Drew Armstrong 2024 arxiv: 2403.06318

"Lattice points and rational q-Catalan numbers"

[inspired by Paul Johnson 2015 arxiv: 1502.07934

"Lattice points and simultaneous core partitions"]

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}$$

standard

Problem: Given $\gcd(a,b) = 1$, we have

$$\mathrm{Cat}(a,b)_q = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \in \mathbb{N}[q]$$

Theorem

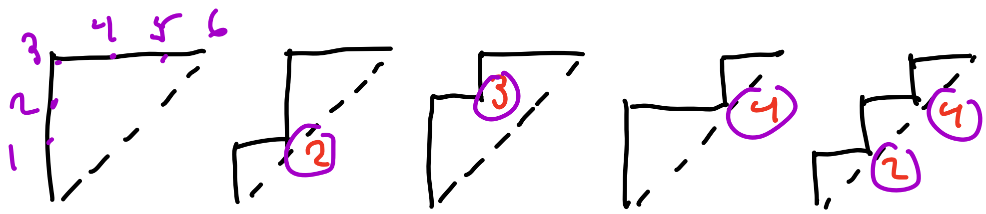
But the coefficients are not well-understood!

Usual approach is via Dyck paths.

For $(a,b) = (n, n+1)$ then

$$\mathrm{Cat}(n,n+1)_q = \mathrm{Cat}(n)_q = \sum_{P \in D(n)} q^{\mathrm{maj}(P)}$$

e.g. $\mathrm{Cat}(3)_q = \frac{1}{[7]_q} \begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6$



$$1 + q^2 + q^3 + q^4 + q^6$$

For $\gcd(a,b) = 1$, $\mathrm{Cat}(a,b) = \# D(a,b)$

where $D(a,b)$ = Dyck paths in $a \times b$ rectangle

but NO GERALIZATION OF MAJOR INDEX

IS KNOWN:

$$\mathrm{Cat}(a,b)_q = \sum_{P \in D(a,b)} q^{?}$$

Everything we know comes from more difficult rational q,t-Catalan:

$$\mathrm{Cat}(a,b)_{q,t} = \sum_{P \in D(a,b)} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}$$

$t = \frac{1}{q}$

$$\mathrm{Cat}(a,b)_q = q^{(a-1)(b-1)/2} \cdot \mathrm{Cat}(a,b)_{q,\frac{1}{q}}$$

$$= \sum_{P \in D(a,b)} q^{\mathrm{area}(P) - \mathrm{dinv}(P) + (a-1)(b-1)/2}$$

not so nice

not so easy

Another issue: For $b_1 < b_2$ and $\gcd(a,b_1) = \gcd(a,b_2) = 1$ we have

$$\mathrm{Cat}(a,b_2)_q - \mathrm{Cat}(a,b_1)_q \in \mathbb{N}[q]$$

conjecture ?

But this does not follow from properties of q,t-Catalan.

Lattice Point Approach is based on the non-symmetric expression

$$\mathrm{Cat}(a,b)_q = \boxed{\frac{1}{[a]_q} \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q}$$

Fix a. Consider the weight & root lattice of type "A_{a-1}"

$\mathcal{L} = \mathbb{Z}^{a-1}$ with basis of "fundamental weights" $\omega_1 = (1,0,\dots,0)$

$\vdots$

$\omega_{a-1} = (0,\dots,0,1)$

$R \leq \mathcal{L}$ sublattice generated by the simple roots, i.e., the columns of the Cartan matrix

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$\alpha_1 = 2\omega_1 - \omega_2$

$= (2,-1,0,\dots,0)$

$\vdots$

$\alpha_{a-1} = -\omega_{a-2} + 2\omega_{a-1}$

$= (0,\dots,0,-1,2)$

Easier to work with:

