Determinant evaluations inspired by Di Francesco's determinant for twenty-vertex configurations

Christian Krattenthaler

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joint work with Christoph Koutschan (RICAM, Linz) and Michael Schlosser (Univ. Wien),
and with Sylvie Corteel (Sorbonne, Paris, UC Berkeley) and Frederick Huang (UC Berkeley)

Di Francesco's determinant for 20V configurations

In 2021, in the context of counting certain configurations in the 20-vertex model, Di Francesco came up with the following conjecture:

Conjecture

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}+\binom{-i+2j+1}{2j+1}\right) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

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More precisely, Di Francesco observed (and showed) that the number of domino tilings of certain regions that he called AZTEC TRIANGLES is the same as the number of these 20-vertex configurations. Furthermore, he proved that the number of domino tilings is given by one half of the above determinant.

Di Francesco's determinant for 20V configurations

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Conjecture

For all positive integers n, we have

$$\underset{0\le i,j\le n-1}{\text{determinant}}\left(2^i\binom{i+2j+1}{2j+1}+\binom{-i+2j+1}{2j+1}\right)$$

$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

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I like determinants with parameters better:

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Di Francesco's determinant, plus a generalisation

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For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1},$$

where $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$, *and* $(\alpha)_0:=1$.

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$$\times\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2},$$

where $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$, *and* $(\alpha)_0:=1$.

Di Francesco's determinant, plus a generalisation

During the (hybrid) Conference on Lattice Paths and Applications at the CIRM, Luminy, in June 2021, Di Francesco presented his work on the 20-vertex model and, in particular, his conjectured determinant evaluation.

Di Francesco's determinant, plus a generalisation

On June 23, 2021 another CK (Christoph Koutschan) received the following email from Doron Zeilberger:

