離散凸解析とマッチングモデル その3：Hatfield-Milgromのモデル

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Contracts (Hatfield-Milgrom model)

- D is a finite set of doctors
- H is a finite set of hospitals
- X is a finite set of contracts:
  - each contract $x \in X$ is associated with a doctor $D(x)$ and a hospital $H(x)$, and includes additional info. e.g., working days and salary between $D(x)$ and $H(x)$, etc.
  - for $k \in D \cup H$ and for $Y \subseteq X$,

$$Y_k = \{x \in Y \mid D(x) = k \text{ or } H(x) = k\}$$

Choice function (H-M model)

- each $k \in D \cup H$ has a choice function C_k with

$$C_k(Y) \subseteq Y_k \qquad (Y \subseteq X)$$

- $C_D(Y) = \bigcup_{i \in D} C_i(Y) \qquad (Y \subseteq X)$ choice fn of D

  $C_H(Y) = \bigcup_{j \in H} C_j(Y) \qquad (Y \subseteq X)$ choice fn of H
- C_D and C_H satisfy consistency:

$$C(Y) \subseteq Z \subseteq Y \Rightarrow C(Z) = C(Y)$$

Substitutability (H-M model)

- rejection fns. R_D and R_H are defined by

$$R_D(Y) = Y - C_D(Y) \qquad (Y \subseteq X)$$
$$R_H(Y) = Y - C_H(Y) \qquad (Y \subseteq X)$$

- C_D and C_H satisfy substitutability:

$$Z \subseteq Y \Rightarrow C(Y) \cap Z \subseteq C(Z)$$
$$\Updownarrow$$
$$Z \subseteq Y \Rightarrow R(Z) \subseteq R(Y)$$

Pairwise Stability (1)

- $Y \subseteq X$ is a pairwise stable allocation if
  - $C_D(Y) = Y$ and $C_H(Y) = Y$
  - for $x \in X - Y$,

$$x \notin C_D(Y \cup \{x\}) \quad \text{or} \quad x \notin C_H(Y \cup \{x\})$$

Pairwise Stability (2)

Lemma A: [Hatfield-Milgrom, 2005]

If $Y_D = X - R_H(Y_H)$ and $Y_H = X - R_D(Y_D)$

then $Y_D \cap Y_H$ is pairwise stable.

Lemma B: [Hatfield-Milgrom, 2005]

If Y is pairwise stable, then there exist Y_D and Y_H s.t.

$$Y_D = X - R_H(Y_H),\ Y_H = X - R_D(Y_D),\ Y = Y_D \cap Y_H$$

半順序集合

- X : 集合（無限集合も可）
- 2項関係 $\prec$ が次の3法則を満たすときXの要素間の**半順序関係**という
  - **反射法則** $x \prec x$
  - **反対称法則** $x \prec y,\ y \prec x \Rightarrow x = y$
  - **推移法則** $x \prec y,\ y \prec z \Rightarrow x \prec z$
- $(X, \prec)$ 半順序集合

束，完備束

- $(X, \prec)$ 半順序集合
- $a \in X$ が $A \subseteq X$ の上界 $\forall x \in A\ [x \prec a]$
  $a \in X$ が $A \subseteq X$ の下界 $\forall x \in A\ [a \prec x]$
- Aの上限: Aの上界全体の最小要素 $\bigvee A$
  Aの下限: Aの下界全体の最大要素 $\bigwedge A$
- $(X, \prec)$ が束とは任意の要素対$\{a,b\}$に対し，上限 $a \vee b$ と下限 $a \wedge b$ が存在する
- $(X, \prec)$ が完備束とは任意の$A \subseteq X$ に対して，上限と下限が存在する

Tarskiの不動点定理

- $(X, \prec)$ 完備束
  $f: X \rightarrow X$ が 単調，すなわち
  $$x \prec y \Rightarrow f(x) \prec f(y)$$
  とする.
- このとき不動点（$f(x)=x$）が存在する
- さらに不動点全体は束をなす

Existence of Stable Outcomes(1)

- partial order $\geq$ on $2^X \times 2^X$ s.t.
  $$(Y, Z) \geq (Y', Z') \Leftrightarrow Y' \subseteq Y \text{ and } Z \subseteq Z'$$
- $(2^X \times 2^X, \geq)$ is a complete lattice
  $$(Y, Z) \vee (Y', Z') = (Y \cup Y', Z \cap Z')$$
  $$(Y, Z) \wedge (Y', Z') = (Y \cap Y', Z \cup Z')$$

Existence of Stable Outcomes(2)

- $F(Y,Z) = (F_1(Z), F_2(F_1(Z)))$ $(Y,Z \subseteq X)$
  where $F_1(Z) = X - R_H(Z)$, $F_2(Z) = X - R_D(Z)$
- F is monotone on $(2^X \times 2^X, \geq)$, i.e.,
  - $(Y, Z) \geq (Y', Z') \Rightarrow Z \subseteq Z'$
    $\Rightarrow R_H(Z) \subseteq R_H(Z') \Rightarrow F_1(Z') \subseteq F_1(Z)$
  - $F_1(Z') \subseteq F_1(Z) \Rightarrow R_D(F_1(Z')) \subseteq R_D(F_1(Z))$
    $\Rightarrow F_2(F_1(Z)) \subseteq F_2(F_1(Z'))$
  - $F(Y, Z) \geq F(Y', Z')$

Existence of Stable Outcomes(3)

- $F(Y,Z) = (F_1(Z), F_2(F_1(Z)))$ $(Y,Z \subseteq X)$
  where $F_1(Z) = X - R_H(Z)$, $F_2(Z) = X - R_D(Z)$
- F is monotone on a complete lattice $(2^X \times 2^X, \geq)$
- by Tarski's fixed point th., there exists (Y, Z) with
  $$(Y, Z) = F(Y, Z) = (X - R_H(Z), X - R_D(Y))$$
- by Lemma A (if $Y_D = X - R_H(Y_H)$ and $Y_H = X - R_D(Y_D)$ then $Y_D \cap Y_H$ is pairwise stable), there exists a pairwise stable outcome