E' is an intermediate step in a series of transpositions transforming E_R to E_S, at this stage using only s_j with $i > N-i$. Similarly

$$\begin{aligned}&\boldsymbol{T}_{N-i-1} \cdots \boldsymbol{T}_k (1 + \boldsymbol{T}_{k-1} + \boldsymbol{T}_{k-1}\boldsymbol{T}_{k-2} + \cdots + \boldsymbol{T}_{k-1} \cdots \boldsymbol{T}_1) M_{\alpha(i),E'} \\ &\qquad = \big(1 + t + t^2 + \cdots + t^{k-1}\big) \boldsymbol{T}_{N-i-1} \cdots \boldsymbol{T}_k M_{\alpha(i),E'} = [k]_t \big\{ M_{\alpha(i+1),E''} + \Sigma \big\}.\end{aligned}$$

Here $\boldsymbol{T}_k$ transforms $M_{\alpha(i),E'}$ to $M_{\alpha(i),E''}$, where $E'' = s_{N-i-1}E'$ (since $c(N-i,E') < 0 < c(N-i-1,E')$ and $\operatorname{inv}(E'') = \operatorname{inv}(E') + 1$. Eventually these steps transform E_R to E_S and λ^- to λ. Each set of m_i (contiguous) λ_i values $\lambda_j = i$ in row 1 of $\lfloor \lambda, E_S \rfloor$ contributes a factor of $[m_i]_t!$. By beginning with E_R the factors appearing in $(\boldsymbol{T}_i + b) M_{\beta,F}$ are always 1 (see (3.3) and (3.2)). ■

Lemma 4.16. *Suppose*

$$F(\alpha, E) := \prod_{\substack{1 \le i < j \le N \\ \alpha_i < \alpha_j,}} g\big(\alpha_j - \alpha_i, c(r_\alpha(j), E) - c(r_\alpha(i, E))\big)$$

for some function g then

$$F(\lambda^-, E) = \prod_{\lambda_i > \lambda_j} g\big(\lambda_i - \lambda_j, c(i,E) - c(j,E)\big).$$

Proof. It is clear that $\lambda_i > \lambda_j$ if and only if $\lambda^-_{N+1-i} > \lambda^-_{N+1-j}$. If $\lambda_{a-1} > \lambda_a = \cdots = \lambda_{a+k-1} > \lambda_{a+k}$ then $[r_{\lambda^-}(b+i)]_{i=1}^k = [a, a+1, \dots, a+k-1]$ and $\lambda^-_{b+i} = \lambda_a$ for $b = N+1-a-k, 1 \le i \le k$. The corresponding contents $[c(r_{\lambda^-}(b+i)), E]_{i=1}^k$ are the same as $[c(a+i-1), E]_{i=1}^k$. So each term in $F_{\lambda^-,E}$ matches one in the stated λ-product. ■

Theorem 4.17. *Suppose $\lfloor \lambda, E_S \rfloor$ is column-strict and $m_i = \#\{j : \lfloor \lambda, E_S \rfloor [1,j] = i\}$ for $0 \le i \le \lambda_1$ then*

$$\begin{aligned}\|p_{\lambda,E_S}\|^2 &= t^{2(N-m-1)+k(\lambda)} [m+1]_t (1-q)^{-|\lambda|} \prod_{i=1}^N \big(qt^{c(i,E_S)}; q\big)_{\lambda_i} \\ &\quad \times \prod_{1 \le i < j \le N} \frac{\big(qt^{c(i,E_S)-c(j,E_S)-1}; q\big)_{\lambda_i-\lambda_j} \big(qt^{c(i,E_S)-c(j,E_S)+1}; q\big)_{\lambda_i-\lambda_j-1}}{\big(1 - q^{\lambda_i-\lambda_j} t^{c(i,E_S)-c(j,E_S)}\big) \big(qt^{c(i,E_S)-c(j,E_S)}; q\big)^2_{\lambda_i-\lambda_j-1}} \\ &\quad \times \frac{[N]_t!}{\prod_{i \ge 0} [m_i]_t!} \mathcal{C}_0\big(E_S\big) \mathcal{C}_1\big(E_R\big).\end{aligned}$$