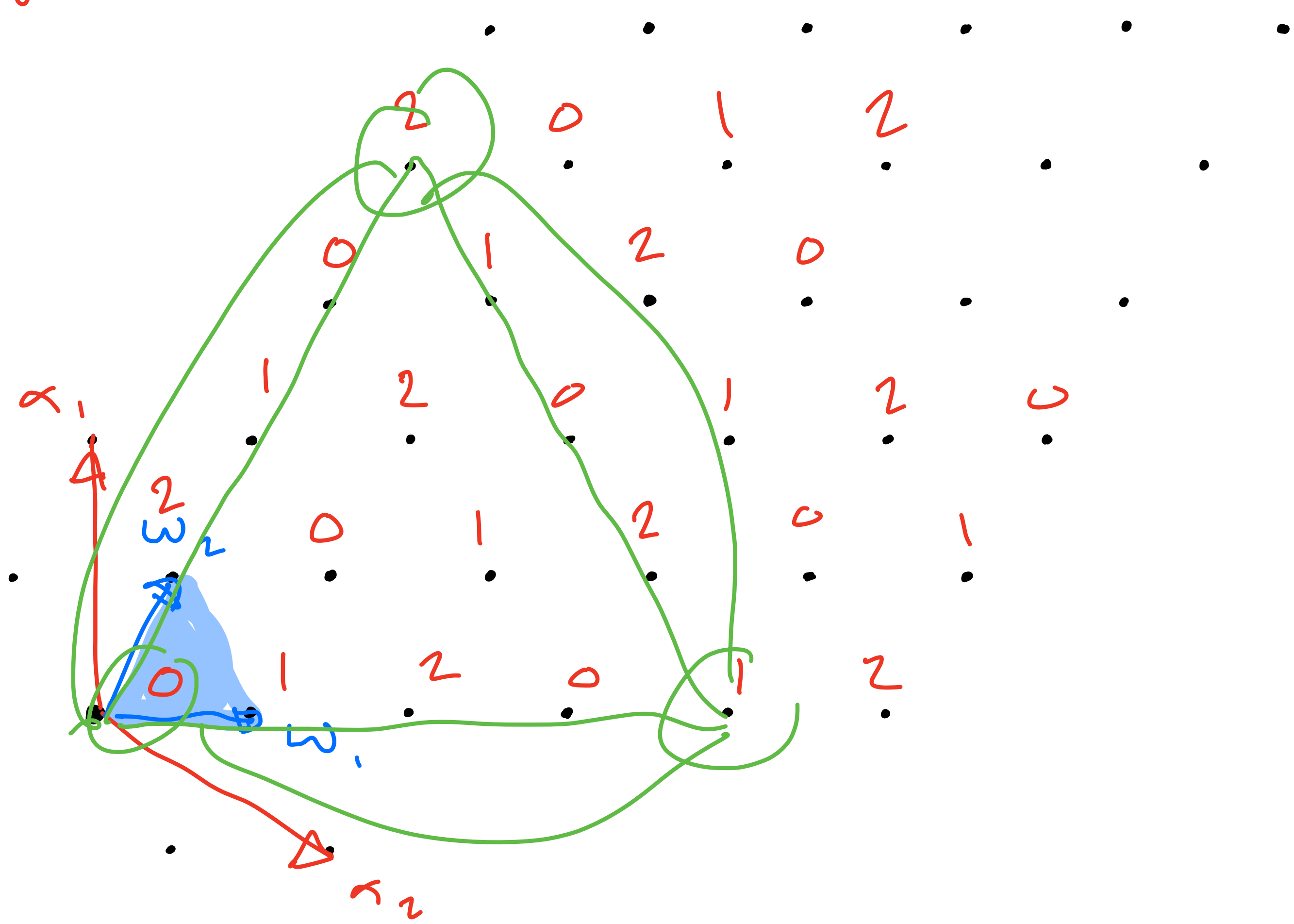
$$R = \{ (x_1, \dots, x_{a-1}) \in \mathcal{L} : x_1 + 2x_2 + \dots + (a-1) x_{a-1} \equiv 0 \bmod a \}$$

