組合せ剛性理論の基礎から先端まで

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組合せ剛性理論

多面体の剛性

- Eulerの予想(1766)
  - 面の接続関係が等しく，対応する面が合同ならば2つの(凸)多面体は合同か？
- Cauchyの剛性定理(1813)
  - Yes!! 対象が凸多面体ならば
  - 凸多面体は(大域)剛堅

組合せ剛性(グラフの剛性)

- Maxwellの条件(1864)
  - 組合せ的必要条件
- Lamanの定理(1970)
  - 2次元ではMaxwellの条件を満たすグラフは剛堅な実現が可能
- 一般剛性(Glűck1975, Asimow and Roth1978)
  - グラフの剛性
  - 2次元一般剛性定理

- Gluckの定理(1975)
  - 多面体は一般的に(局所)剛堅
  - 極大平面グラフの殆ど全ての3次元実現は剛堅

- Connellyの反例(1977)
  - (局所)剛堅ではない多面体が存在

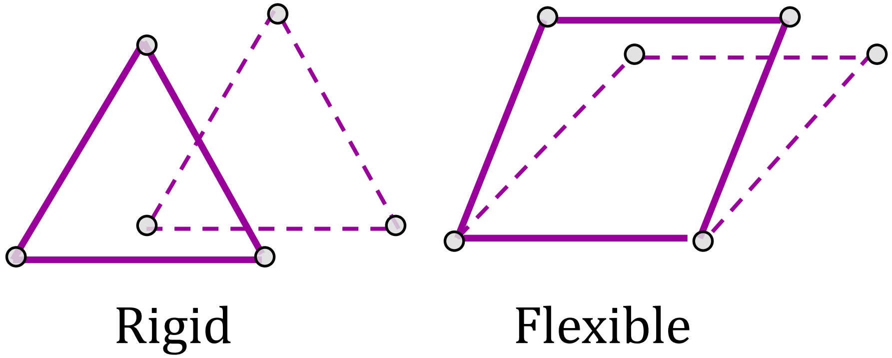
⋮

- 3次元一般剛性定理(20??)

⋮

Bar-joint Frameworks

- d-dimensional bar-joint framework: (G, p)
  - $G = (V, E)$: グラフ
  - p: ジョイント配置;$V \to \mathbb{R}^d$



- 動き(motion): 辺長一定制約を満たす頂点の連続的移動
- 自明な動き: 合同なフレームワークへの移動
- d次元剛堅(d-rigid): 全ての可能な動きが自明なフレームワーク

Maxwellのルール (modern form)

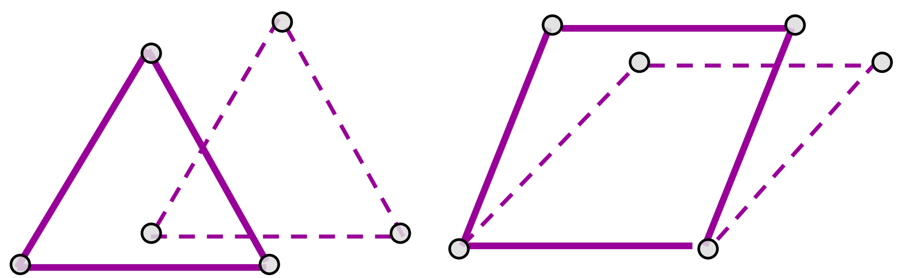
- (J.C.Maxwell1864) もしbar-joint framework (G,p) が一般的(generic)なジョイント配置 p において剛堅ならば,

$$|E| \geq d|V| - \binom{d+1}{2}$$

全自由度

$\mathbb{R}^d$の等長変換の次元

- p が 一般的(generic) $\overset{\text{def}}{\iff}$ 座標値の集合が $\mathbb{Q}$ 上で代数的に独立(i.e., $\mathbb{Q}$の要素を係数とする非ゼロ多項式を満たしていない.)