Proof. By (4.1) and Theorem 4.15 $c = \prod_{i \ge 0} [m_i]_t!$,

$$\begin{aligned}\|p_{\lambda,E_S}\|^2 &= \frac{[N]_t!}{\prod_{i \ge 0} [m_i]_t!} \frac{\mathcal{C}_0\big(E_S\big) \mathcal{R}_0\big(\lambda^-, E_R\big)}{\mathcal{C}_0\big(E_R\big)} \big\| M_{\lambda^-,E_R} \big\|^2 \\ &= \frac{[N]_t!}{\prod_{i \ge 0} [m_i]_t!} \frac{\mathcal{C}_0\big(E_S\big) \mathcal{R}_0\big(\lambda^-, E_R\big)}{\mathcal{C}_0\big(E_R\big) \mathcal{E}\big(\lambda^-, E_R\big)} \big\| M_{\lambda,E_R} \big\|^2\end{aligned}$$

and

$$\begin{aligned}\big\| M_{\lambda,E_R} \big\|^2 &= t^{\ell(\lambda)} \|\tau_{E_R}\|^2 (1-q)^{-|\lambda|} \prod_{i=1}^N \big(qt^{c(i,E_R)}; q\big)_{\lambda_i} \\ &\quad \times \prod_{1 \le i < j \le N} \frac{\big(qt^{c(i,E_R)-c(j,E_R)-1}; q\big)_{\lambda_i-\lambda_j} \big(qt^{c(i,E_R)-c(j,E_R)+1}; q\big)_{\lambda_i-\lambda_j}}{\big(qt^{c(i,E_R)-c(j,E_R)}; q\big)^2_{\lambda_i-\lambda_j}}.\end{aligned}$$

Also $\|\tau_{E_R}\|^2 = t^{2(N-m-1)}[m+1]_t\mathcal{C}\big([c(i,E_R)]_{i=1}^N\big) = t^{2(N-m-1)}[m+1]_t\mathcal{C}_0\big(E_R\big)\mathcal{C}_1\big(E_R\big)$. Since $\mathcal{E}\big(\lambda^-,E_R\big) = \mathcal{R}_0\big(\lambda^-,E_R\big)\mathcal{R}_1\big(\lambda^-,E_R\big)$ the terms $\mathcal{R}_0\big(\lambda^-,E_R\big)$ and $\mathcal{C}_0\big(E_R\big)$ cancel out. By the lemma

$$\mathcal{R}_1\big(\lambda^-,E_S\big) = \prod_{1\le i<j\le N} u_1\big(q^{\lambda_i-\lambda_j}t^{c(i,E_R)-c(j,E_R)}\big) = \prod_{1\le i<j\le N} \frac{1-q^{\lambda_i-\lambda_j}t^{c(i,E_R)-c(j,E_R)+1}}{1-q^{\lambda_i-\lambda_j}t^{c(i,E_R)-c(j,E_R)}}$$

and dividing by this product changes two of the $(*;q)_{\lambda_i-\lambda j}$ terms to $(*;q)_{\lambda_i-\lambda j-1}$. ∎

Note that E_R can be replaced by E_S in the first two lines of the formula for $\|p_{\lambda,E_S}\|^2$. By using the M map the formulas produce supersymmetric polynomials in $\mathcal{P}_{m,1}$: consider the polynomials $M(p_{\lambda,E_S})$, where $p_{\lambda,E_S}\in\mathcal{P}_{m-1,0}$. This is why we do not go into detail about the $E\in\mathcal{E}_1$ case. The norm formula implies the identity

$$\sum_{\lfloor\alpha,F\rfloor=\lfloor\lambda,E\rfloor} \frac{\mathcal{C}_1(F)\mathcal{R}_0(\alpha,F)}{\mathcal{C}_0(F)\mathcal{R}_1(\alpha,F)} = \frac{[N]_t!}{\prod_{i\ge0}[m_i]_t!}\frac{\mathcal{C}_1\big(E_R\big)}{\mathcal{C}_0\big(E_S\big)\mathcal{R}_1\big(\lambda^-,E_R\big)}.$$

