

Equivalent Integrable Metrics on the Sphere with Quartic Invariants

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Received March 31, 2022, in final form December 04, 2022; Published online December 06, 2022

<https://doi.org/10.3842/SIGMA.2022.094>

Abstract. We discuss canonical transformations relating well-known geodesic flows on the cotangent bundle of the sphere with a set of geodesic flows with quartic invariants. By adding various potentials to the corresponding geodesic Hamiltonians, we can construct new integrable systems on the sphere with quartic invariants.

Key words: integrable metrics; canonical transformations; two-dimensional sphere

2020 Mathematics Subject Classification: 37J35; 70H06; 70H45

1 Introduction

In the study of metric spaces, there are various notions of two metrics on the same underlying space Q being “the same”, or equivalent. For instance, there are topologically equivalent metrics and strong equivalent metrics [3]. In the Riemannian geometry two metrics are projectively equivalent if their geodesics coincide, in Kähler geometry two metrics are c-projectively equivalent if their J -planar curves coincide and so on, see [9, 10, 12] and references within.

In symplectic geometry, it is natural to say that two metrics g and g' on the configuration space Q are equivalent if the corresponding geodesic Hamiltonians

$$T = \sum_{i,j=1}^n g_{ij}(q)p_i p_j \quad \text{and} \quad T' = \sum_{i,j=1}^n g'_{ij}(q)p_i p_j \quad (1.1)$$

are related by some transformation of the phase space

$$\rho: T^*Q \rightarrow T^*Q \quad (1.2)$$

preserving canonical symplectic form $\omega = dp \wedge dq$. Here $q = q_1, \dots, q_n$ are coordinates on Q and $p = p_1, \dots, p_n$ are fibrewise coordinates with respect to the cotangent vectors dq_1, \dots, dq_n . Well-known examples of such canonical transformations are point transformations and non-point transformations in $T^*\mathbb{R}^n$

$$\rho: q_i \rightarrow p_i \quad \text{and} \quad p_i \rightarrow -q_i, \quad i = 1, \dots, n,$$

relating two geodesic Hamiltonians (1.1) when both metrics are the homogeneous polynomials of second order in coordinates. We aim to construct and classify other non-point canonical transformations relating to two polynomials of the second order in momenta (1.1).

Canonical transformation preserves the form of canonical Poisson brackets, which allows us to obtain new integrable geodesic flows by using the following algorithm:

- take some known integrable geodesic flow with Hamiltonian $T = T_1$ and independent integrals of motion T_2, \dots, T_n in the involution

$$\{T_i, T_j\} = 0, \quad i, j = 1, \dots, n;$$

- take non-point canonical transformation ρ (1.2), which maps geodesic Hamiltonian T to geodesic Hamiltonian T' , and calculate a set of independent functions $\rho(T_k)$ in the involution on T^*Q with respect to the same canonical Poisson brackets

$$\{\rho(T_i), \rho(T_j)\} = 0, \quad i, j = 1, \dots, n;$$

- compute $n - 1$ functions K_m on integrals of motion $\rho(T_k)$, so that functions K_m are polynomials in momenta, which simplifies all further calculations;
- find potential $V(q)$ solving equations

$$\{H_i, H_j\} = 0, \quad i, j = 1, \dots, n,$$

where $H_1 = T' + V(q)$ and $H_m = K_m + W_m(p, q)$, with respect to V and polynomials in momenta W_m ;

- calculate new integrable metric \tilde{g} on Q by using Maupertuis principle

$$\tilde{H} = \frac{T'}{h - V} = \sum_{i,j=1}^n \tilde{g}_{ij}(q) p_i p_j. \quad (1.3)$$

The main unsolved problems in this method are the construction of the non-point canonical transformations ρ (1.2) relating a given quadratic polynomial T with other quadratic polynomial T' and computation of the applicable to the Maupertuis principle polynomials in momenta K_m . Several canonical transformations ρ were obtained in the framework of algebraic geometry for the 2D Euclidean space in [16, 22, 24, 25], for the 2D sphere in [17, 18, 20, 21] and for the 2D ellipsoid in [19].

In this note, we present canonical transformation ρ (1.2) on the cotangent bundle to $(n - 1)$ -dimensional sphere $S^{(n-1)}$ using globally defined coordinates on the ambient space \mathbb{R}^n . At $n = 3$ this transformation was obtained in [20, 21] in terms of the locally defined coordinates on the sphere. Because we only want to prove the existence of such non-point canonical transformations ρ (1.2) and their applicability to the construction of new integrable metrics and so-called magnetic Hamiltonians $H_1 = T' + V$ with generalized potential V depending on velocities

$$V = \sum_{i=1}^n u_i(q) p_i + U(q),$$

we do not discuss the properties of obtained integrable systems, the curvature of the metrics, etc.

2 Non-point canonical transformations

Let us consider Cartesian coordinates $x = (x_1, \dots, x_n)$ in Euclidean space \mathbb{R}^n and the conjugated momenta p_{x_i} on $T^*\mathbb{R}^n$, so that

$$\{x_i, x_j\}' = \{p_{x_i}, p_{x_j}\}' = 0, \quad \{x_i, p_{x_j}\}' = \delta_{ij}, \quad i, j = 1, \dots, n.$$

The unit $(n - 1)$ -dimensional sphere $\mathbb{S}^{(n-1)} \subset \mathbb{R}^n$ and its cotangent bundle $T^*\mathbb{S}^{(n-1)} \subset T^*\mathbb{R}^n$ are defined via constraints

$$F_1 = x_1^2 + \dots + x_n^2 = 1, \quad F_2 = x_1 p_{x_1} + \dots + x_n p_{x_n} = 0. \quad (2.1)$$

Induced symplectic structure on $T^*\mathbb{S}^{n-1}$ is given by the Dirac–Poisson bracket

$$\{f, g\} = \{f, g\}' - \frac{\{F_1, f\}'\{F_2, g\}' - \{F_1, g\}'\{F_2, f\}'}{\{F_1, F_2\}'},$$

which reads as

$$\{x_i, x_j\} = 0, \quad \{x_i, p_{x_j}\} = \delta_{ij} - x_i x_j, \quad \{p_{x_i}, p_{x_j}\} = x_j p_{x_i} - x_i p_{x_j}. \quad (2.2)$$

Images of these variables (x, p_x) we denote as $y = (y_1, y_2, y_3)$ and $p_y = (p_{y_1}, p_{y_2}, p_{y_3})$

Proposition 2.1. *Consider the following mapping of the cotangent bundle $T^*\mathbb{S}^{(n-1)}$*

$$\rho_b: (x_i, p_{x_i}) \rightarrow (y_i, p_{y_i}), \quad i = 1, \dots, n,$$

defined by equations

$$y_i = \sqrt{\frac{b_i}{\mathcal{H}}} p_{x_i} \quad \text{and} \quad x_i p_{x_i} + y_i p_{y_i} = 0, \quad (2.3)$$

where

$$\mathcal{H} = b_1 p_{x_1}^2 + \dots + b_n p_{x_n}^2 \quad \text{and} \quad b_i > 0.$$

This mapping preserves constraints

$$\begin{aligned} F_1 &= x_1^2 + \dots + x_n^2 = 1 = y_1^2 + \dots + y_n^2, \\ F_2 &= x_1 p_{x_1} + \dots + x_n p_{x_n} = 0 = y_1 p_{y_1} + \dots + y_n p_{y_n}, \end{aligned}$$

the form of Hamiltonian

$$\mathcal{H} = b_1 p_{x_1}^2 + \dots + b_n p_{x_n}^2 = b_1 p_{y_1}^2 + \dots + b_n p_{y_n}^2, \quad (2.4)$$

and the form of induced Poisson brackets (2.2).