Di Francesco's determinant, plus a generalisation

On June 23, 2021 another CK (Christoph Koutschan) received the following email from Doron Zeilberger:

```
Dear Christoph,

Philippe Di Francesco just gave a great talk
at the Lattice path conference mentioning,
inter alia, a certain conjectured
determinant.  It is
Conj. 8.1 (combined with Th. 8.2) in
https://arxiv.org/pdf/2102.02920.pdf

I am curious if you can prove it by the
Koutschan-Zeilberger-Aek holonomic ansatz
method.  If you can do it before Friday, June
25, 2021, 17:00 Paris time, I will mention it
in my talk in that conference.

Best wishes
         Doron
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Christoph Koutschan succeeded to prove Di Francesco's determinant evaluation using Zeilberger's holonomic Ansatz (and heavy computer calculations). However, today's computers seem to be not powerful enough to prove the conjectures for the "x-determinants."

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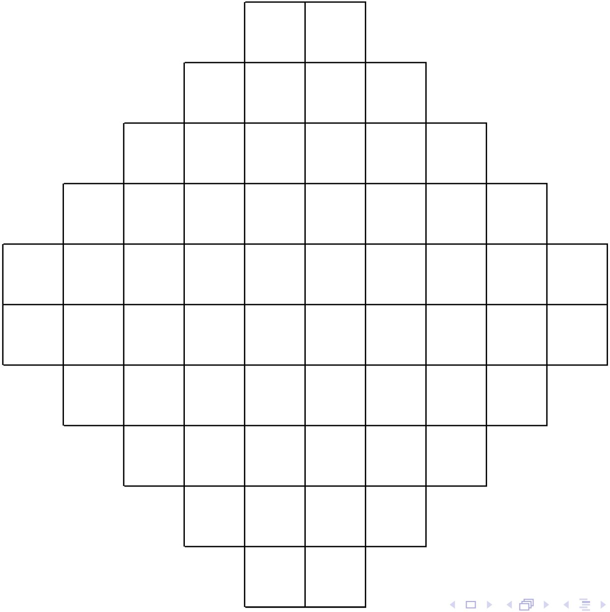
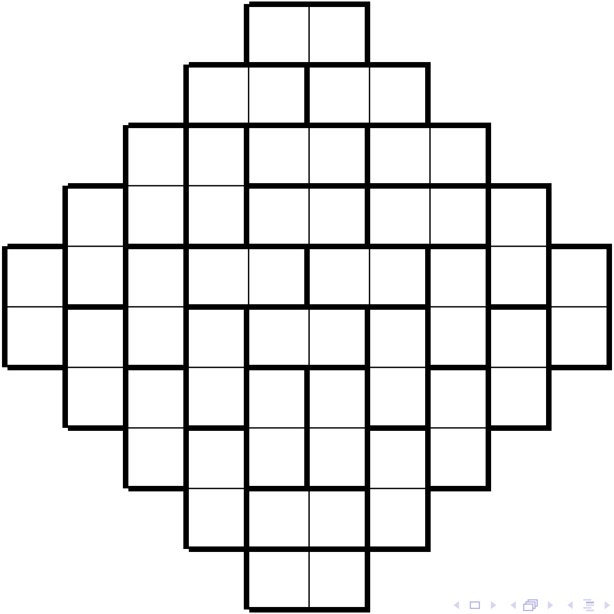
The Aztec diamond



Domino tilings of the Aztec diamond



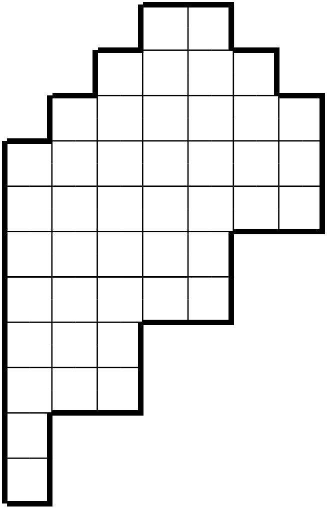
The Aztec diamond theorem

Theorem (ELKIES, KUPERBERG, LARSEN, PROPP 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

The Aztec triangle of Di Francesco



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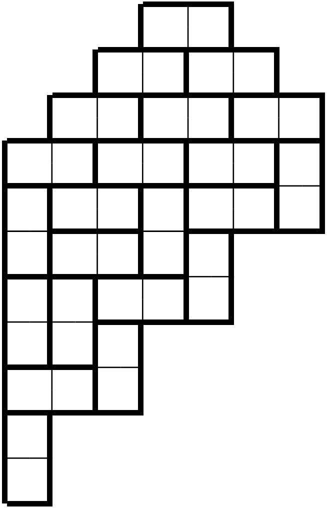
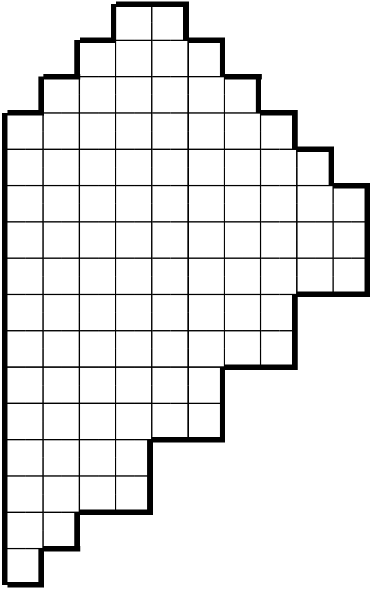


Theorem (Di Francesco + Koutschan)

The number of domino tilings of the Aztec triangle of size n is

$$2^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!}.$$

A generalised Aztec triangle



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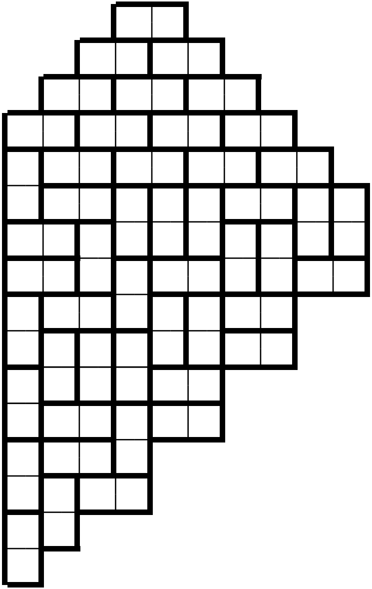

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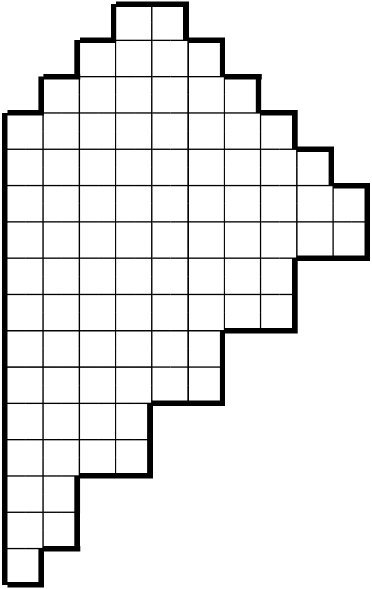

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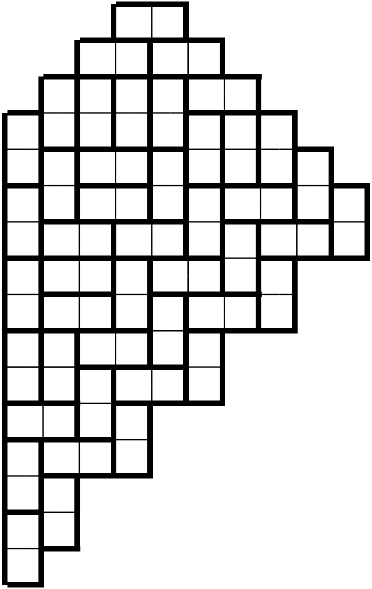

(Sylvie Corteel and Frederick Huang)

Enumeration of generalised Aztec triangles

Conjecture (Corteel, Huang)

The number of domino tilings of the (n,k)-Aztec triangle of type I is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

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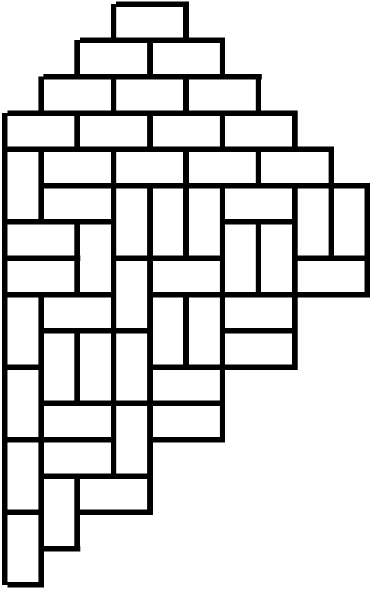
Conjecture (Corteel, Huang)

The number of domino tilings of the (n,k)-Aztec triangle of type II is

$$\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

Generalised Aztec triangles and non-intersecting paths



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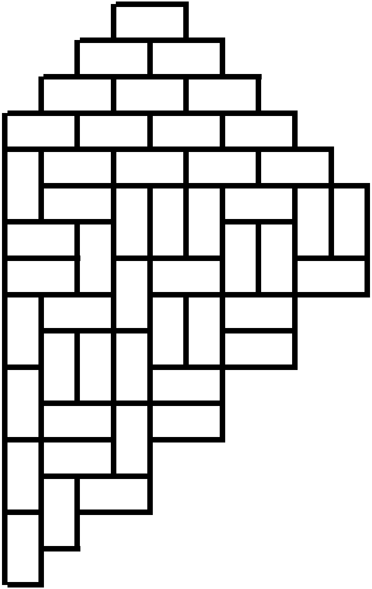

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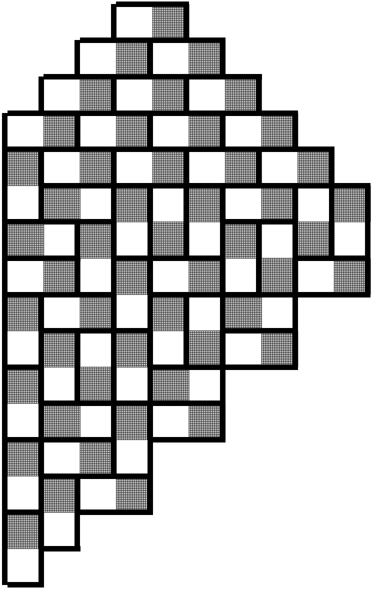

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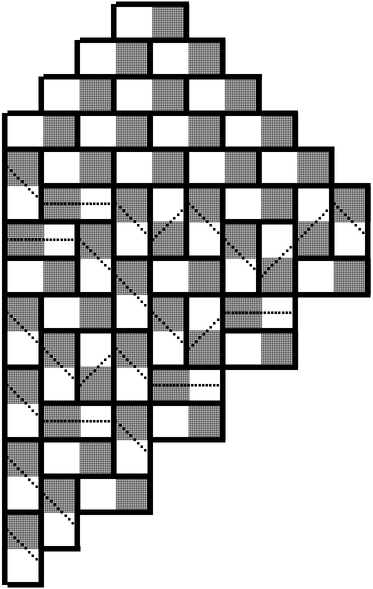

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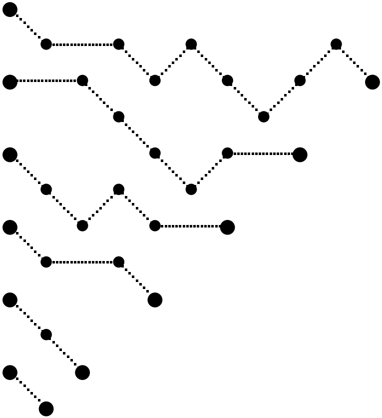

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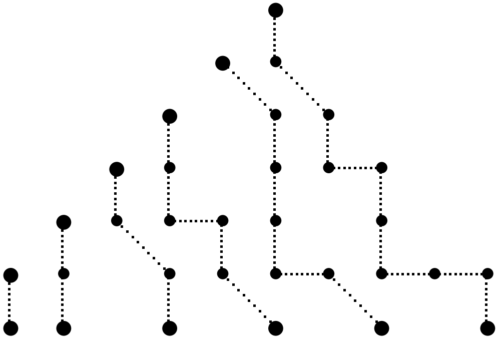

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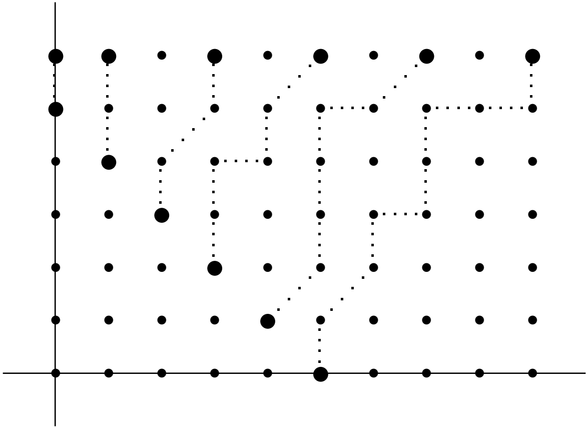

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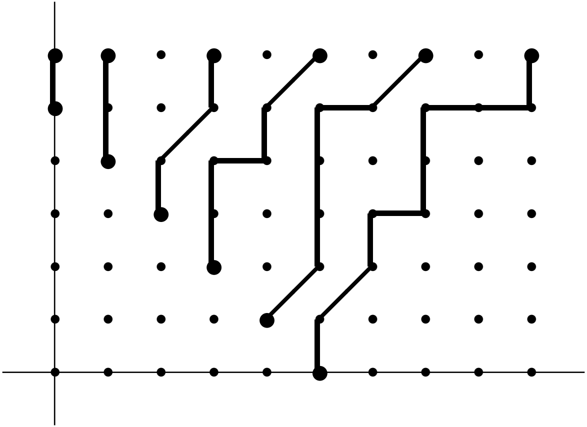

Hence, by the Lindström–Gessel–Viennot Theorem for non-intersecting lattice paths:
The number of domino tilings of the (n, k)-Aztec triangle of type I equals $\det D_1(k; n)$*, where*

$$D_1(k; n) = (D(2j - i, i + n - k - 1))_{1 \le i,j \le k},$$

with $D(m, n)$ a Delannoy number, i.e., the number of paths from $(0, 0)$ to (m, n) consisting of steps $(1, 0)$, $(0, 1)$, and $(1, 1)$.

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Furthermore, the number of domino tilings of the (n,k)-Aztec triangle of type II equals $\det D_2(k;n)$, *where*

$$D_2(k;n) \\ = \left(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\right)_{1\le i,j\le k}.$$

A determinant evaluation

We need to show:

$$\det\left(D(2j-i,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i},$$

and also:

$$\det\left(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

The number of domino tilings of the generalised Aztec triangle of type I

The evaluation of these two determinants is a longer story and is left out here for time considerations.

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Theorem

The number of domino tilings of the (n,k)-Aztec triangle of type I is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

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Theorem

The number of domino tilings of the (n, k)-Aztec triangle of type II is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

The "x-determinant"

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Back to:

Conjecture

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$\stackrel{?}{=} 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

The relation between two determinants

Define

$$D_1(k;n) := \det\left(D(2j-i, i+n-k-1)\right)_{1\le i,j\le k},$$

or, after the shifts $i \mapsto i+1$ and $j \mapsto j+1$, equivalently

$$D_1(k;n) := \det\left(D(2j-i+1, i+n-k)\right)_{0\le i,j\le k-1},$$

Furthermore, define

$$D_3(k;x) := \det_{0\le i,j\le k-1}\left(2^i\binom{x+i+2j+1}{2j+1} + \binom{x-i+2j+1}{2j+1}\right).$$

Theorem

We have

$$D_1(k;y+k) = \tfrac{1}{2}D_3(k;2y).$$

The relation between two determinants

For the proof, we follow a sequence of transformations that Di Francesco had designed to relate the two determinants in the special case he was looking at (i.e., $x = 0$). Only at the beginning an additional transformation has to be inserted (which reduces to the identity for $x = 0$), subsequently Di Francesco's transformations do the job.

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The details are left out here in the interest of time.

The "x-determinants": proved

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For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

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$$=2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

and

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j}{2j}+\binom{x-i+2j}{2j}\right)$$
$$=2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}$$
$$\times\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2}.$$

Determinant evaluations: variations on a theme

Recall:

Theorem

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Consider the determinants

$$\det_{0\le i,j\le n-1}\left(A^{i+\beta}\binom{i+Bj+\gamma}{Bj+\alpha}+\binom{-i+Bj+\delta}{Bj+\alpha}\right).$$