This formula was checked by computer algebra for a "small" example, $N=5$, $m=2$, $\lambda=(2,2,1,1,0)$ with

$$\big\lfloor\lambda,E_S\big\rfloor = \begin{bmatrix} 0 & 1 & 2\\ \cdot & 1 & 2\end{bmatrix}, \qquad E_R=\{2,4,5\}, \qquad E_S=\{1,3,5\};$$

there are 120 labels (β,F) with $\lfloor\beta,F\rfloor = \big\lfloor\lambda,E_S\big\rfloor$, that is $\dim\mathcal{M}\big(\lambda,E_S\big)=120$.

4.3 Special values

In the scalar Macdonald polynomial situation there are formulas for special values. There is one such fairly simple formula here. Let $F=\{1,2,\dots,m,N\}$. (In the $\mathcal{Y}_1$-case use E_1.) Refer to Proposition 2.34 for useful facts about τ_F and $\|\tau_F\|^2$.

Proposition 4.18. *Suppose $p\in s\mathcal{P}_m$ is symmetric and*

$$z := \big(z_1,z_2,\dots,z_m,t^{N-m-1},\dots,t^2,t,1\big)$$

then

$$p(z;\theta) = \prod_{1\le i<j\le m}(z_i-tz_j)\prod_{k=1}^m\big(z_k-t^{N-m}\big)p_0(z_1,\dots,z_m)\tau_F,$$

where p_0 is $\mathcal{S}_m$-symmetric.

Proof. First we show that if $\boldsymbol{T}_ip(x;\theta)=tp(x;\theta)$ then $T_ip\big(x^{(i)};\theta\big)=tp\big(x^{(i)};\theta\big)$, where $x_i^{(i)} = tx_{i+1}^{(i)}$. By hypothesis

$$(1+\boldsymbol{T}_i)p(x;\theta) = (1+t)p(x;\theta) = p(x;\theta)+(1-t)x_{i+1}\frac{p(x;\theta)-p(xs_i;\theta)}{x_i-x_{i+1}}+T_ip(xs_i;\theta).$$

Substitute $x_i=tx_{i+1}$ in the equations:

$$(1+t)p(x;\theta) = p(x;\theta)-(p(x;\theta)-p(xs_i;\theta))+T_ip(xs_i;\theta) = (1+T_i)p(xs_i;\theta)$$

and this shows $(t-T_i)p(x;\theta)=0$ at $x=x^{(i)}$. By hypothesis on z this shows $T_ip(z;\theta)=tp(z;\theta)$ for $m+1\le i<N$ and this implies $\omega_ip(z;\theta)=t^{N-i}p(z;\theta)$ for $m+1\le i\le N$. There is only

one τ_E which has these $\{\omega_i\}$-eigenvalues, namely τ_F. Thus $p(z;\theta) = \widetilde{p}(z_1,\ldots,z_m)\tau_F$ for some polynomial $\widetilde{p}$ and $\boldsymbol{T}_i\widetilde{p}(z)\tau_F = t\widetilde{p}(z)\tau_F$ for $1 \le i < m$. In this range $T_i\tau_F = -\tau_F$ thus $\widetilde{p}$ satisfies the equation

$$\begin{aligned}
t\widetilde{p}(z)\tau_E &= (1-t)z_{i+1}\frac{\widetilde{p}(z)-\widetilde{p}(zs_i)}{z_i - z_{i+1}}\tau_E - \widetilde{p}(zs_i)\tau_E,\\
\widetilde{p}(zs_i) &= \frac{z_{i+1}-tz_i}{z_i - tz_{i+1}}\widetilde{p}(z).
\end{aligned}$$

Thus $z_i - tz_{i+1}$ is a factor of $\widetilde{p}(z)$ because $\widetilde{p}(zs_i)$ is polynomial. Furthermore $\widetilde{p}(z)/(z_i - tz_{i+1})$ is s_i-invariant. We claim by induction that $(z_i - tz_{i+k})$ is a factor of $\widetilde{p}(z)$ for $1 \le i < i+k \le m$: this is valid for $k=1$ so consider that $(z_i - tz_{i+k})$ is a factor of $\widetilde{p}(z)$ and $\widetilde{p}(z)/(z_{i+k} - tz_{i+k+1})$ is s_{i+k}-invariant thus $(z_i - tz_{i+k+1})$ is a factor (where $i+k+1 \le m$).