More generally, let

$$R_k = \{ \vec{x} \in \mathcal{L} : x_1 + 2x_2 + \dots + (a-1)x_{a-1} \equiv k \bmod a \}$$

Then $R_0, R_1, \dots, R_{a-1}$ are the cosets of R in $\mathcal{L}$, and $\mathcal{L}/R \cong \mathbb{Z}/a\mathbb{Z}$.

e.g. $a = 3$



Fundamental Alcove

Δ = Convex hull of
$\{0, \omega_1, \omega_2, \ldots, \omega_{a-1}\}$

For all $b \in \mathbb{N}$, $\#(b\Delta \cap \mathcal{L}) = \binom{a-1+b}{a-1}$.

If $\gcd(a,b) = 1$ then an easy action of $\mathbb{Z}/a\mathbb{Z}$ on $b\Delta \cap \mathcal{L}$ shows that

$$\#(b\Delta \cap R) = \frac{1}{a}\#(b\Delta \cap \mathcal{L}) = \frac{1}{a}\binom{a-1+b}{a-1}.$$

In particular, $a \,\Big|\, \binom{a-1+b}{a-1}$.

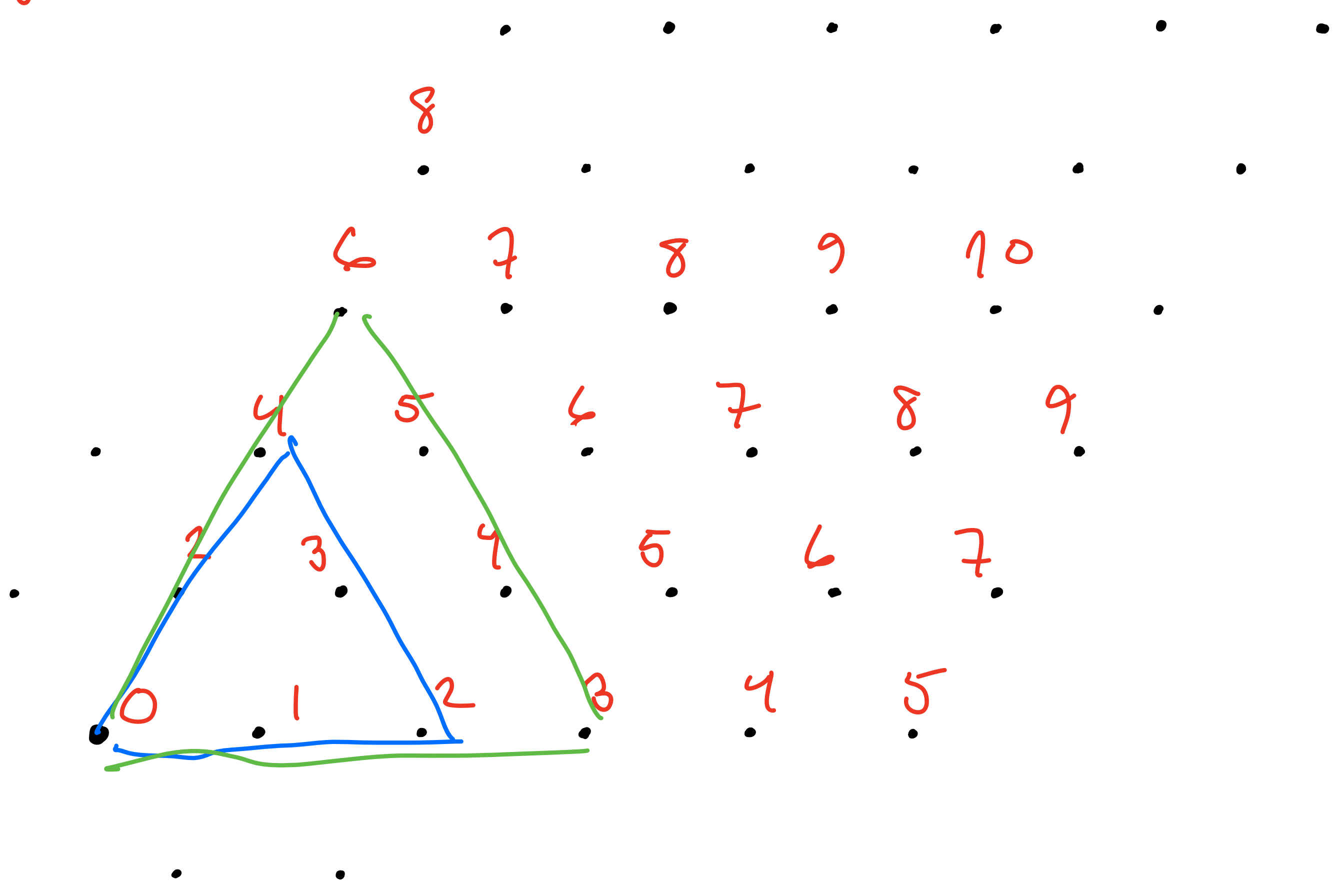
Now bring in q: We want to explain how and why $[a]_q \,\Big|\, \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q$