The proof is a straightforward verification of the Poisson bracket, the forms of constraints, and the form of Hamiltonian.

Below we also consider composition of ρ_b (2.3) and similar map ρ_c

$$\rho_c: y_i = \sqrt{\frac{c_i}{\sum_{i=1}^n c_i \tilde{p}_{x_i}^2}} \tilde{p}_{x_i} \quad \text{and} \quad \tilde{x}_i \tilde{p}_{x_i} + y_i p_{y_i} = 0, \quad c_i \in \mathbb{R},$$

which is the canonical transformation

$$\sigma_{bc}: (x, p_x) \rightarrow (\tilde{x}, \tilde{p}_x) \quad (2.5)$$

depending on $2n$ parameters $b_i, c_i, i = 1, \dots, n$. This composition also preserves canonical Poisson brackets (2.2) and the form of Hamiltonian

$$\mathcal{H} = p_{x_1}^2 + \dots + p_{x_n}^2 = \tilde{p}_{x_1}^2 + \dots + \tilde{p}_{x_n}^2.$$

Here \tilde{p}_{x_i} are momenta corresponding to coordinates \tilde{x}_i .

We can construct a family of equivalent integrable metrics on the sphere using these canonical transformations ρ_b and σ_{bc} . For instance, applying mapping (2.3) to the geodesic Hamiltonian on $T^*\mathbb{S}^{(n-1)}$

$$T = \sum_{i=1}^n a_i p_{y_i}^2 + \mathcal{H} \sum_{i=1}^n c_i y_i^2, \quad \mathcal{H} = \sum_{i=1}^n b_i p_{y_i}^2, \quad (2.6)$$

we obtain geodesic Hamiltonian of the similar form

$$\rho_b(T) = \sum_{i=1}^n b_i c_i p_{x_i}^2 + \mathcal{H} \sum_{i=1}^n a_i b_i^{-1} x_i^2, \quad \mathcal{H} = \sum_{i=1}^n b_i p_{x_i}^2. \quad (2.7)$$

When $b_i = 1$, we have a simple permutation of parameters $a_i \leftrightarrow c_i$ in the original Hamiltonian (2.6).

This permutation of parameters is not as trivial as it seems. Let us take Hamiltonian T (2.6) and polynomial of the second order in momenta

$$K = \sqrt{w(y)}\mathcal{H}, \quad w(y) = \sum_{i \geq j} e_{ij} y_i^2 y_j^2, \quad (2.8)$$

where $w(y)$ is a polynomial of second order in squares y_j^2 , which is not a full square. If we substitute T (2.6) and K (2.8) into

$$\{T, K\} = 0$$

and solve the resulting system of algebraic equations for b_i , c_i , and d_i , we obtain a geodesic flow with two integrals of motion which are polynomials of second order in momenta.

Mapping ρ_b (2.3) relates second order polynomial in momenta T to the second order polynomial in momenta $\rho_b(T)$ (2.7) commuting with $\rho_b(K)$,

$$\{\rho_b(T), \rho_b(K)\} = 0, \quad \rho_b(K) = \rho(\sqrt{w(y)}\mathcal{H}) = \sqrt{\sum_{i \geq j} e_{ij} b_i b_j p_{x_i}^2 p_{x_j}^2}, \quad (2.9)$$

and with its square $\rho^2(K)$, which is a polynomial of the fourth order in momenta.

For instance, when $n = 3$ and $b_i = 1$, the following Hamiltonian

$$\begin{aligned} T &= a_1 p_{y_1}^2 + a_2 p_{y_2}^2 + a_3 p_{y_3}^2 \\ &\quad - \frac{1}{2}((a_2 + a_3)y_1^2 + (a_1 + a_3)y_2^2 + (a_1 + a_2)y_3^2)(p_{y_1}^2 + p_{y_2}^2 + p_{y_3}^2) \end{aligned} \quad (2.10)$$

commutes with the polynomial of second order in momenta

$$K = \sqrt{w(y)}(p_{y_1}^2 + p_{y_2}^2 + p_{y_3}^2),$$

where

$$w(y) = ((a_2 - a_3)y_1^2 + (a_3 - a_1)y_2^2 - (a_1 - a_2)y_3^2)^2 + 4(a_3 - a_2)(a_3 - a_1)y_1^2 y_2^2.$$

After transformation (2.3), we obtain geodesic Hamiltonian (2.7)

$$\begin{aligned} \rho_b(T) &= -\frac{1}{2}((a_2 + a_3)p_{x_1}^2 + (a_1 + a_3)p_{x_2}^2 + (a_1 + a_2)p_{x_3}^2) \\ &\quad + (a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2)(p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2) \end{aligned}$$

commuting with a square root $\rho_b(K)$ (2.9) and with its square $\rho^2(K)$

$$\rho^2(K) = ((a_2 - a_3)p_{x_1}^2 + (a_3 - a_1)p_{x_2}^2 - (a_1 - a_2)p_{x_3}^2)^2 + 4(a_3 - a_2)(a_3 - a_1)p_{x_1}^2 p_{x_2}^2,$$

which is the quartic polynomial in momenta.

So, on the two-dimensional sphere, we have at least one non-trivial example of equivalent geodesic flows with quadratic and quartic polynomial invariants T , K and $\rho_b(T)$, $\rho^2(K)$, respectively. An application of the Maupertuis principle to the construction of the corresponding nonequivalent metrics (1.3) is discussed in Section 3.

In the next subsection, we rewrite canonical transformations ρ_b (2.3) and σ_{bc} (2.5) in other variables on cotangent bundle $T^*\mathbb{S}^2$ to the two-dimensional sphere \mathbb{S}^2 and study properties of these transformations. It allows us to construct other examples of equivalent metrics and understand how to construct similar ones on the $(n-1)$ -dimensional sphere.

For brevity, below we will drop ρ_b and σ_{bc} which do not affect understanding, and simply write H instead of $\rho_b(H)$ or $\sigma_{bc}(H)$.

2.1 Euler flow on two-dimensional sphere

The three-dimensional Euler top on the phase space $\mathfrak{so}(3)$ is defined by Hamiltonian

$$\mathcal{H}_e = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 \quad (2.11)$$

commuting with any component M_1 , M_2 and M_3 of the angular momentum vector

$$M = (M_1, M_2, M_3) \in \mathfrak{so}(3).$$

Many implicit and explicit maps preserve a form of this Hamiltonian, see [23] and references within.

We consider another Hamiltonian system defined by the same Hamiltonian (2.11) but on the six-dimensional phase space $T^*\mathbb{S}^2$, when vector $M = x \times p_x$ is a cross product of two vectors x and p_x so that

$$M_1 = x_3 p_{y_2} - x_2 p_{y_3}, \quad M_2 = x_1 p_{y_3} - x_3 p_{y_2}, \quad M_3 = x_2 p_1 - x_1 p_{y_2}.$$

We denote the similar cross product of the vectors y and p_y from (2.3) as $L = y \times p_y$.