Suppose $z'_m = t^{N-m} = tz'_{N-m+1}$ then $T_m\widetilde{p}(z')\tau_F = t\widetilde{p}(z')\tau_F$ but this implies $\widetilde{p}(z') = 0$ or $\omega_m\tau_F = t^{N-m}\tau_F$ which is impossible. Thus $\left(z_m - t^{N-m}\right)$ is a factor of $\widetilde{p}(z)$. The symmetry properties imply $\left(z^i - t^{N-m}\right)$ is a factor of $\widetilde{p}(z)$ for $1 \le i \le m$. ■

González and Lapointe [10] proved an evaluation formula for the version of supersymmetric Macdonald polynomials constructed in [3], with

$$z = \left(t^{N-1}q^{-m}, t^{N-2}q^{1-m}, \ldots, t^{N-m}q^{-1}, t^{N-m-1}, \ldots, t, 1\right).$$

An example appears to show there is no such general result in our version. However there may be one for the special case where $\lfloor\lambda, E_S\rfloor[1,j] = 0$ for $1 \le j \le N-m$. At this point we offer no conjecture, but some very small examples with $N = 3$, 4 and $|\lambda| \le 4$ suggest there is something to be found.

4.4 Minimal symmetric polynomial

For given N, m, isotype $(N-m, 1^m)$ there is a unique column-strict tableau with minimum sum of entries, namely

$$\lfloor\lambda, E\rfloor = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ \cdot & 1 & 2 & \cdots & m\end{bmatrix},$$

thus $\lambda = (m, m-1, \ldots, 2, 1, 0, \ldots, 0)$ and $E_R = E_S = \{1, 2, \ldots, m, N\}$. There is non-trivial multiplicity $m_0 = N - m$. Thus $p_{\lambda,E}$ is the symmetric polynomial in $\mathcal{P}_{m,0}$ of minimum bosonic degree $\frac{m(m+1)}{2}$ There is a concise formula for $\|p_{\lambda,E}\|^2$.

Theorem 4.19. *Suppose $\lambda = (m, m-1, \ldots, 1, 0, \ldots, 0) \in \mathbb{N}_0^{N,+}$ and $E = \{1, 2, \ldots, m, N\}$ then $\lfloor\lambda, E\rfloor$ is column-strict and*

$$\|p_{\lambda,E}\|^2 = \frac{[N]_t![N]_t t^\gamma}{[N-m]_t![N-m]_t}\left(qt^{-N};q\right)_m \prod_{j=2}^{m}\left(qt^{-j};q\right)_{j-1},$$

with $\gamma = (N-m-1)\left(1 + \frac{(m+1)(m+2)}{2}\right) + \frac{m(m+1)(m+2)}{6}$.

Proof. The exponent on t is

$$\begin{aligned}
\gamma &= 2(N-m-1) + k(\lambda) + m(N-m-1)\\
&= (m+2)(N-m-1) + \sum_{i=1}^{m}(N-2i+1)(m+1-i).
\end{aligned}$$