Are there other instances of "nice" evaluations?

Determinant evaluations: variations on a theme; $A = B = 2$

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Let

$$D_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

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In this notation, the previous two determinant evaluations read as follows.

Theorem

For all positive integers n, we have

$$D_{1,0,x+1,x+1}(n) = 2^{\binom{n}{2}+1} \prod_{i=0}^{n-1} \frac{i!}{(2i+1)!} \prod_{i=0}^{\lfloor n/2 \rfloor} (x+4i+1)_{n-2i} \times \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (x-2i+3n)_{n-2i-1}.$$

Determinant evaluations: variations on a theme; $A = B = 2$

Let

$$D_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

In this notation, the previous two determinant evaluations read as follows.

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In this notation, the previous two determinant evaluations read as follows.

Theorem

For all positive integers n, we have

$$D_{0,0,x,x}(n) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1} \times \prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2}.$$

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In the search space

$$\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 9 \text{ and } -9 \le \gamma,\delta \le 9\}$$

we looked for further "nice" evaluations.

Determinant evaluations: variations on a theme; $A = B = 2$

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Theorem

The following determinant evaluations hold for all $n \ge 1$:

$$D_{-2,0,-1,-1}(n) = -2\prod_{i=2}^{n}\frac{8(2i-3)(2i-1)\,\Gamma(4i-5)\,\Gamma\left(\frac{i+1}{2}\right)}{i\,\Gamma(3i-2)\,\Gamma\left(\frac{3i-3}{2}\right)},$$

$$D_{0,2,3,-1}(n) = \prod_{i=1}^{n}\frac{3(2i-1)\,\Gamma(4i+3)\,\Gamma\left(\frac{i+1}{2}\right)}{4(i+2)\,\Gamma(3i+1)\,\Gamma\left(\frac{3i+5}{2}\right)},$$

$$D_{1,1,0,-2}(n) = -2\prod_{i=1}^{n}\frac{(2i-1)\,\Gamma(4i-3)\,\Gamma\left(\frac{i}{2}\right)}{2\,\Gamma(3i-2)\,\Gamma\left(\frac{3i}{2}\right)},$$

$$D_{1,1,1,-1}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(4i-1\right)\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(3i\right)\Gamma\left(\frac{3i-1}{2}\right)},$$

$$D_{2,1,2,0}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(4i\right)\Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(3i\right)\Gamma\left(\frac{3i+2}{2}\right)},$$

$$D_{0,1,1,-1}(n) = 3\prod_{i=2}^{n} \frac{\Gamma\left(4i\right)\Gamma\left(\frac{i-1}{2}\right)}{\Gamma\left(3i+1\right)\Gamma\left(\frac{3i-3}{2}\right)}.$$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 4$*:*

$$D_{2,1,2,0}(n) = \frac{1}{8}\, D_{1,1,-1,-3}(n+1) = \frac{1}{40}\, D_{0,1,-4,-6}(n+2)$$
$$= -\frac{1}{24576}\, D_{1,2,-4,-8}(n+2),$$

$$\begin{aligned}
D_{1,1,1,-1}(n) &= D_{2,1,1,-1}(n) = \frac{1}{3}\,D_{0,1,-2,-4}(n+1)\\
&= -\frac{1}{32}\,D_{1,1,-2,-4}(n+1) = -\frac{1}{224}\,D_{1,2,-2,-6}(n+1)\\
&= -\frac{1}{168}\,D_{0,1,-5,-7}(n+2) = -\frac{1}{3696}\,D_{0,2,-5,-9}(n+2)\\
&= -\frac{1}{337920}\,D_{1,2,-5,-9}(n+2),\\
D_{1,1,0,-2}(n) &= \frac{1}{5}\,D_{0,1,-3,-5}(n+1) = \frac{1}{1008}\,D_{1,2,-3,-7}(n+1),\\
D_{-2,1,0,-2}(n) &= D_{0,2,3,-1}(n-1),\\
D_{2,1,1,-1}(n) &= D_{4,2,4,0}(n-1),\\
D_{1,1,-2,-4}(n) &= -\frac{16}{5}\,D_{3,2,1,-3}(n-1) = \frac{64}{3}\,D_{5,3,4,-2}(n-2)\\