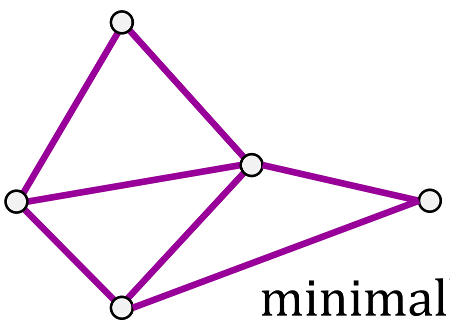


Rigid　　　Flexible

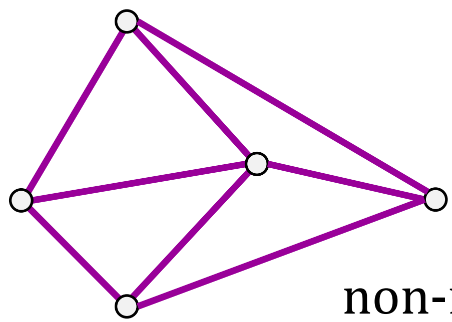
Maxwellのルール (dual form)

- 一般的pにおいて，もし(G, p)が$\mathbb{R}^d$内において極小剛堅(minimally rigid)ならば
  - $|E| = d|V| - \binom{d+1}{2}$
  - $\forall F \subseteq E\ with\ |V(F)| \geq d,\ |F| \leq d|V(F)| - \binom{d+1}{2}$

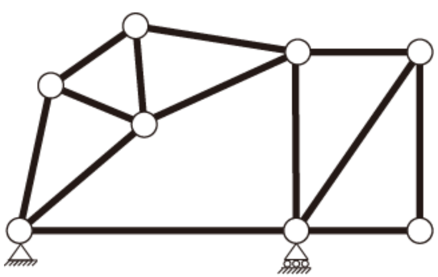
  (ここで$V(F)$は F に接続している頂点の集合)



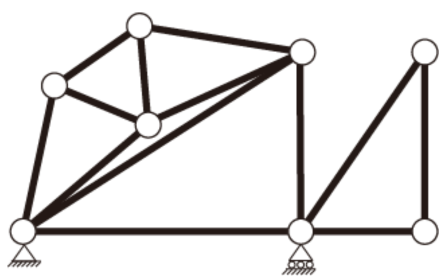
minimally rigid



non-minimally rigid



minimally rigid



flexible

$m = 13,$
$n = 8,$
$2n - 3 = 13$

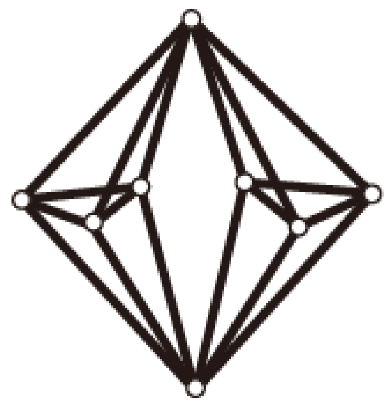
Lamanの定理 (modern form)

- (Laman1970) 一般的pにおいて, (G, p) が2次元極小剛堅(minimally rigid) ⇔
  - $|E| = 2|V| - 3$
  - $\emptyset \neq \forall F \subseteq E, |F| \leq 2|V(F)| - 3$
  - 右辺はE上の劣モジュラ関数を定める
  - $\Rightarrow O(n^2)$時間でテスト可能

  - 3つの証明法:
    - (Lovasz&Yemini82) グラフ的マトロイドの2合併から剛性マトロイドの構築
    - (Tay94) Crapoの3Tree2-分解
    - (Laman71,Tay&Whhiteley85,Whiteley96) 帰納的グラフ構築法

高次元フレームワーク

- 3次元においてMaxwellの条件は十分ではない：



|E|=3|V|-6
(18=3･8-6)
|F|≦3|V(F)|-6

- 未解決問題：3次元において Lamanの定理に対応する一般剛性の組合せ的特徴付けを与えよ

- 組合せ剛性理論
  - d-次元剛性の研究（主に特殊ケースの研究）
  - Lamanの定理の一般化
  - (大域剛性の組合せ論的性質の研究)

概要

- Bar-joint Frameworks
  - 基礎
  - Lamanの定理
- 一般剛性マトロイド
  - 組合せ的背景((k, l)-疎性を通して)
  - 剛性マトロイドと組合せ的マトロイド
  - (アルゴリズム)
- 高次元の剛性
  - 3-dimensional Bar-joint Frameworks
  - 特殊構造モデル
- 最近の進展
  - Molecular フレームワーク
  - 周期フレームワーク