The spectral vector is $\zeta_{\lambda,E}=\left[q^m t^{-m},q^{m-1}t^{1-m},\dots,qt^{-1},t^{N-m-1},\dots,t,1\right]$. Consider the content product (part of $\mathcal{C}$) Π_k for the pairs $(m+1-k,m+j)$ with $1\le k\le m$ and $j=1,\dots,N-m-1$:

$$\Pi_k=\prod_{j=1}^{N-m-1}u\left(t^{-k-j}\right)=\prod_{j=1}^{N-m-1}\frac{1-t^{-k-j+1}}{1-t^{-k-j}}\frac{t-t^{-k-j}}{1-t^{-k-j}}=t^{N-m-1}\frac{1-t^{-k}}{1-t^{-k-N+m+1}}\frac{1-t^{-k-N+m}}{1-t^{-k-1}},$$

by telescoping the products (and $t-t^{-k-j}=t(1-t^{-k-j-1})$) then

$$\prod_{k=1}^{m}\Pi_k=t^{m(N-m-1)}\frac{1-t^{-1}}{1-t^{-m-1}}\frac{1-t^{-N}}{1-t^{-N+m}}=t^{m(N-m-1)}\frac{[N]_t}{[m+1]_t[N-m]_t}.$$

Consider the q-factors for the pairs $(m+1-k,m+j)$ with $1\le k\le m$ and $j=1,\dots,N-m-1$ (use telescoping)

$$P_{1,k}=\prod_{j=1}^{N-m-1}\frac{(qt^{-k-j-1};q)_k\left(qt^{-k-j+1};q\right)_{k-1}}{\left(qt^{-k-j};q\right)_k\left(qt^{-k-j};q\right)_{k-1}}=\frac{\left(qt^{-k-N+m};q\right)_k\left(qt^{-k};q\right)_{k-1}}{(qt^{-m-1};q)_m\left(qt^{-k-N+m+1};q\right)_{k-1}},$$

then

$$P_1:=\prod_{k=1}^{m}P_{1,k}=\frac{\left(qt^{-N};q\right)_m}{\left(qt^{-m-1};q\right)_m}.$$

Next consider $(m+1-k,m+1-i)$ with $k>i\ge 1$ together with $(m+1-k,N)$:

$$P_{2,k}=\prod_{i=0}^{k-1}\frac{\left(qt^{-k+i-1};q\right)_{k-i}\left(qt^{-k+i+1};q\right)_{k-i-1}}{\left(qt^{-k+i};q\right)_{k-i}\left(qt^{-k+i};q\right)_{k-i-1}}=\frac{\left(qt^{-k-1};q\right)_k}{\left(qt^{-k};q\right)_k}.$$

Combine

$$\begin{aligned}\prod_{i=1}^{N}\left(qt^{c(i,E)};q\right)_{\lambda_i}P_1\prod_{k=1}^{m}P_{2,k}&=\frac{\left(qt^{-N};q\right)_m}{\left(qt^{-m-1};q\right)_m}\prod_{k=1}^{m}\frac{\left(qt^{-k};q\right)_k\left(qt^{-k-1};q\right)_k}{\left(qt^{-k};q\right)_k}\\&=\left(qt^{-N};q\right)_m\prod_{k=1}^{m-1}\left(qt^{-k-1};q\right)_k.\end{aligned}$$

This concludes the proof. ■

The formula has a hook length interpretation: $\prod_{(i,j)\in Y_E}\left(qt^{-\mathrm{hook}(i,j)};q\right)_{\mathrm{leg}(i,j)}$; here $\mathrm{leg}(1,j)=0$ for $2\le j\le N-m$; $\mathrm{hook}(i,1)=m+2-i$, $\mathrm{leg}(i,1)=m+1-i$, $2\le i\le m+1$ and $\mathrm{hook}(1,1)=N$, $\mathrm{leg}(1,1)=m$ (see [9, Theorem 6.22]).

4.5 Antisymmetric superpolynomials

Consider $p_a=\sum_{\lfloor\beta,F\rfloor=\lfloor\lambda,E_R\rfloor}A(\beta,F)M_{\beta,F}$, where $\lfloor\lambda,E\rfloor$ is row-strict and $\boldsymbol{T}_i p_a=-p_a$ for $1\le i<N$. Recall the action of $\boldsymbol{T}_i$ on the sum, which decomposes into pairs and singletons. Suppose $\lfloor\beta,F\rfloor=\lfloor\lambda,E_R\rfloor$ for some β with $\beta_i<\beta_{i+1}$, some i. Let $z=\zeta_{\beta,F}(i+1)/\zeta_{\beta,F}(i)$ then by (3.3)

$$(\boldsymbol{T}_i+1)\left(A(\beta,F)M_{\beta,F}+A(s_i\beta,F)M_{s_i\beta,F}\right)=0$$

implies $A(\beta,F)=-\frac{1-zt}{1-z}A(s_i\beta,F)$. Recall $\sigma(n)=(-1)^n$ (Definition 2.9).