&= -128\,D_{7,4,7,-1}(n-3),
\end{aligned}$$

Determinant evaluations: variations on a theme; $A = B = 2$

$$D_{1,1,-1,-3}(n) = -4\, D_{3,2,2,-2}(n-1) = 16\, D_{5,3,5,-1}(n-2),$$
$$D_{1,1,0,-2}(n) = -2\, D_{3,2,3,-1}(n-1).$$

Determinant evaluations: variations on a theme; $A = B = 2$

$$D_{1,1,-1,-3}(n) = -4\,D_{3,2,2,-2}(n-1) = 16\,D_{5,3,5,-1}(n-2),$$
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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

Determinant evaluations: variations on a theme; $A = B = 3$

Let

$$E_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(3^{i+\beta}\binom{i+3j+\gamma}{3j+\alpha}+\binom{-i+3j+\delta}{3j+\alpha}\right).$$

Determinant evaluations: variations on a theme; $A = B = 3$

Let

$$E_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(3^{i+\beta}\binom{i+3j+\gamma}{3j+\alpha} + \binom{-i+3j+\delta}{3j+\alpha}\right).$$

In the search space

$$\begin{aligned}&\big\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 6 \text{ and } -8 \le \gamma,\delta \le 8\big\}\\ \cup&\big\{(\alpha,\beta,\gamma,\delta) : 6 \le \alpha \le 10 \text{ and } 0 \le \beta \le 10 \text{ and } -10 \le \gamma,\delta \le 10\big\}\end{aligned}$$

we looked for "nice" evaluations.

Determinant evaluations: variations on a theme; $A = B = 3$

Let

$$E_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(3^{i+\beta}\binom{i+3j+\gamma}{3j+\alpha}+\binom{-i+3j+\delta}{3j+\alpha}\right).$$

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Theorem

The following determinant evaluations hold for all $n \ge 1$:

$$E_{-3,0,-1,-1}(n) = 2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma(4i-5)\,\Gamma\left(\frac{i+2}{3}\right)}{i(i+1)\,\Gamma(3i-5)\,\Gamma\left(\frac{4i-1}{3}\right)},$$

$$E_{-3,1,0,-2}(n) = -2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma(4i-4)\,\Gamma\left(\frac{i}{3}\right)}{i(i+1)^2\,\Gamma(3i-5)\,\Gamma\left(\frac{4i-3}{3}\right)},$$

$$E_{0,3,5,-1}(n) = \prod_{i=1}^{n}\frac{2^{i+1}(3i-2)(3i-1)\,\Gamma(4i+4)\,\Gamma\left(\frac{i+2}{3}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(3i+1)\,\Gamma\left(\frac{4i+5}{3}\right)},$$

Determinant evaluations: variations on a theme; $A = B = 3$

$$E_{0,1,1,-1}(n) = \prod_{i=1}^{n} \frac{2^{i+1}\,\Gamma(4i-2)\,\Gamma\left(\frac{i+2}{3}\right)}{i\,\Gamma(3i-2)\,\Gamma\left(\frac{4i-1}{3}\right)},$$

$$E_{1,1,2,0}(n) = \prod_{i=1}^{n} \frac{2^{i}\,\Gamma(4i)\,\Gamma\left(\frac{i+1}{3}\right)}{3i\,\Gamma(3i-1)\,\Gamma\left(\frac{4i+1}{3}\right)},$$

$$E_{3,2,3,-1}(n) = \prod_{i=1}^{n} \frac{2^{i}\,\Gamma(4i+1)\,\Gamma\left(\frac{i+2}{3}\right)}{\Gamma(3i+1)\,\Gamma\left(\frac{4i+2}{3}\right)},$$

$$E_{1,0,1,1}(n) = 2\prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)},$$

$$E_{2,0,2,2}(n) = 2\prod_{i=1}^{n} \frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i+2}{3}\right)}{\Gamma(3i+1)\,\Gamma\left(\frac{4i+2}{3}\right)}.$$

Determinant evaluations: variations on a theme; $A = B = 3$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 3$:

$$
\begin{aligned}
E_{0,0,0,0}(n) &= \frac{1}{2}\, E_{0,1,-1,-3}(n) = \frac{1}{5}\, E_{0,2,-2,-6}(n),\\
E_{1,0,1,1}(n) &= -\frac{1}{84}\, E_{1,3,-2,-8}(n) = 2\, E_{4,2,4,0}(n-1)\\
&= \frac{6}{5}\, E_{4,3,3,-3}(n-1),\\