By definition

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1, & x_1 M_1 + x_2 M_2 + x_3 M_3 &= 0, \\ y_1^2 + y_2^2 + y_3^2 &= 1, & y_1 L_1 + y_2 L_2 + y_3 L_3 &= 0, \end{aligned} \quad (2.12)$$

and the symplectic structure on $T^*\mathbb{S}^2$ is given by the bracket

$$\begin{aligned} \{M_i, M_j\} &= \varepsilon_{ijk} M_k, & \{M_i, x_j\} &= \varepsilon_{ijk} x_k & \{x_i, x_j\} &= 0, \\ \{L_i, L_j\} &= \varepsilon_{ijk} L_k, & \{L_i, y_j\} &= \varepsilon_{ijk} y_k, & \{y_i, y_j\} &= 0, \end{aligned} \quad (2.13)$$

where ε_{ijk} is the skew-symmetric tensor.

Let us rewrite map (2.3) in these variables on $T^*\mathbb{S}^2$.

Proposition 2.2. *When $b_1 = b_2 = b_3 = 1$, map ρ_b (2.3) on $T^*\mathbb{S}^2$ has the following form*

$$L_k = M_k, \quad \text{and} \quad y_k^2 + x_k^2 + \frac{M_k^2}{M_1^2 + M_2^2 + M_3^2} = 1, \quad k = 1, 2, 3. \quad (2.14)$$

This map preserves the angular momentum vector and the Poisson brackets (2.13).

Proof: From (2.3) and (2.12), we have

$$\begin{aligned} x_1(x_3 M_2 - x_2 M_3) + y_1(y_3 L_2 - y_2 L_3) &= 0, \\ x_2(x_1 M_3 - x_3 M_1) + y_2(y_1 L_3 - y_3 L_1) &= 0, \end{aligned}$$

$$x_3(x_2M_1 - x_1M_2) + y_3(y_2L_1 - y_1L_2) = 0,$$

and

$$\frac{(y_iL_j - y_jL_i)^2}{L_1^2 + L_2^2 + L_3^2} = 1 - y_k^2 - \frac{L_k^2}{L_1^2 + L_2^2 + L_3^2}, \quad i \neq j \neq k \neq i.$$

Solving these equations for x_i and M_i we obtain

$$\begin{aligned} x_1 &= \frac{(y_2L_3 - y_3L_2)}{\sqrt{\mathcal{H}}}, & M_1 &= \frac{x_2y_3(y_1L_2 - y_2L_1)}{x_3} - \frac{x_3y_2(y_3L_1 - y_1L_3)}{x_2}, \\ x_2 &= \frac{(y_3L_1 - y_1L_3)}{\sqrt{\mathcal{H}}}, & M_2 &= \frac{x_3y_1(y_2L_3 - y_3L_2)}{x_1} - \frac{x_1y_3(y_1L_2 - y_2L_1)}{x_3}, \\ x_3 &= \frac{(y_1L_2 - y_2L_1)}{\sqrt{\mathcal{H}}}, & M_3 &= \frac{x_1y_2(y_3L_1 - y_1L_3)}{x_2} - \frac{x_2y_1(y_2L_3 - y_3L_2)}{x_1}, \end{aligned} \quad (2.15)$$

where

$$\mathcal{H} = L_1^2 + L_2^2 + L_3^2 = M_1^2 + M_2^2 + M_3^2$$

is the square of the angular momentum vector. After that, we can directly verify that

$$L_k - M_k = 0, \quad k = 1, 2, 3,$$

when constraints (2.12) hold. Using (2.15), we can also directly check the Poisson brackets (2.13).

2.2 Magnetic flow on the sphere

Let us take geodesic Hamiltonian on the sphere (2.4)

$$\mathcal{H} = a_1p_{x_1}^2 + a_2p_{x_2}^2 + a_3p_{x_3}^2,$$

which in (x, M) coordinates reads as

$$\begin{aligned} \mathcal{H} &= a_1(x_2M_3 - x_3M_2)^2 + a_2(x_1M_3 - x_3M_1)^2 + a_3(x_2M_1 - x_1M_2)^2 \\ &= a_1M_1^2 + a_2M_2^2 + a_3M_3^2 + (a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - a_1 - a_2 - a_3)(M_1^2 + M_2^2 + M_3^2). \end{aligned}$$

After the shift of momenta

$$p_x \rightarrow p_x + \beta x, \quad \beta \in \mathbb{R},$$

we obtain the magnetic flow on the sphere defined by the Hamiltonian

$$\mathcal{H} = a_1(p_{x_1} + \beta x_1)^2 + a_2(p_{x_2} + \beta x_2)^2 + a_3(p_{x_3} + \beta x_3)^2, \quad (2.16)$$

with linear terms in momenta [2, 8, 11].

An integrable map preserving this flow is given by mapping (2.15) after the shift of momenta.

Proposition 2.3. *Let us consider mapping on $T^*\mathbb{S}^2$*

$$\rho_\beta: (y, L) \rightarrow (x, M)$$

defined as

$$x_1 = \frac{(y_2L_3 - y_3L_2 + \beta y_1)}{\sqrt{\mathcal{H}}}, \quad x_2 = \frac{(y_3L_1 - y_1L_3 + \beta y_2)}{\sqrt{\mathcal{H}}}, \quad x_3 = \frac{(y_1L_2 - y_2L_1 + \beta y_3)}{\sqrt{\mathcal{H}}},$$

and

$$\begin{aligned} M_1 &= \frac{x_2 y_3 (y_1 L_2 - y_2 L_1 + \beta y_3)}{x_3} - \frac{x_3 y_2 (y_3 L_1 - y_1 L_3 + \beta y_2)}{x_2}, \\ M_2 &= \frac{x_3 y_1 (y_2 L_3 - y_3 L_2 + \beta y_1)}{x_1} - \frac{x_1 y_3 (y_1 L_2 - y_2 L_1 + \beta y_3)}{x_3}, \\ M_3 &= \frac{x_1 y_2 (y_3 L_1 - y_1 L_3 + \beta y_2)}{x_2} - \frac{x_2 y_1 (y_2 L_3 - y_3 L_2 + \beta y_1)}{x_1}. \end{aligned} \quad (2.17)$$

This mapping preserves the form of Hamiltonian \mathcal{H} (2.16), Poisson bracket (2.13) and values of its Casimir functions (2.12).

In contrast with the previous transformation (2.14), this transformation changes the angular momentum vector so that $L_i \neq M_i$.

Thus, we rewrite map ρ_b (2.3) using the entries of the angular momentum vector (2.15) and construct its trivial generalisation ρ_β (2.17). Below we apply these maps to construct equivalent geodesic and non-equivalent potential flows on the two-dimensional sphere.