Define "tilted height"

$$T(x_1, \ldots, x_{a-1}) = x_1 + 2x_2 + \cdots + (a-1)x_{a-1}$$

Then $\begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q = \sum_{x \in b\Delta \cap \mathcal{L}} q^{T(\vec{x})}$

e.g. $a = 3$



$\begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$

$\begin{bmatrix} 2+3 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$

⋮

etc.

Proof : Use the "tilted basis"

$$\{w_1, w_2, \dots, w_{a-1}\} \longrightarrow \{w_1, w_2 - w_1, \dots, w_{a-1} - w_{a-2}\}$$

$$(x_1, \dots, x_{a-1}) \longrightarrow (y_1, \dots, y_{a-1})$$

$$y_1 = x_1 + \dots + x_{a-1}$$
$$y_2 = x_2 + \dots + x_{a-1}$$
$$\vdots$$
$$y_{a-1} = x_{a-1}$$

$\vec{x} \in b\Delta \cap \mathbb{Z} \longrightarrow$ Partitions in $(a-1) \times b$ rectangle :

$$b \geq y_1 \geq y_2 \geq \dots \geq y_{a-1} \geq 0$$

$$T(\vec{x}) = x_1 + 2x_2 + \dots + (a-1)x_{a-1}$$
$$= y_1 + y_2 + \dots + y_{a-1} = |\vec{y}|$$