E_{2,0,2,2}(n) &= 2\, E_{5,2,5,1}(n-1) = 18\, E_{8,4,8,0}(n-2)\\
&= \frac{162}{5}\, E_{8,5,7,-3}(n-2),\\
E_{-3,2,1,-3}(n) &= E_{0,3,5,-1}(n-1),\\
E_{0,1,-1,-3}(n) &= 4\, E_{3,2,3,-1}(n-1),\\
E_{1,1,0,-2}(n) &= -2\, E_{4,2,4,0}(n-1),
\end{aligned}
$$

Determinant evaluations: variations on a theme; $A = B = 3$

$$\begin{aligned}
E_{1,2,-1,-5}(n) &= -12\, E_{4,3,3,-3}(n-1) = -180\, E_{7,4,7,-1}(n-2),\\
E_{2,1,1,-1}(n) &= E_{5,2,5,1}(n-1),\\
E_{2,2,0,-4}(n) &= \frac{15}{2}\, E_{5,3,4,-2}(n-1) = -45\, E_{8,4,8,0}(n-2),\\
E_{2,3,-1,-7}(n) &= 36\, E_{5,4,3,-5}(n-1) = -\frac{13608}{5}\, E_{8,5,7,-3}(n-2).
\end{aligned}$$

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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

Determinant evaluations: variations on a theme; $A = B = 3$

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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

A closer analysis of these individual evaluations plus some further experimental computations leads us to the following conjecture.

Determinant evaluations: variations on a theme; $A = B = 3$

Conjecture

Let

$$\Xi(x) := \prod_{i=2}^{x} \frac{3\,\Gamma(i)\,\Gamma(4i-3)\,\Gamma(4i-2)}{2\,\Gamma(3i-2)^2\,\Gamma(3i-1)} \text{ and } \mu_m(x) := \begin{cases} 2, & \text{if } 3 \mid (x-m), \\ 1, & \text{otherwise.} \end{cases}$$

Then, for all non-negative integers x and for all $n \ge x$, we have

$$E_{0,x,-x,-3x}(n) = 2\mu_1(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-1}\,\Gamma(4i-3)\,\Gamma\left(\frac{i+1}{3}\right)}{\Gamma(3i-2)\,\Gamma\left(\frac{4i-2}{3}\right)},$$

$$E_{1,x,1-x,1-3x}(n) = 2\mu_2(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x+2}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)},$$

$$E_{2,x,2-x,2-3x}(n) = \frac{\mu_0(x)}{n}\,\Xi(x)\,(-1)^{\lfloor \frac{x+1}{3} \rfloor} \prod_{i=2}^{n} \frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i-1}{3}\right)}{9\,\Gamma(3i)\,\Gamma\left(\frac{4i+2}{3}\right)}.$$

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Let

$$\Xi(x) := \prod_{i=2}^{x} \frac{3\,\Gamma(i)\,\Gamma(4i-3)\,\Gamma(4i-2)}{2\,\Gamma(3i-2)^2\,\Gamma(3i-1)} \text{ and } \mu_m(x) := \begin{cases} 2, & \text{if } 3 \mid (x-m), \\ 1, & \text{otherwise.} \end{cases}$$

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$$E_{1,x,1-x,1-3x}(n) = 2\mu_2(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x+2}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)},$$

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Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

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$$F_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(4^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

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In the search space

$$\left\{(\alpha,\beta,\gamma,\delta) : -6\le\alpha,\beta\le 9 \text{ and } -9\le\gamma,\delta\le 9\right\}$$

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Theorem

The following determinant evaluations hold for all $n \ge 1$:

$$F_{1,0,1,1}(n) = 2\prod_{i=1}^{n}\frac{3^{i-1}\,\Gamma(3i-1)\,\Gamma\left(\frac{i+1}{2}\right)}{\Gamma(2i)\,\Gamma\left(\frac{3i-1}{2}\right)},$$