3 Main example of equivalent metrics on the sphere

Elliptic coordinate system u_1, \dots, u_{n-1} on the sphere with parameters $a_1 < \dots < a_n$ is defined through equation

$$\sum_{i=1}^n \frac{x_i^2}{\lambda - a_i} = \frac{\prod_{k=1}^{n-1} (\lambda - u_k)}{\prod_{i=1}^n (\lambda - a_i)}, \quad (3.1)$$

that implies $\sum x_i^2 = 1$. Elliptic coordinates are orthogonal and locally defined, they take values in the intervals

$$a_1 < u_1 < a_2 < u_2 < \dots < u_{n-1} < a_n.$$

The Poisson bracket between elliptic coordinates u_k and their conjugated momenta p_{u_k} is the canonical Poisson bracket

$$\{u_i, u_j\} = 0, \quad \{p_{u_i}, p_{u_j}\} = 0, \quad \{u_i, p_{u_j}\} = \delta_{ij}, \quad i, j = 1, \dots, n-1.$$

When $n = 3$ six variables x_i and M_i are expressed via four variables $u_{1,2}$ and $p_{u_{1,2}}$ in the following way

$$\begin{aligned} x_i &= \sqrt{\frac{(u_1 - a_i)(u_2 - a_i)}{(a_j - a_i)(a_k - a_i)}}, \quad i \neq j \neq k \neq i, \\ M_i &= \frac{2\varepsilon_{ijk} x_j x_k (a_j - a_k)}{u_1 - u_2} ((a_i - u_1)p_{u_1} - (a_i - u_2)p_{u_2}). \end{aligned} \quad (3.2)$$

Similar second pair of elliptic coordinates $v_{1,2}$ on the sphere together with the conjugated momenta $p_{v_{1,2}}$

$$\{v_1, v_2\} = \{v_1, p_{v_2}\} = \{v_2, p_{v_1}\} = \{p_{v_1}, p_{v_2}\} = 0, \quad \{v_1, p_{v_1}\} = \{v_2, p_{v_2}\} = 1,$$

determine the second set of variables on $T^*\mathbb{S}^2$

$$y_i = \sqrt{\frac{(v_1 - a_i)(v_2 - a_i)}{(a_j - a_i)(a_k - a_i)}}, \quad i \neq j \neq k \neq i,$$

$$L_i = \frac{2\varepsilon_{ijk}y_jy_k(a_j - a_k)}{v_1 - v_2} ((a_i - v_1)p_{v_1} - (a_i - v_2)p_{v_2}).$$

In elliptic coordinates, the square of the angular momentum is equal to

$$\mathcal{H} = \frac{4(a_1 - u_1)(a_2 - u_1)(a_3 - u_1)p_{u_1}^2}{u_1 - u_2} + \frac{4(a_1 - u_2)(a_2 - u_2)(a_3 - u_3)p_{u_1}^2}{u_2 - u_1} \quad (3.3)$$

and mapping (2.15)

$$\rho_b: (u, p_u) \rightarrow (v, p_v)$$

can be rewritten using a pair of the so-called Abel polynomials on auxiliary variable z defining intersection divisor on the hyperelliptic curve C [20, 21]:

$$\begin{aligned} \mathcal{P}(z) &= \frac{(z - v_2)\varphi(v_1)p_{v_1}}{v_1 - v_2} + \frac{(z - v_1)\varphi(v_2)p_{v_2}}{v_2 - v_1} \\ &= -\frac{(z - u_2)\varphi(u_1)p_{u_1}}{u_1 - u_2} - \frac{(z - u_1)\varphi(u_2)p_{u_2}}{u_2 - u_1} \end{aligned} \quad (3.4)$$

and

$$\psi(z) = \mathcal{H}(z - v_1)(z - v_2)(z - u_1)(z - u_2) = \varphi(z)(z\mathcal{H} + \mathcal{H}_e) - \mathcal{P}(z)^2, \quad (3.5)$$

where $\varphi(z) = -(a_1 - z)(a_2 - z)(a_3 - z)$, \mathcal{H} is the square of the angular momentum vector (3.3) and \mathcal{H}_e is the Hamiltonian of the Euler top (2.11)

$$\begin{aligned} \mathcal{H}_e &= a_1M_1^2 + a_2M_2^2 + a_3M_3^2 = \frac{4u_2\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{4u_1\varphi(u_2)p_{u_2}^2}{u_2 - u_1} \\ &= a_1L_1^2 + a_2L_2^2 + a_3L_3^2 = \frac{4v_2\varphi(v_1)p_{v_1}^2}{v_1 - v_2} + \frac{4v_1\varphi(v_2)p_{v_2}^2}{v_2 - v_1}. \end{aligned}$$

These equations (3.4) and (3.5) should be interpreted as an identity for z and each set of elliptic variables u, p_u and v, p_v .

Transformation ρ_b (2.14) is a partial case of the integrable maps associated with the non-holonomic Chaplygin and Veselova systems on the sphere [20, 21].

3.1 Integration of the original integrable flow

Let us come back to the geodesic Hamiltonian (2.10), which becomes additive separable Hamiltonian in elliptic variables

$$H = \mathcal{S}_1 + \mathcal{S}_2, \quad \mathcal{S}_k = -(a_1 - v_k)(a_2 - v_k)(a_3 - v_k)p_{v_k}^2, \quad (3.6)$$

commuting with linear integrals of motion

$$I_1 = \sqrt{(a_1 - v_1)(a_2 - v_1)(a_3 - v_1)}p_{v_1} \quad \text{and} \quad I_2 = \sqrt{(a_1 - v_2)(a_2 - v_2)(a_3 - v_2)}p_{v_2},$$

quadratic integral of motion $J = \mathcal{S}_1 - \mathcal{S}_2$ (2.8) and any other functions $f(\mathcal{S}_1, \mathcal{S}_2)$ on $\mathcal{S}_{1,2}$.

The corresponding diagonal metric

$$g(v_1, v_2) = \begin{pmatrix} \varphi(v_1) & 0 \\ 0 & \varphi(v_2) \end{pmatrix}, \quad \varphi(z) = -(a_1 - z)(a_2 - z)(a_3 - z) \quad (3.7)$$

has a non-trivial isometry group. Integrals of motion $f(\mathcal{S}_1, \mathcal{S}_2)$ are in involution with respect to the Poisson brackets associated with the canonical Poisson bivector P ,

$$P = \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial p_{v_1}} + \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial p_{v_2}} \quad (3.8)$$

and second compatible Poisson bivector

$$P' = \mathcal{S}_1 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial p_{v_1}} + \mathcal{S}_2 \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial p_{v_2}}. \quad (3.9)$$

This pair of compatible Poisson bivectors determines the bi-Hamiltonian vector field

$$X = P'dH = PdH', \quad \text{where } H' = \frac{\mathcal{S}_1^2 + \mathcal{S}_2^2}{2}.$$

The Hamilton–Jacobi equation $H = E$ (3.6) admits additive separation

$$H = E_1 + E_2, \quad \mathcal{S}_1(v_1, p_{v_1}) = E_1, \quad \mathcal{S}_2(v_2, p_{v_2}) = E_2.$$

Because

$$\frac{dv_k}{dt} = \{H, v_k\} = 2(a_1 - v_k)(a_2 - v_k)(a_3 - v_k)p_{v_k}, \quad k = 1, 2, \quad (3.10)$$

and

$$p_{v_k}^2 = -\frac{E_k}{(a_1 - v_k)(a_2 - v_k)(a_3 - v_k)}, \quad (3.11)$$

we have the following separate equations

$$\left(\frac{dv_k}{dt}\right)^2 + 4(a_1 - v_k)(a_2 - v_k)(a_3 - v_k)E_k = 0, \quad k = 1, 2. \quad (3.12)$$

Standard substitution

$$v_k = \frac{a_1 + a_2 + a_3}{3} + \frac{w}{E_k}, \quad \frac{dv_1}{dt} = \frac{1}{E_k} \frac{dw}{dt}, \quad (3.13)$$

reduces equations (3.10) to equations for the elliptic Weierstrass function

$$\left(\frac{dw}{dt}\right)^2 = 4w^3 - g_2w + g_3,$$

where

$$g_2 = \frac{4(a_1^2 - a_1a_2 - a_1a_3 + a_2^2 - a_2a_3 + a_3^2)E_k^2}{3},$$

$$g_3 = \frac{4(2a_1 - a_2 - a_3)(a_1 - 2a_2 + a_3)(a_1 + a_2 - 2a_3)E_k^3}{27}.$$

Thus, we can express variables v_1, v_2 (3.13) and p_{v_1}, p_{v_2} (3.11) via two elliptic \wp -functions on time.