Lemma 4.20. *If $\boldsymbol{T}_i p_a = -p_a$ for $1 \le i < N$ then*

$$A(\beta, F) = \sigma(\mathrm{inv}(\beta))\mathcal{R}_1(\beta, F)A(\beta^+, F).$$

Proof. Suppose $\beta_i > \beta_{i+1}$ then

$$\frac{\mathcal{R}_1(s_i\beta, F)}{\mathcal{R}_1(\beta, F)} = u_1\left(\frac{\zeta_{\beta,F}(i)}{\zeta_{\beta,F}(i+1)}\right). \qquad \blacksquare$$

Consider the possibilities when $\beta_i = \beta_{i+1}$ and $j = r_\beta(i)$: (i) if $c(j,F) = c(j+1,F) - 1$, that is, j and $j+1$ are in adjacent cells of column 1 of Y_F then $\boldsymbol{T}_i M_{\beta,F} = -M_{\beta,F}$, imposing no conditions on $A(\beta,F)$; (ii) if $c(j,F) = c(j+1,F) + 1$ then $\boldsymbol{T}_i M_{\beta,F} = tM_{\beta,F}$ but this occurs only if there are adjacent equal values (β_i) in row 1 of $\lfloor \lambda, E_R \rfloor$, ruled out by hypothesis; (iii) $c(j,F) < 0 < c(j+1,F)$. In this case we relate $M_{\beta,F}$ to M_{β,s_jF}, where $\mathrm{inv}(s_jF) = \mathrm{inv}(F) - 1$: the formulas similar to (3.2) with $z = \zeta_{\beta,F}(i)/\zeta_{\beta,F}(i+1) = t^{c(j,F)-c(j+1,F)}$ appear here: then

$$(\boldsymbol{T}_i + 1)\big(A(\beta,F)M_{\beta,F} + A(\beta,s_jF)M_{\beta,s_jF}\big) = 0$$

implies $A\big(\beta, s_jF\big) = -\frac{1-tz}{1-z}A(\beta,F)$.

Lemma 4.21. $\sigma(\mathrm{inv}(F))\dfrac{A(\beta,F)}{\mathcal{C}_1(F)} = \sigma\big(\mathrm{inv}(E_S)\big)\dfrac{A\big(\beta,E_S\big)}{\mathcal{C}_1\big(E_S\big)}$.

Thus

$$\begin{aligned} A(\beta,F) &= \sigma(\mathrm{inv}(\beta))\mathcal{R}_1(\beta,F)A(\beta^+,F) \\ &= \sigma\big(\mathrm{inv}(\beta) + \mathrm{inv}(F) + \mathrm{inv}\left(E_S\right)\big)A\big(\lambda,E_S\big)\frac{\mathcal{R}_1(\beta,F)\mathcal{C}_1\big(E_S\big)}{\mathcal{C}_1(F)}. \end{aligned}$$

Theorem 4.22. *Suppose $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{Y}_0$, and $\lfloor \lambda, E \rfloor$ is row-strict then*

$$p^a_{\lambda,E} = \sum_{\lfloor \alpha,F \rfloor \in \mathcal{T}(\lambda,E)} \sigma\big(\mathrm{inv}(\beta) + \mathrm{inv}(F) + \mathrm{inv}\left(E_S\right)\big)\frac{\mathcal{C}_1\big(E_S\big)\mathcal{R}_1(\alpha,F)}{\mathcal{C}_1(F)}M_{\alpha,F}$$

is the antisymmetric polynomial in $\mathcal{M}(\lambda,E)$, unique when the coefficient of M_{λ,E_S} is 1.