$$F_{1,0,2,2}(n) = 2\prod_{i=1}^{n}\frac{3^{i-1}\,\Gamma(3i)\,\Gamma\left(\frac{i}{2}\right)}{2\,\Gamma(2i)\,\Gamma\left(\frac{3i}{2}\right)},$$

$$F_{1,0,3,3}(n) = 2\prod_{i=1}^{n}\frac{3^{i}\,\Gamma(3i-1)\,\Gamma\left(\frac{i+1}{2}\right)}{\Gamma(2i)\,\Gamma\left(\frac{3i-1}{2}\right)}.$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \ge 4$:

$$
\begin{aligned}
F_{1,0,1,1}(n) &= \frac{2}{3}\, F_{1,1,-1,-3}(n) = \frac{1}{21}\, F_{1,2,-3,-7}(n),\\
F_{1,0,2,2}(n) &= -2\, F_{1,1,0,-2}(n) = \frac{2}{7}\, F_{1,2,-2,-6}(n),\\
F_{1,0,3,3}(n) &= 2\, F_{1,1,1,-1}(n) = \frac{2}{5}\, F_{1,2,-1,-5}(n) = \frac{1}{99}\, F_{1,3,-3,-9}(n),\\
F_{1,1,-1,-3}(n) &= -6\, F_{3,2,2,-2}(n-1) = 24\, F_{5,3,5,-1}(n-2),\\
F_{1,1,0,-2}(n) &= -2\, F_{3,2,3,-1}(n-1).
\end{aligned}
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Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 4$:

$$F_{1,0,1,1}(n) = \frac{2}{3}\, F_{1,1,-1,-3}(n) = \frac{1}{21}\, F_{1,2,-3,-7}(n),$$

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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Theorem

Let x be an indeterminate. Then, for all integers $n \geq 1$, we have

$$\det_{0 \le i,j \le n-1} \left(4^i \binom{x+i+2j+1}{2j+1} + \binom{x-i+2j+1}{2j+1} \right)$$
$$= 2^{\binom{n+1}{2}+1} 3^{\binom{n}{2}} \prod_{i=1}^{n} \frac{i!}{(2i)!} \prod_{i=0}^{n-1} (x+3i+1)_{n-i}.$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

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Let x be an indeterminate. Then, for all integers $n \geq 1$, we have

$$\det_{0\le i,j\le n-1}\left(4^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom{n+1}{2}+1}3^{\binom{n}{2}}\prod_{i=1}^{n}\frac{i!}{(2i)!}\prod_{i=0}^{n-1}(x+3i+1)_{n-i}.$$

Proof.

Even if the computer takes much longer, also this determinant evaluation can be proved by means of Zeilberger's holonomic Ansatz. □

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There are also two infinite families of evaluations.

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Theorem

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(4^i\binom{x+i+2j+3}{2j+3}+\binom{x-i+2j+3}{2j+3}\right)$$
$$=\left(2\cdot 6^{\binom n2}\prod_{i=0}^{n-1}\frac{i!}{(2i+3)!}\right)\left((x+2)(x+3)\prod_{i=0}^{n-1}(x+3i+1)_{n-i}\right)$$
$$\times \operatorname{Pol}_n(x),$$

where $\operatorname{Pol}_n(x)$ *is a monic polynomial in* x *of degree* $2n-2$.

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Theorem

For all positive integers n, we have

$$\det_{0 \le i,j \le n-1} \left(4^i \binom{x+i+2j+3}{2j+3} + \binom{x-i+2j+3}{2j+3} \right)$$
$$= \left(2 \cdot 6^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{i!}{(2i+3)!} \right) \left((x+2)(x+3) \prod_{i=0}^{n-1} (x+3i+1)_{n-i} \right)$$
$$\times \operatorname{Pol}_n(x),$$

where $\operatorname{Pol}_n(x)$ is a monic polynomial in x of degree $2n-2$.

Proof.

We give a lengthy proof by identification of factors. □

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

What is the polynomial $\mathrm{Pol}_n(x)$?

Determinant evaluations: variations on a theme; $A=4$, $B=2$

What is the polynomial $\operatorname{Pol}_n(x)$?

Conjecture

The polynomial $\operatorname{Pol}_n(x)$ *in the previous theorem is given by the recurrence*

$$3\operatorname{Pol}_{n+3}(x)-2\big(18n^2+9nx+72n-3x^2-3x+49\big)\operatorname{Pol}_{n+2}(x)$$