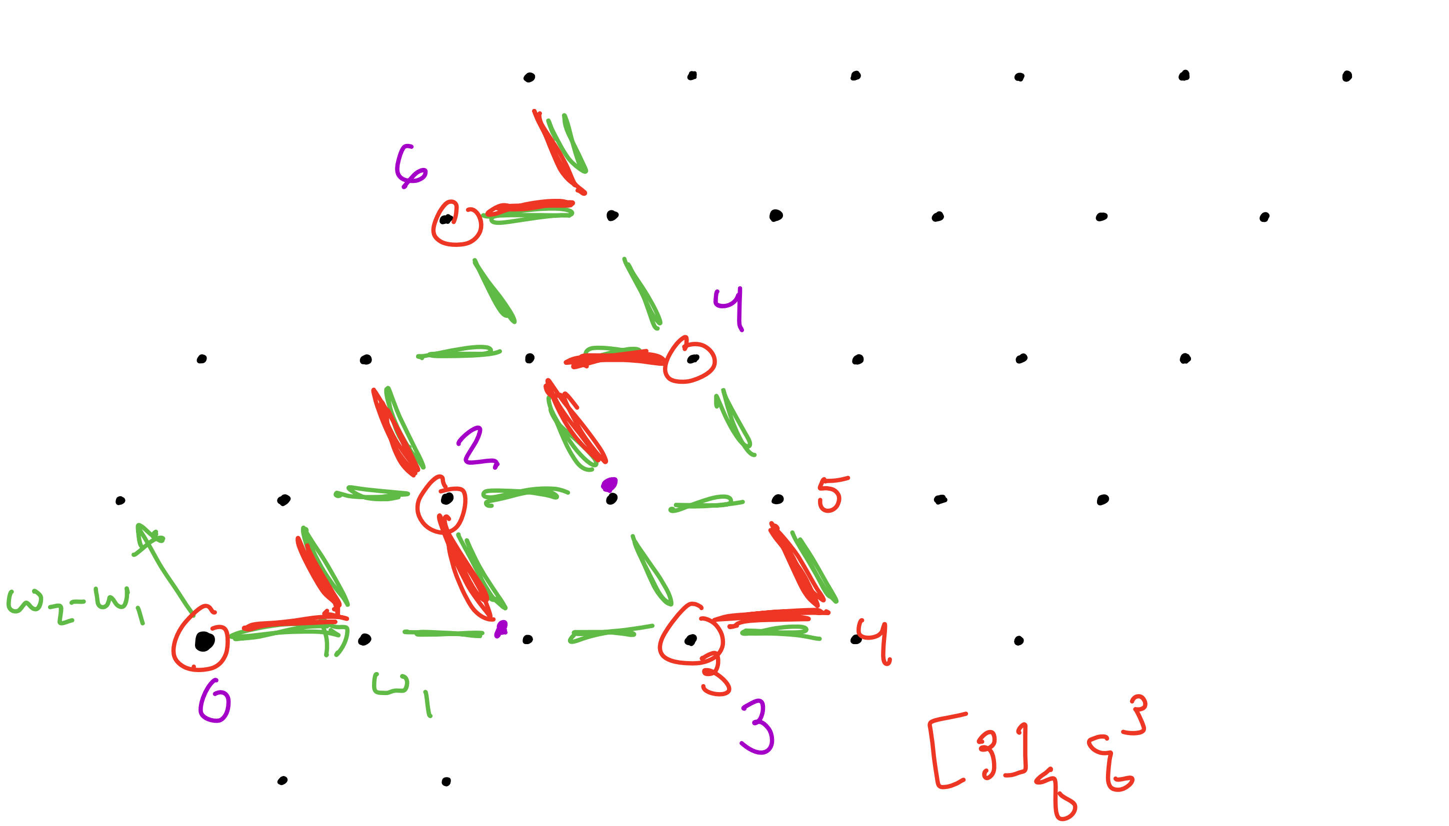
Let $Y(a,b)$ = set of Young diagrams then

$$\sum_{\vec{x} \in b\Delta \cap \mathbb{Z}} q^{T(\vec{x})} = \sum_{\vec{y} \in Y(a,b)} q^{|\vec{y}|} = \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q$$

well known

In fact, we can think of $\mathcal{L}$ and $b\Delta \cap \mathcal{L} = Y(a,b)$ as a poset of Young diagrams under inclusion:

e.g. $a=3$



If we can find a decomposition of this poset into "ribbons" (saturated chains of length a) then we will prove that

$$[a]_q \;\Big|\; \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q$$

Furthermore, if for any $\vec{x} \in b\Delta \cap R$ we let

$J(\vec{x})$ = tilted height of the lowest point in the ribbon containing $\vec{x}$

then we will have

$$\mathrm{Cat}(a,b)_q = \frac{1}{[a]_q} \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q$$

$$= \sum_{\vec{x} \in b\Delta \cap R} q^{J(\vec{x})}.$$

Why the letter "J" ?

Let $\mathrm{Box} = \{ \vec{x} \in \mathcal{L} : 0 \le x_i \le a-1 \}$

$$\# \mathrm{Box} = a^{a-1}$$

$$\sum_{\vec{x} \in \mathrm{Box}} q^{T(x)} = [a]_q [a]_{q^2} \cdots [a]_{q^{a-1}}$$

we observe that

$$\#(\mathrm{Box} \cap R) = a^{a-2}.$$

A "Johnson statistic" is a function

$$J : R \to \mathbb{Z}$$

with the following properties

① $$\sum_{\vec{x} \in R \cap \mathrm{Box}} q^{J(\vec{x})} = [a]_{q^2} [a]_{q^3} \cdots [a]_{q^{a-1}}$$

② $$J(\vec{x} + a\vec{y}) = J(\vec{x}) + a\,T(\vec{x})$$

③ $$\sum_{\vec{x} \in b\Delta \cap R} q^{J(\vec{x})} = \mathrm{Cat}(a,b)_q$$

for any b coprime to a.