The antisymmetrizing operator is defined analogously to S.

Definition 4.23. For $n \ge 1$ let $X^a_0 = 1$ and $X^a_n = 1 - \frac{1}{t}\boldsymbol{T}_n X_{n-1}$, and $A^{(n)} = X^a_1 X^a_2 \cdots X^a_n$.

Equivalently $X^a_n = 1 - \frac{1}{t}\boldsymbol{T}_n + \frac{1}{t^2}\boldsymbol{T}_n\boldsymbol{T}_{n-1} + \cdots + \frac{(-1)^n}{t^n}\boldsymbol{T}_n \cdots \boldsymbol{T}_2\boldsymbol{T}_1$.

Theorem 4.24. *If $1 \le j \le n$ then $(\boldsymbol{T}_j + 1)A^{(n)} = 0$.*

Proof. The operators $\left\{-\frac{1}{t}\boldsymbol{T}_i\right\}$ satisfy the braid relations so the same approach as in Theorem 4.11 works here, and the proof then follows from $(\boldsymbol{T}_i + 1)\big(1 - \frac{1}{t}\boldsymbol{T}_i\big) = 0$. $\blacksquare$

Similarly to Corollary 4.13 one can show that

$$A^{(N-1)}A^{(N-1)} = t^{-N(N-1)/2}[N]_t!A^{(N-1)}.$$

There is a result analogous to Proposition 4.18.

Lemma 4.25. *Suppose for some i that $\boldsymbol{T}_i p(x;\theta) = -p(x;\theta)$ then $T_i p\left(x^{(i)};\theta\right) = -p\left(x^{(i)};\theta\right)$, where $x^{(i)}_{i+1} = tx^{(i)}_i$.*

Proof. By hypothesis

$$(t-\boldsymbol{T}_i)p(x;\theta)=(t+1)p(x;\theta)=tp(x;\theta)-(1-t)x_{i+1}\frac{p(x;\theta)-p(xs_i;\theta)}{x_i-x_{i+1}}T_ip(xs_i;\theta).$$

Substitute $x_{i+1}=tx_i$ in the equations:

$$(t+1)p(x;\theta)=tp(x;\theta)-t(p(x;\theta)-p(xs_i;\theta))-T_ip(xs_i;\theta)=(t-T_i)p(xs_i;\theta)$$

and this shows $(1+T_i)p(x;\theta)=0$ at $x=x^{(i)}$. ■

Suppose $p^a_{\lambda,E}$ is antisymmetric and $E_0=\{N-m,N-m+1,\dots,N\}$ and consider $p^a_{\lambda,E}(z)$, where $z=\left(z_1,z_2,\dots,z_{N-m-1},t^{-m},\dots,t^{-2},t^{-1},1\right)$, then by the lemma $T_ip(z;\theta)=-p(z;\theta)$ for $N-m\le i<N$ which $\omega_ip(z;\theta)=t^{i-N}p(z;\theta)$ for $N-m\le i\le N$. The eigenvalues determine τ_{E_0} and thus $p(z;\theta)=\widetilde{p}(z)\tau_{E_0}(\theta)$. If range $1\le i\le N-m-2$ then $T_i\tau_F=t\tau_F$ thus $\widetilde{p}$ satisfies the equation

$$\begin{aligned}
&-\widetilde{p}(z)\tau_{E_0}=(1-t)z_{i+1}\frac{\widetilde{p}(z)-\widetilde{p}(zs_i)}{z_i-z_{i+1}}\tau_{E_0}+t\widetilde{p}(zs_i)\tau_{E_0},\\
&\widetilde{p}(zs_i)=\frac{z_i-tz_{i+1}}{z_{i+1}-tz_i}\widetilde{p}(z).
\end{aligned}$$