$$+\big(135n^4+108n^3x+810n^3-54n^2x^2+108n^2x+1395n^2-52nx^3-510nx^2$$
$$-1100nx+120n-9x^4-152x^3-855x^2-1780x-1020\big)\operatorname{Pol}_{n+1}(x)$$
$$-6(n+1)(n-x-2)(n+x+2)(3n+x+3)(3n+x+5)(3n+x+7)\operatorname{Pol}_n(x)$$
$$=0$$

and initial values

$$\operatorname{Pol}_1(x)=1,$$
$$\operatorname{Pol}_2(x)=\tfrac{1}{3}\big(3x^2+31x+60\big),$$
$$\operatorname{Pol}_3(x)=\tfrac{1}{9}\big(9x^4+234x^3+2061x^2+6956x+7680\big).$$

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

Let

$$G_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+4j+\gamma}{4j+\alpha} + \binom{-i+4j+\delta}{4j+\alpha}\right).$$

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

Let

$$G_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+4j+\gamma}{4j+\alpha}+\binom{-i+4j+\delta}{4j+\alpha}\right).$$

In the search space

$$\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 9 \text{ and } -9 \le \gamma,\delta \le 9\}$$

we looked for "nice" evaluations.

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Conjecture

The following determinant evaluations hold for all $n \ge 1$:

$$G_{0,2,3,-1}(n) = \prod_{i=1}^{n}\frac{(2i-1)(4i-3)(4i-1)\,\Gamma(6i)\,\Gamma\left(\frac{i+3}{4}\right)}{i(i+1)(i+2)(3i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i+3}{4}\right)},$$

$$G_{1,3,6,0}(n) = \prod_{i=1}^{n}\frac{8(2i-1)(2i+1)^2(4i-1)(4i+1)\,\Gamma(6i+2)\,\Gamma\left(\frac{i+2}{4}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(5i+2)\,\Gamma\left(\frac{5i+6}{4}\right)},$$

$$G_{1,1,0,-2}(n) = -4\prod_{i=1}^{n}\frac{(3i-2)\,\Gamma(6i-5)\,\Gamma\left(\frac{i}{4}\right)}{8\,\Gamma(5i-4)\,\Gamma\left(\frac{5i}{4}\right)},$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

$$G_{3,0,3,3}(n) = 2\prod_{i=1}^{n} \frac{\Gamma\left(6i-1\right)\Gamma\left(\frac{i+3}{4}\right)}{\Gamma\left(5i\right)\Gamma\left(\frac{5i-1}{4}\right)},$$

$$G_{2,1,2,0}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(6i-1\right)\Gamma\left(\frac{i+2}{4}\right)}{2(2i-1)\,\Gamma\left(5i-1\right)\Gamma\left(\frac{5i-2}{4}\right)}.$$

Moreover, the following identities are conjectured to hold for all $n \geq 3$:

$$\begin{aligned} G_{3,0,3,3}(n) &= \frac{2}{3}\,G_{0,1,-2,-4}(n+1) = -\frac{1}{672}\,G_{1,3,-2,-8}(n+1) \\ &= \frac{1}{63}\,G_{5,4,3,-5}(n) = \frac{4}{1002001}\,G_{6,6,3,-9}(n) \\ &= -\frac{8}{5}\,G_{9,5,8,-2}(n-1), \end{aligned}$$

Determinant evaluations: variations on a theme; $A=4$, $B=2$

$$G_{1,1,0,-2}(n) = -\frac{1}{49}\, G_{2,3,0,-6}(n) = -\frac{2}{7}\, G_{6,4,5,-3}(n-1)$$
$$= -\frac{4}{5577}\, G_{7,6,5,-7}(n-1),$$
$$G_{2,1,2,0}(n) = 2\, G_{7,4,7,-1}(n-1).$$

Determinant evaluations: variations on a theme; $A = 4,\ B = 2$

$$G_{1,1,0,-2}(n) = -\frac{1}{49}\, G_{2,3,0,-6}(n) = -\frac{2}{7}\, G_{6,4,5,-3}(n-1)$$
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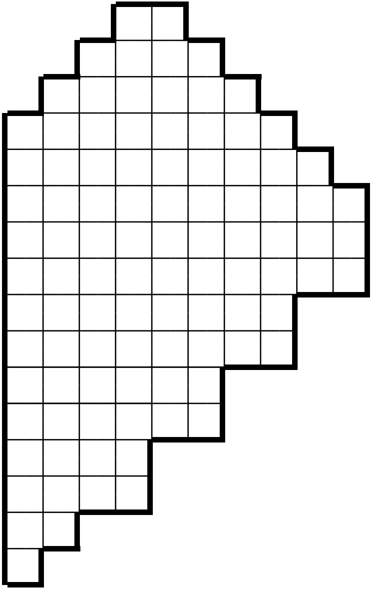
$$G_{2,1,2,0}(n) = 2\, G_{7,4,7,-1}(n-1).$$

In principle, Zeilberger's holonomic Ansatz should work for all of these. However, our computer capacities are not big enough such that the corresponding calculations can be carried out.

A generalised Aztec triangle



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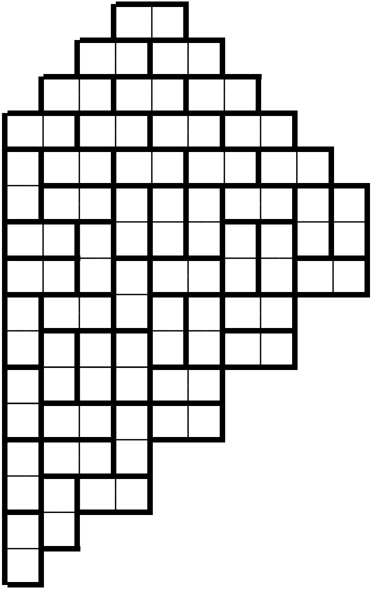

A generalised Aztec triangle