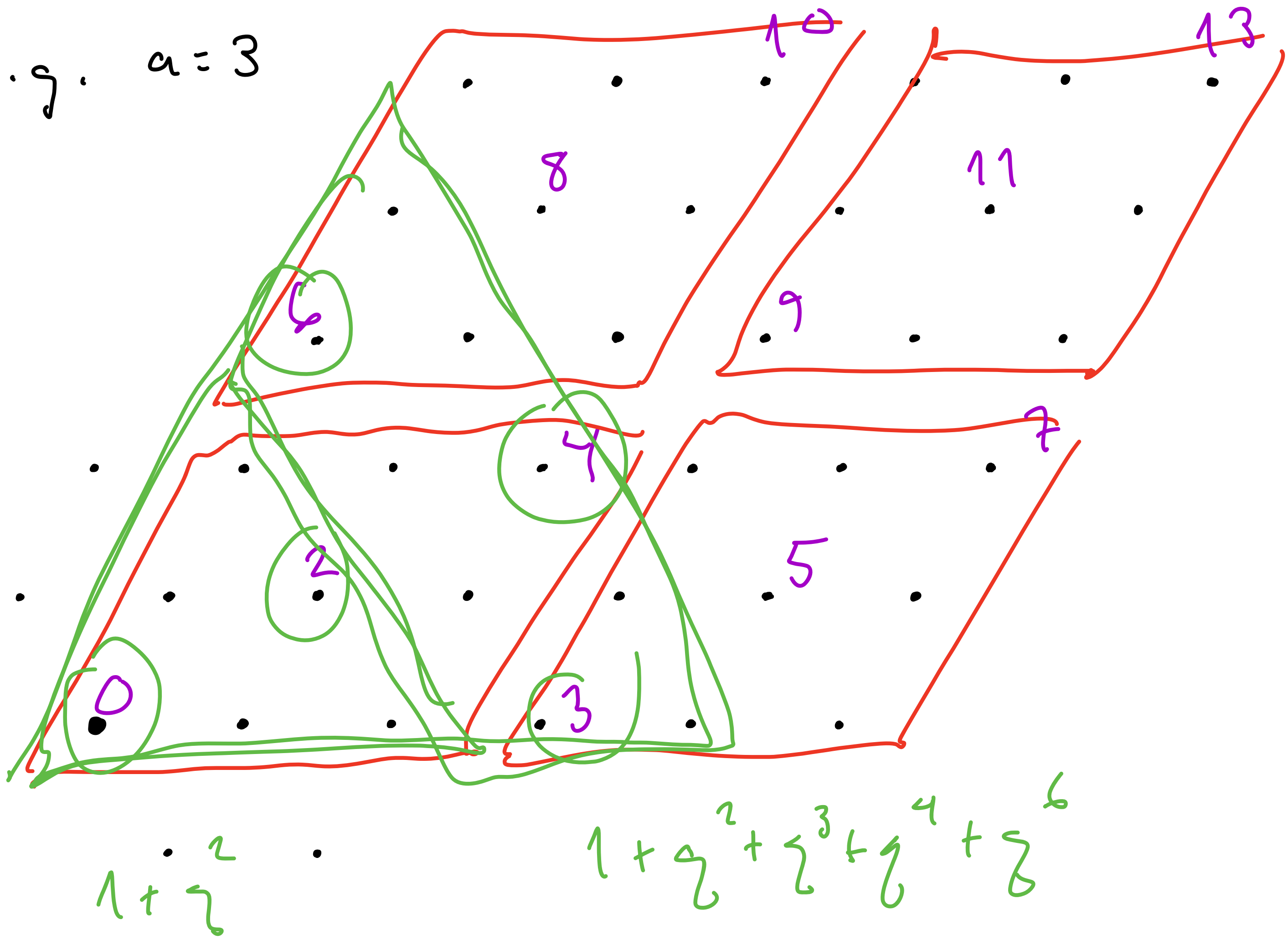
Note that this implies

$$\mathrm{Cat}(a,b_2)_q - \mathrm{Cat}(a,b_1)_q$$

$$= \sum_{\vec{x} \in (b_2\Delta - b_1\Delta) \cap R} q^{J(\vec{x})} \in \mathbb{N}[q].$$

Johnson statistics exist for $a = 3, 4$.

e.g. $a = 3$



This Johnson statistic comes from a ribbon partition of R.

This method could be used to produce Johnson statistics in general.

Let

$$\mathrm{Box}[m,n] = \{ \vec{x} \in \mathrm{Box} : m \leq \sum x_i \leq n \}$$

Let $1 = c_1 < c_2 < \cdots < c_r = (a-1)^2$ be all integers $\leq (a-1)^2$ and coprime to a. $[r = (a-2)\phi(a) + 1]$

And let $\mathrm{Slice}_i := \mathrm{Box}[c_{i-1}+1, c_i]$.

Then for all $\gcd(a,b) = 1$,

$$b\Delta \cap \mathcal{L} = \coprod \text{translations of slices}$$

Define Catalan germs:

$$\mathrm{Cat}((a;c_i))_q = \frac{1}{[a]_q} \sum_{\vec{x} \in \mathrm{Slice}_i} q^{T(\vec{x})}$$

Theorem: $\in \mathbb{Z}[q]$

Conjecture: $\in \mathbb{N}[q]$

if we can find this

$$\mathrm{Cat}((a;c_i))_q = \sum_{\vec{x} \in R \cap \mathrm{Slice}_i} q^{J(\vec{x})}$$

Theorem (by inclusion-exclusion):