This implies $(z_{i+1}-tz_i)$ is a factor of $\widetilde{p}(z)$ and $\frac{\widetilde{p}(z)}{z_{i+1}-tz_i}$ is s_i-invariant. Furthermore (tz_i-z_j) is a factor of $\widetilde{p}(z)$ for $1\le i<j\le N-m-1$. Also $z_{N-m-1}=t^{-m-1}$ implies $\widetilde{p}(z)=0$ (or else $\omega_{N-m-1}\tau_{E_0}=t^{-m-1}\tau_{E_0}$, contra) and so $\left(t^{m+1}z_{N-m-1}-1\right)$ is a factor of $\widetilde{p}(z)$. Thus

$$p^a(z;\theta)=\prod_{1\le i<j\le N-m-1}(tz_i-z_j)\prod_{k=1}^{N-m-1}\left(t^{m+1}z_k-1\right)p_0(z_1,\dots,z_{N-m-1})\tau_{E_0}$$

and p_0 is $\mathcal{S}_{N-m-1}$-symmetric. With methods similar to those of Theorem 4.19 and by use of the antisymmetrizing operator $A^{(N-1)}$ one can derive a formula for $\|p^a_{\lambda,E}\|^2$.

5 Conclusion

We constructed a representation of the Hecke algebra $\mathcal{H}_N(t)$ on superpolynomials and applied the theory of vector-valued nonsymmetric Macdonald polynomials to this situation. The basic facts such as orthogonal bases for irreducible representations on fermionic variables, the partial order on compositions used in expressions for the Macdonald polynomials, and a sketch of the Yang–Baxter graph technique for constructing the polynomials starting from degree zero were presented. The polynomials are mutually orthogonal with respect to a bilinear form in which the generators of the Hecke algebra are self-adjoint. The ideas of Baker and Forrester were used to construct symmetric polynomials and to determine their squared norms.

There are some topics which deserve further investigation. What can be proven about values of the nonsymmetric Macdonald polynomials at special points such as $\left(t^{N-1},t^{N-2},\dots,t,1\right)$ (see Remark 3.14)? Are there special values of the symmetric and anti-symmetric polynomials? A minimal factorization was proven in Proposition 4.18.

Characterizing singular values of the parameters q, t and the corresponding polynomials is another important problem: this means that for a specific value of (q,t) there is a polynomial annihilated by $\boldsymbol{D}_i$ for $1\le i\le N$ (see Definition 3.12). This problem is connected with the existence of maps between different modules. Also there should be interesting factorizations. Here are two examples with $N=5$.

Let $\alpha = (2,0,0,0,0)$, $m = 2$, $E = \{3,4,5\} \in \mathcal{Y}_0$ then $\boldsymbol{D}_i M_{\alpha,E} = 0$ for $1 \le i \le 5$ when $q^2t^5 = 1$ or $qt = -1$ (that is $q^2t^2 = 1$, $qt \ne 1$), and

$$M_{\alpha,E}\big(x_1, x_2, tx_2, t^2x_2, t^3x_2\big) = t^{10}(tx_1 - x_2)(qtx_1 - x_2)\tau_E,$$
$$\tau_E = t^4\theta_3\theta_4 - t^3\theta_3\theta_5 + t^2\theta_4\theta_5,$$

when (q,t) takes on a singular value (note if $q = -1/t$ then $qtx_1 - x_2 = -(x_1 + x_2)$).

Let $\alpha = (2,0,0,0,0)$, $m = 3$, $E = \{1,2\} \in \mathcal{Y}_1$ then $\boldsymbol{D}_i M_{\alpha,E} = 0$ for $1 \le i \le 5$ when $q^2t^{-5} = 1$ or $q = -t$ (that is $q^2t^{-2} = 1$ and $qt^{-1} \ne 1$), and

$$M_{\alpha,E}\big(x_1, x_2, t^{-1}x_2, t^{-2}x_2, t^{-3}x_2\big) = t^6\big(t^{-1}x_1 - x_2\big)\big(qt^{-1}x_1 - x_2\big)\tau_E,$$
$$\tau_E = \theta_1\theta_2(\theta_3 + \theta_4 + \theta_5),$$

when $q^2 = t^5$ (set $q = u^5$, $t = u^2$) or $q = -t$.

Obviously there are delicate interactions among α, E, q, t, x for such factorizations to hold.

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