第1部 Bar-joint Frameworks

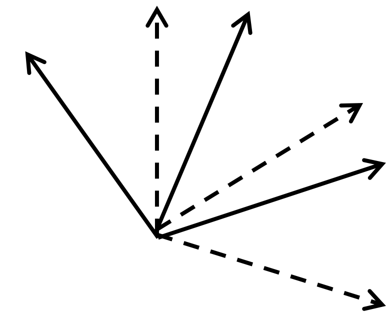
- 基礎
  - 等長変換
  - 剛性
  - 無限小剛性
  - 一般剛性
- Lamanの定理
  - Henneberg構築
  - Lamanの定理

Isometries

- isometry (等長変換) $I: \mathbb{R}^d \to \mathbb{R}^d$

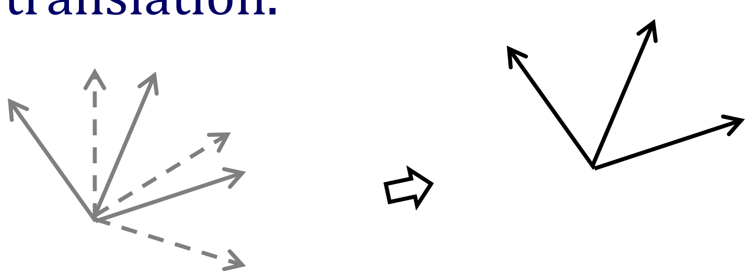
$$\forall p, q \in \mathbb{R}^d, \qquad \|p - q\| = \|Ip - Iq\|$$

- Euclidean group $E(d)$: the set of all isometries of $\mathbb{R}^d$



  - the set of translations
    - T: $p \in \mathbb{R}^d \mapsto p + t \in \mathbb{R}^d$ for some $t \in \mathbb{R}^d$
  - $\cong$ the set of d-dimensional vectors ($\cong \mathbb{R}^d$)
  - the set of rotations and reflections (orthogonal transformations)
    - R: (standard basis of $\mathbb{R}^d$) $\mapsto$ (orthogonal unit-vector basis)
  - $\cong$ the set of $d \times d$ orthogonal matrices (orthogonal group $O(d)$)

- Observation. Any isometry is an orthogonal transformation followed by a translation.



Isometries

- Prop. I is isometry iff it can be written as

$$I: p \in \mathbb{R}^d \mapsto Mp + t \in \mathbb{R}^d$$

for some orthogonal matrix $M \in O(d)$ and $t \in \mathbb{R}^d$

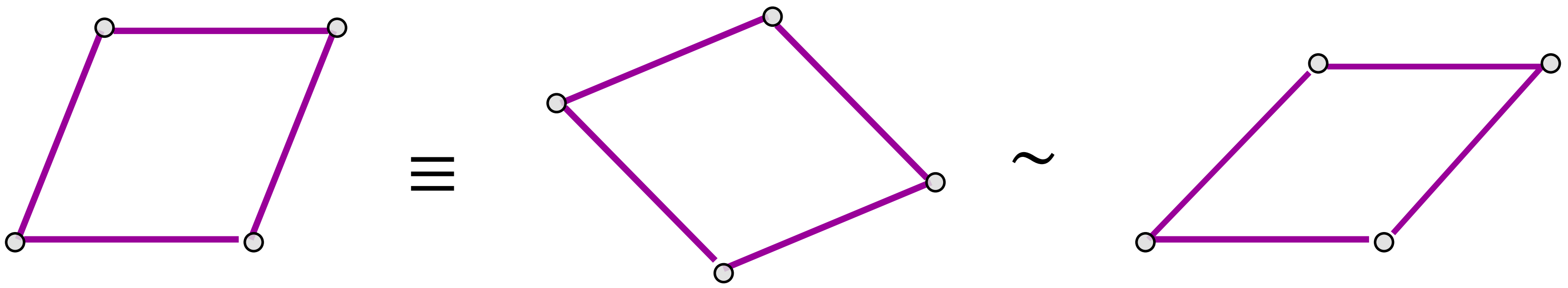
  - "$\Leftarrow$"
    $\langle (Mp+t)-(Mq+t),(Mp+t)-(Mq+t)\rangle =$
    $(p-q)^\top M^\top M(p-q) = (p-q)^\top (p-q) = \langle p-q, p-q\rangle$