$$\mathrm{Cat}(a,b)_q = \sum_i \begin{bmatrix} a-1+\left\lfloor \frac{b-c_i}{a} \right\rfloor \\ a-1 \end{bmatrix}_{q^a} \mathrm{Cat}((a;c_i))_q$$

e.g. $a = 3$ $\quad (c_1, c_2, c_3) = (1, 2, 4)$

$\leq (3-1)^2$ and coprime to 3

$$\mathrm{Cat}((3;1))_q = 1$$

$$\mathrm{Cat}((3;2))_q = q^2$$

$$\mathrm{Cat}((3;4))_q = q^4$$

$$\mathrm{Cat}(3,b)_q = 1 \begin{bmatrix} 2+\left\lfloor \frac{b-1}{3} \right\rfloor \\ 2 \end{bmatrix}_{q^3}$$

$$+ \; q^2 \begin{bmatrix} 2+\left\lfloor \frac{b-1}{3} \right\rfloor \\ 2 \end{bmatrix}_{q^3}$$

$$+ \; q^4 \begin{bmatrix} 2+\left\lfloor \frac{b-4}{3} \right\rfloor \\ 2 \end{bmatrix}_{q^3}$$

One approach to find a Jons. Stat.

Find a ribbon partition for each poset $\mathrm{Box}[c_{i-1}+1, c_i]$.

More generally, consider the poset of partions under componentwise order (inclusion of diagrams)

$$\mathrm{Box}[m,n] = \{ y_1 \geq y_2 \geq \cdots \geq y_{a-1} \geq 0 \\ m \leq y_1 \leq n, \\ 0 \leq y_i - y_{i+1} < a, \\ 0 \leq y_{a-1} < a \}$$

Conjecture: If $\gcd(a, m-1) = \gcd(a,n) = 1$ then $\mathrm{Box}[m,n]$ has a decomposition into ribbons (saturated chains of length a).