Below we apply transformation ρ_b (2.14) to this simple geodesic flow (3.12).

3.2 Some properties of the equivalent metrics

Let us introduce the diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

and its spectral characteristics

$$b = a_1 + a_2 + a_3, \quad c = a_1a_2 + a_1a_3 + a_2a_3, \quad d = a_1a_2a_3. \quad (3.14)$$

It allows us to rewrite Hamiltonian H (3.6) using the angular momentum vector

$$H = \frac{1}{2}(L, \mathbf{A}L) + \frac{(x, \mathbf{A}x) - b}{4}(L, L).$$

After transformation ρ_b (2.14) this Hamiltonian has the following form

$$H = \frac{1}{4}(M, \mathbf{A}M) - \frac{1}{4}(x, \mathbf{A}x)(M, M), \quad (3.15)$$

so in elliptic coordinates (3.2) we have

$$H = g_{11}p_{u_1}^2 + g_{22}p_{u_2}^2 = \frac{(u_1 + 2u_2 - b)\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{(2u_1 + u_2 - b)\varphi(u_2)p_{u_2}^2}{u_2 - u_1}, \quad (3.16)$$

which allows us to calculate the corresponding diagonal metric on \mathbb{S}^2

$$g(u_1, u_2) = \begin{pmatrix} \frac{(u_1 + 2u_2 - b)\varphi(u_1)}{u_1 - u_2} & 0 \\ 0 & \frac{(2u_1 + u_2 - b)\varphi(u_2)}{u_2 - u_1} \end{pmatrix}. \quad (3.17)$$

For metric space (\mathbb{S}^2, g) we can define a vector space of symmetric $(m, 0)$ Killing tensors K , which are solutions of the Killing equations

$$[[g, K]] = 0, \quad (3.18)$$

where $[[\cdot, \cdot]]$ is a Schouten bracket.

When $m = 1$, solutions K are said to be infinitesimal isometries that form an isometry group. According to [7]: “everybody knows that isometry group $\text{Isom}(M, g) = \text{Id}$ for generic Riemannian or pseudo-Riemannian metrics g for $\dim M \geq 2$ ”. Our metric is no exception.

Proposition 3.1. *Metric g (3.17) on a two-dimensional sphere \mathbb{S}^2 has the trivial isometry group and trivial vector spaces of Killing tensors of valency two and three when $m = 2, 3$.*

The proof is a straightforward solution of the Killing equation (3.18) at $m = 1, 2, 3$.

Transformation ρ_b (2.14) maps integral of motion

$$K = \mathcal{S}_1 \mathcal{S}_2$$

to the following polynomial of fourth order in momenta

$$K = K_1^2 \cdot K_2 = \frac{1}{16}(M, \mathbf{A}x)^2(M, M).$$

In elliptic coordinates (3.2) these factors are equal to

$$K_1 = \frac{p_{u_1} - p_{u_2}}{u_1 - u_2}, \quad K_2 = \frac{\varphi(u_1)\varphi(u_2)(\varphi(u_1)p_{u_1}^2 - \varphi(u_2)p_{u_2}^2)}{u_1 - u_2}, \quad (3.19)$$

which allows us to determine the Killing tensor K of valency $m = 4$ on the sphere \mathbb{S}^2 , which satisfies the Killing equation (3.18).

Transformation ρ_b (2.14) preserves the form of the canonical Poisson bivector P (3.8)

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (3.20)$$

and changes the form of the second Poisson bivector P' (3.9)

$$P' = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & A_4 & A_5 \\ -A_2 & -A_4 & 0 & A_6 \\ -A_3 & -A_5 & -A_6 & 0 \end{pmatrix}, \quad (3.21)$$

where

$$\begin{aligned} A_1 &= \frac{2\varphi_1\varphi_2(p_{u_1} - p_{u_2})}{(u_1 - u_2)^2}, & A_2 &= -g_{11}p_{u_1}^2 - \frac{\varphi_1\varphi_2(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^3}, \\ A_5 &= -g_{22}p_{u_2}^2 + \frac{\varphi_1\varphi_2(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^3}, & A_3 &= \frac{\varphi_1^2(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^3} - g_{11}p_{u_2}(2p_{u_1} - p_{u_2}), \\ A_4 &= -\frac{\varphi_2^2(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^3} + g_{22}p_{u_1}(p_{u_1} - 2p_{u_2}), \\ A_6 &= -\frac{(p_{u_1} - p_{u_2})p_{u_1}p_{u_2}}{(u_1 - u_2)^2} \left(\frac{\partial}{\partial u_1} \frac{\varphi_1\varphi_2}{(u_1 - u_2)^3} - \frac{\partial}{\partial u_2} \left(g_{22} - \frac{\varphi_2^2}{(u_1 - u_2)^3} \right) \right), \end{aligned}$$

and $\varphi_k = \varphi(u_k)$ for brevity. The structure of this Poisson bivector P' is completely different from the structure of the so-called natural Poisson bivectors on the sphere [15, 26].

Both bivectors P (3.8) and P' (3.21) are invertible, which allows us to introduce a hereditary or recursion operator defined as

$$N = P'P^{-1}.$$

The spectral curve of N has the form

$$\det(N - \lambda \text{Id}) = (\lambda^2 + H\lambda - K)^2,$$

where H and K are given by (3.16)–(3.19). Thus, the following equation holds

$$P'dH = PdJ, \quad J = \frac{H^2}{2} - K = \frac{\mathcal{S}_1^2 + \mathcal{S}_2^2}{2},$$

where P and P' are given by (3.20) and (3.21) and

$$dH = \begin{pmatrix} \frac{\partial H}{\partial u_1} \\ \frac{\partial H}{\partial u_2} \\ \frac{\partial H}{\partial p_{u_1}} \\ \frac{\partial H}{\partial p_{u_2}} \end{pmatrix} \quad \text{and} \quad dH' = \begin{pmatrix} \frac{\partial H'}{\partial u_1} \\ \frac{\partial H'}{\partial u_2} \\ \frac{\partial H'}{\partial p_{u_1}} \\ \frac{\partial H'}{\partial p_{u_2}} \end{pmatrix}.$$

So, we have a geodesic flow on the bi-Hamiltonian manifold $(T^*\mathbb{S}^2, P, P')$ defined by the coefficients of the Casimir functions of the Poisson pencil $P_\lambda = P + \lambda P'$.

Proposition 3.2. *On the cotangent bundle $T^*\mathbb{S}^2$ Hamiltonian H (3.16)*

$$H = \frac{1}{4}(M, \mathbf{A}M) - \frac{1}{4}(x, \mathbf{A}x)(M, M) = \frac{(u_1 + 2u_2 - b)\varphi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{(2u_1 + u_2 - b)\varphi(u_2)p_{u_2}^2}{u_2 - u_1}$$

yields bi-Hamiltonian vector field

$$X = PdH = (N^{-1}P)dJ, \quad N = P'P^{-1},$$

where P and P' are given by (3.20) and (3.21).

The proof is a straightforward calculation.

3.3 Potential motion

Let us discuss potentials $V(x)$ which can be added to the geodesic Hamiltonian H (3.15). For instance, starting with the following separable Hamiltonian

$$H_1 = \mathcal{S}_1 + \mathcal{S}_2 + \beta \frac{v_1 \mathcal{S}_1 - v_2 \mathcal{S}_2}{\mathcal{S}_1 - \mathcal{S}_2},$$

so that

$$H_1 = E \Rightarrow \mathcal{S}_1^2 + \beta v_1 \mathcal{S}_1 - E \mathcal{S}_1 = \mathcal{S}_2^2 + \beta v_2 \mathcal{S}_2 - E \mathcal{S}_2,$$

we obtain Hamiltonian

$$H_1 = (M, \mathbf{A}M) - (x, \mathbf{A}x)(M, M) + V(x), \quad V(x) = \beta(\mathbf{A}x, x) = \beta(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2),$$

commuting with the second integral of motion

$$H_2 = (M, \mathbf{A}x)^2((M, M) - \beta).$$

Other potentials may be obtained using the following substitution

$$\begin{aligned} H_1 &= (M, \mathbf{A}M) - (x, \mathbf{A}x)(M, M) + V(x), \\ H_2 &= ((M, \mathbf{A}x)^2 + (M, \mathbf{A}x)U_1(x) + U_2(x))((M, M) + U_3(x)). \end{aligned} \quad (3.22)$$

The equation $\{H_1, H_2\} = 0$ has several solutions from which we single out the following polynomial ‘‘cubic’’ potential

$$V(x) = \alpha \left((x, \mathbf{A}x) - \frac{b}{3} \right)^3, \quad \alpha \in \mathbb{R},$$

where $b = a_1 + a_2 + a_3$, and

$$\begin{aligned} U_1(x) &= 0, \quad U_2(x) = 0, \quad U_3(x) = -\alpha W(x), \\ W(x) &= (a_1 - a_3)(a_1 - a_2)x_1^4 + (a_2 - a_1)(a_2 - a_3)x_2^4 + (a_3 - a_1)(a_3 - a_2)x_3^4 \\ &\quad - (a_1^2 + 2a_2a_3)x_1^2 - (a_2^2 + 2a_1a_3)x_2^2 - (a_3^2 + 2a_1a_2)x_3^2 + \frac{b^2}{3}. \end{aligned}$$

In this case polynomial H_1 (3.22) becomes a rational function in v -variables

$$H_1 = (\mathcal{S}_1 + \mathcal{S}_2) + \alpha \left(\frac{(3v_1 - b)\mathcal{S}_1 - (3v_2 - b)\mathcal{S}_2}{\mathcal{S}_1 - \mathcal{S}_2} \right)^3$$

and we know nothing about the separation of variables and the bi-Hamiltonian structure of the corresponding vector field when $\alpha \neq 0$.

Following [6, 13, 14], we can use the Maupertuis principle to construct a new metric on the sphere associated with the Hamiltonian

$$\tilde{H} = \frac{g_{11}p_{u_1}^2 + g_{22}p_{u_2}^2}{E - V(x)}.$$

The corresponding additional integral of motion is the polynomial of the fourth order in momenta.

4 One family of equivalent metrics

As an example, we take another separable Hamiltonian

$$H = \hat{\lambda}_1 + \hat{\lambda}_2, \quad \hat{\lambda}_k = v_k \mathcal{S}_k = -v_k(a_1 - v_k)(a_2 - v_k)(a_3 - v_k)p_{v_k}^2,$$

which in (y, L) variables reads as

$$H = a_1(y_2L_3 - y_3L_2)^2 + a_2(y_1L_3 - y_3L_1)^2 + a_3(y_2L_1 - y_1L_2)^2,$$

up to the factor $1/4$. In Cartesian coordinates, it has the form

$$\hat{H} = a_1p_{y_1}^2 + a_2p_{y_2}^2 + a_3p_{y_3}^2.$$

Below we apply transformations ρ_b (2.3) with different values of b_i to this Hamiltonian and present equivalent metrics on the sphere related to canonical transformation σ (2.5).

Case 1. Using transformation ρ_b (2.3) with

$$b_1 = b_2 = b_3 = 1,$$

we obtain the following integrals of motion

$$\begin{aligned} H^{(1)} &= \left(\frac{2\varphi(u_1)}{(u_1 - u_2)^2} - u_1 + \frac{(b^2 - bu_2 - 2c)}{u_1 - u_2} \right) \varphi_1 p_{u_1}^2 - \frac{4\varphi_1 \varphi_2 p_{u_1} p_{u_2}}{(u_1 - u_2)^2} \\ &\quad + \left(\frac{2\varphi(u_2)}{(u_1 - u_2)^2} - u_2 + \frac{b^2 - bu_1 - 2c}{u_2 - u_1} \right) \varphi_2 p_{u_2}^2, \\ K^{(1)} &= \varphi(u_1)\varphi(u_2)K_1^2 K_2, \quad \text{where } K_1 = \frac{p_1 - p_{y_2}}{u_1 - u_2}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} K_2 &= \left(\frac{\varphi(u_2)}{(u_1 - u_2)^2} - u_2 - \frac{u_2(b - 2u_1)}{u_1 - u_2} \right) \varphi(u_1)p_{u_1}^2 - \frac{2\varphi(u_1)\varphi(u_2)p_{u_1}p_{u_2}}{(u_1 - u_2)^2} \\ &\quad + \left(\frac{\varphi(u_1)}{(u_1 - u_2)^2} - u_1 - \frac{u_1(b - 2u_2)}{u_2 - u_1} \right) \varphi(u_2)p_{u_2}^2, \end{aligned}$$

and $\varphi(z)$ is given by (3.7).

Case 2. Using transformation ρ_b (2.3) with

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3,$$

we obtain the following integrals of motion

$$\begin{aligned} H^{(2)} &= \frac{(d - cu_1 + (u_1 - 2u_2 + b)u_1u_2)\varphi(u_1)p_{u_1}^2}{(u_1 - u_2)^2} + \frac{2\varphi(u_1)\varphi(u_2)p_{u_1}p_{u_2}}{(u_1 - u_2)^2} \\ &\quad + \frac{(d - cu_2 + (u_2 - 2u_1 + b)u_1u_2)\varphi(u_2)p_{u_2}^2}{(u_1 - u_2)^2}, \\ K^{(2)} &= K_1^2 K_2 = \left(\frac{u_1 p_{u_1} - u_2 p_{u_2}}{u_1 - u_2} \right)^2 \cdot \frac{\varphi(u_1)\varphi(u_2)(\varphi(u_1)p_{u_1}^2 - \varphi(u_2)p_{u_2}^2)}{u_1 u_2 (u_1 - u_2)}. \end{aligned} \quad (4.2)$$

Case 3. Using transformation ρ_b (2.3) with

$$b_1 = a_1^2, \quad b_2 = a_2^2, \quad b_3 = a_3^2,$$

we obtain the following integrals of motion

$$\begin{aligned} H^{(3)} &= \frac{u_1(2u_1 + u_2 - \tilde{b}u_1u_2)\varphi(u_1)p_1^2}{u_1 - u_2} + \frac{u_2(u_1 + 2u_2 - \tilde{b}u_1u_2)\varphi(u_2)p_2^2}{u_2 - u_1}, \\ K^{(3)} &= K_1^2 K_2 = \left(\frac{u_1^2 p_1 - u_2^2 p_2}{u_1 - u_2} \right)^2 \cdot \frac{\varphi(u_1)\varphi(u_2)(u_1\varphi(u_1)p_1^2 - u_2\varphi(u_2)p_2^2)}{u_1 - u_2}, \end{aligned} \quad (4.3)$$

where $\tilde{b} = a_1^{-1} + a_2^{-1} + a_3^{-1}$.

Proposition 4.1. *Geodesic Hamiltonians $H^{(1)}$ (4.1), $H^{(2)}$ (4.2) and $H^{(3)}$ (4.3) are related to each other by canonical transformations σ (2.5) with the suitable set of parameters.*

Thus, we have three equivalent metrics on the two-dimensional sphere.

4.1 Potential motion

We can try to destroy this equivalence of the geodesic flows by adding potentials to the geodesic Hamiltonian, for instance, changing Hamiltonians in the following way

$$H_1 = H + V \quad \text{and} \quad H_2 = K_1^2(K_2 + U).$$

Let us present some potentials explicitly:

$$\begin{aligned} V^{(1)} &= \frac{1}{b - u_1 - u_2} \left(\alpha \left(b - \frac{c}{b - u_1 - u_2} - \frac{d}{(b - u_1 - u_2)^2} \right) + \beta \right), \\ U^{(1)} &= \frac{\varphi(u_1)\varphi(u_2)}{b - u_1 - u_2} \left(\alpha \frac{u_2\varphi(u_1) - u_1\varphi(u_2)}{(b - u_1 - u_2)^2(u_1 - u_2)} + \beta \right), \end{aligned}$$

and

$$\begin{aligned} V^{(2)} &= u_1 u_2 (\alpha (u_1^2 u_2^2 - cu_1 u_2 + bd) + \beta), \\ U^{(2)} &= \varphi(u_1)\varphi(u_2) (\alpha (u_1^2 u_2^2 - cu_1 u_2 + d(u_1 + u_2)) + \beta). \end{aligned}$$

In the third case, we have

$$V^{(3)} = \alpha \frac{(u_1 + u_2)(u_1 + u_2 + \tilde{b}u_1u_2)^2}{u_1^3 u_2^3} + \beta \frac{u_1 + u_2}{u_1 u_2},$$

$$U^{(3)} = \frac{\varphi(u_1)\varphi(u_2)}{4u_1u_2(u_1 - u_2)^2} \left(\alpha \frac{d(u_1^2 + u_1u_2 + u_2^2) - cu_1u_2(u_1 + u_2)}{u_1^2u_2^2} + \beta \right).$$

Using relations

$$u_1 + u_2 = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - b, \quad u_1u_2 = a_2a_3x_1^2 + a_1a_3x_2^2 + a_1a_2x_3^2,$$

we can rewrite these potentials in terms of Cartesian coordinates.

As above, the Maupertuis principle allows the construction of a new metric on the sphere associated with the Hamiltonian

$$\tilde{H} = \frac{g_{11}p_{u_1}^2 + g_{22}p_{u_2}^2}{E - V},$$

where g and V are metrics and potentials associated with $H^{(1)}$ (4.1), $H^{(2)}$ (4.2) and $H^{(3)}$ (4.3).

4.2 Hamiltonians with linear in momenta terms

Let us come back to the separable Hamiltonian (3.6)

$$H = \mathcal{S}_1 + \mathcal{S}_2 = \varphi(v_1)p_{v_1}^2 + \varphi(v_2)p_{v_2}^2.$$

Using transformation ρ_b (2.3) with

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3,$$

we obtain

$$H = \left(2a_1 - \frac{a_2 + a_3}{a_2a_3}w \right) (x_2M_3 - x_3M_2)^2 + \left(2a_2 - \frac{a_1 + a_3}{a_1a_3}w \right) (x_1M_3 - x_3M_1)^2 \\ + \left(2a_3 - \frac{a_1 + a_2}{a_1a_2}w \right) (x_2M_1 - x_1M_2)^2,$$

where

$$w = u_1u_2 = a_2a_3x_1^2 + a_1a_3x_2^2 + a_1a_2x_3^2.$$

In elliptic coordinates, this Hamiltonian has the following form

$$H = g_{11}p_{u_1}^2 + 2g_{12}p_{u_1}p_{u_2} + g_{22}p_{u_2}^2, \tag{4.4}$$

where the metric is

$$g = \begin{pmatrix} \frac{d(2u_1 - u_2) - u_1^2u_2^2 + bu_1u_2^2 - cu_1u_2}{(u_1 - u_2)^2} & \frac{u_1u_2\varphi(u_1)\varphi(u_2)}{(u_1 - u_2)^2} \\ \frac{u_1u_2\varphi(u_1)\varphi(u_2)}{(u_1 - u_2)^2} & \frac{d(u_1 - 2u_2) - u_1^2u_2^2 + bu_1^2u_2 - cu_1u_2}{(u_1 - u_2)^2} \end{pmatrix},$$

and b , c and d are combinations of a_1 , a_2 and a_3 (3.14). The corresponding quartic invariant is a product of two polynomials in momenta

$$K = \left(\frac{u_1p_{u_1} - u_2p_{u_2}}{u_1 - u_2} \right)^2 \frac{\varphi(u_1)\varphi(u_2)(u_1\varphi(u_1)p_{u_1}^2 - u_2\varphi(u_2)p_{u_2}^2)}{u_1 - u_2}. \tag{4.5}$$

These integrals of motion H and K coincide with integrals H (3.16) and K (3.19) after canonical transformation σ_{bc} (2.5) depending on parameters $b_i = 1$ and $c_k = a_k$.

The main difference is that canonical transformation $p_k \rightarrow p_k + \beta_k$ acts trivially when $b_i = 1$

$$\mathcal{H} = \sum (p_k + \beta x_k)^2 = \sum p_k^2 + 2\beta \sum x_k p_k + \beta^2 \sum x_k^2 = \sum p_k^2 + \beta^2$$

according to constraints (2.1). When $b_i = a_i$, this transformation adds nontrivial term to the Hamiltonian \mathcal{H} (2.4), which is linear polynomial in momenta

$$\mathcal{H} = \sum a_k (p_k + \beta x_k)^2 = \sum a_k p_k^2 + 2\beta \sum a_k x_k p_k + \beta^2 \sum a_k x_k^2.$$

As a result, applying a transformation (2.17) to the Hamiltonian

$$\tilde{H} = \tilde{\mathcal{S}}_1 + \tilde{\mathcal{S}}_2, \quad \tilde{\mathcal{S}}_k = \mathcal{S}_k + \beta^2 v_k = \varphi(v_k) p_{v_k}^2 + \beta^2 v_k,$$

we obtain Hamiltonian on the $T^*\mathbb{S}^2$

$$\tilde{H} = H - \frac{2\beta u_1 u_2 (u_2 \varphi(u_1) p_{u_1} - u_1 \varphi(u_2) p_{u_2})}{u_1 - u_2} - \beta^2 \left(\frac{u_2^2 \varphi(u_1) - u_1^2 \varphi(u_2)}{u_1 - u_2} + d(u_1 + u_2) \right)$$

involving linear terms in velocity. Here H is given by (4.4) and the corresponding second integral of motion is equal to

$$\tilde{K} = K + \beta^4 B_4 + \beta_3 B_3 + \beta_2 B_2 + \beta B_1,$$

where K is given by (4.5) and

$$\begin{aligned} B_4 &= -b u_1^2 u_2^2 - d(u_1^2 + 2u_1 u_2 + u_2^2) + c u_1 u_2 (u_1 + u_2), \\ B_3 &= -\frac{2u_1 u_2 (u_2^2 \varphi(u_1) p_{u_1} - u_1^2 \varphi(u_2) p_{u_2})}{u_1 - u_2}, \\ B_1 &= -\frac{2u_1 u_2 \varphi(u_1) \varphi(u_2) p_{u_1} p_{u_2} (u_1 p_{u_1} - u_2 p_{u_2})}{u_1 - u_2}, \\ B_2 &= \frac{(d(2u_1^2 + u_1 u_2 - u_2^2) - 2c u_2 u_1^2 + 2b u_1^2 u_2^2 + u_1 u_2^3 (u_2 - 3u_1)) u_1 \varphi(u_1) p_{u_1}^2}{(u_1 - u_2)^2} \\ &\quad + \frac{2u_1 u_2 (u_1 + u_2) \varphi(u_1) \varphi(u_2) p_{u_1} p_{u_2}}{(u_1 - u_2)^2} \\ &\quad + \frac{(d(2u_2^2 + u_1 u_2 - u_1^2) - 2c u_1 u_2^2 + 2b u_1^2 u_2^2 + u_2 u_1^3 (u_1 - 3u_2)) u_2 \varphi(u_2) p_{u_2}^2}{(u_1 - u_2)^2}. \end{aligned}$$

According [2, 8, 11] this Hamiltonian defines magnetic flow on the sphere.

5 Conclusion

We discuss a relatively simple map ρ_b (2.3) preserving the form of Hamiltonian

$$\mathcal{H} = b_1 p_{x_1}^2 + \dots + b_n p_{x_n}^2, \quad b_i \in \mathbb{R},$$

and the Dirac–Poisson bracket (2.2) on cotangent bundle $T^*\mathbb{S}^{(n-1)}$ to the sphere.

Applying this map to the following Hamiltonian, which in terms of elliptic coordinates (3.1) has the form

$$T = \mathcal{S}_1 + \dots + \mathcal{S}_{n-1}, \quad \mathcal{S}_k = u_k^m \prod_{i=1}^n (u_k - a_i) p_{u_k}^2, \quad m = 0, 1, \dots,$$

we obtain polynomials of the second order in momenta at $b_i = a_i^\ell$, $\ell = 0, 1, \dots$,

$$\rho_b(T) = \sum_{i,j=1}^n g_b^{ij}(x) p_{x_i} p_{x_j}.$$

Because these polynomials commute with n independent, non-polynomial functions $\rho_b(\mathcal{S}_k)$, they determine a set of equivalent metrics $g_b(x)$ on the sphere. By adding various potentials V_b to these equivalent geodesic Hamiltonians $\rho_b(T)$ we can construct different integrable flows and different metrics (1.3) on the sphere.

The main problem is how to get a set of functions on $\rho_b(\mathcal{S}_k)$ which are polynomials in momenta. In this note, we study two-dimensional sphere when $n = 3$ and prove that second-order polynomial $\rho_b(\mathcal{S}_1 + \mathcal{S}_2)$ commutes with a polynomial of fourth order in momenta $\rho_b(\mathcal{S}_1 \mathcal{S}_2)$. In further publications, we will present a similar result for equivalent metrics on the three-dimensional 3D sphere.

Another interesting problem is to consider canonical transformations preserving Hamiltonian of the form

$$\mathcal{H} = b_1 p_{x_1}^2 + \dots + b_n p_{x_n}^2 + V(x), \quad b_i \in \mathbb{R},$$

which were obtained for different partial cases in [17, 18, 19, 20, 21].

A The Maupertuis principle

In modern invariant, coordinate-free Hamiltonian mechanics [1, 27], an integrable system is defined as a Lagrangian submanifold in which n parameters are considered as functions on $2n$ -dimensional symplectic manifold. In a generic case, the Lagrangian submanifold depends on $m > n$ parameters and gives rise to a family of C_m^n integrable systems with common trajectories.

In traditional Hamiltonian mechanics, there are several coordinate-dependent descriptions of the integrable system with common trajectories [4], and the Maupertuis principle is the oldest of them. Roughly speaking, the Maupertuis or Jacobi–Maupertuis principle says that trajectories of the natural Hamiltonian systems are geodesics for the suitable metrics on configuration space, see [5, 6, 13, 14] and references within.

Below we present known technical construction of the geodesic Hamiltonians in a suitable to our purpose form. Let us take the Hamilton function in the so-called natural form

$$H = T + V(q), \quad T = \sum_{i,j} g_{ij}(q) p_i p_j,$$

where potential $V(q)$ is a function on coordinates q and c . Suppose that H commutes with a sum of the homogeneous polynomials of m -order in momenta

$$K = \sum_{m=0}^N K_m,$$

where N is an arbitrary integer number, all terms in the polynomial K have the same parity.

From $\{H, K\} = 0$ follows that geodesic Hamiltonian

$$\tilde{T} = \sum_{i,j} \tilde{g}_{ij}(q) p_i p_j = \frac{T}{h - V}, \quad \tilde{g}(q) = \frac{g(q)}{h - V},$$

where h is a constant, commutes $\{\tilde{T}, \tilde{K}\} = 0$ with a sum of the homogeneous polynomials of m -order in momenta

$$\tilde{K} = K_m + \tilde{T} K_{m-2} + \tilde{T}^2 K_{m-4} + \dots$$

Indeed, we can rewrite equation $\{H, K\} = 0$ as a set of equations

$$\{T, K_j\} + \{V, K_{j+2}\} = 0, \quad j = m, m-2, \dots, \quad K_{m+2} = K_{-1} = K_{-2} = 0$$

by using Euler's homogeneous function theorem. Substituting these equations into

$$\begin{aligned} \{\tilde{T}, \tilde{K}\} &= \{\tilde{T}, K_m\} + \tilde{T}\{\tilde{T}, K_{m-2}\} + \tilde{T}^2\{\tilde{T}, K_{m-4}\} + \dots \\ &= \frac{\{T, K_m\}}{h-V} + \left(\frac{T}{(h-V)^2} \{V, K_m\} + \frac{\tilde{T}}{h-V} \{T, K_{m-2}\} \right) + \dots \\ &= 0 + \frac{T}{(h-V)^2} (\{V, K_m\} + \{T, K_{m-2}\}) + \dots = 0, \end{aligned}$$

and grouping terms of the same order in momenta we directly verify that \tilde{T} commutes with \tilde{K} .

Acknowledgements

We are very grateful to the referees for thorough analysis of the manuscript, constructive suggestions and proposed corrections, which certainly lead to a more profound discussion of the results. The work was supported by the Russian Science Foundation (project 21-11-00141).

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