- Prop. $O(d)$ forms a smooth $\binom{d}{2}$-dimensional submanifold of $\mathbb{R}^{d^2}$
- Coro. $E(d)$ forms a smooth $\binom{d+1}{2}$-dimensional submanifold of $\mathbb{R}^{d^2+d}$

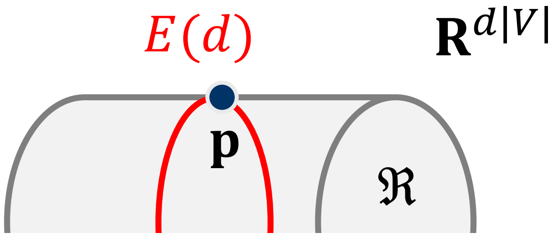
Rigidity (剛性)

- d-dimensional bar-joint framework (G,p)
  - $G = (V,E)$: a graph with $n = |V|$ and $m = |E|$;
  - p: joint-configuration, i.e., $p: V \rightarrow \mathbb{R}^d$ (or $p \in \mathbb{R}^{dn}$)

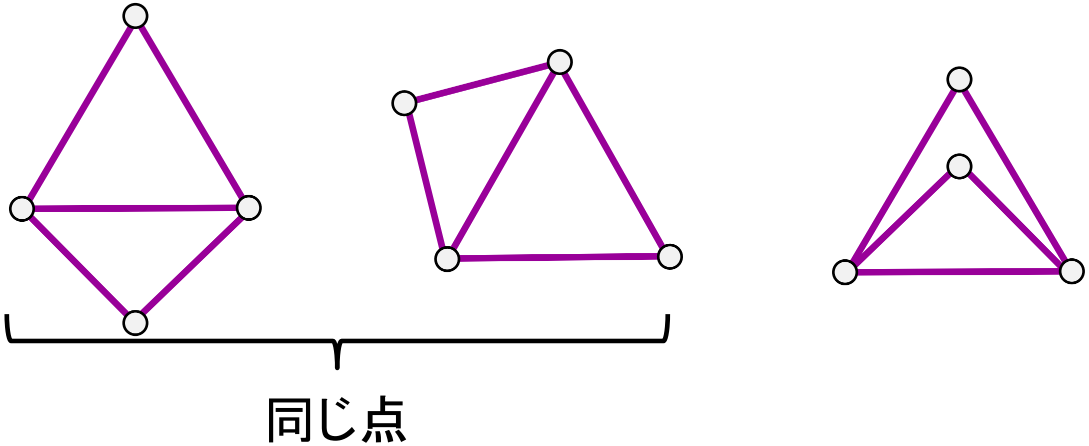
- $(G,p) \sim (G,q) \overset{\text{def}}{\Leftrightarrow} \forall uv \in E, ||p(u) - p(v)|| = ||q(u) - q(v)||$
- $(G,p) \equiv (G,q) \overset{\text{def}}{\Leftrightarrow} \forall u,v \in V \times V, ||p(u) - p(v)|| = ||q(u) - q(v)||$



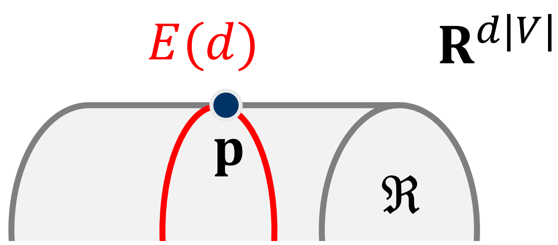
- Realization space $\mathfrak{R} := \{q \in \mathbb{R}^{dn} \mid (G,q) \sim (G,p)\}$
- $\{\, q \in \mathbb{R}^{dn} \mid (G,q) \equiv (G,p)\,\} \cong E(d)$



- (G,p) is rigid (剛堅) $\overset{\text{def}}{\Leftrightarrow}$ $[p]$ is an isolated point in $\mathfrak{R}/E(d)$
  - NP-hard in general (?)
- (G,p) is globally rigid (大域剛堅) $\overset{\text{def}}{\Leftrightarrow}$ $\mathfrak{R}/E(d)$ consists of a single point $[p]$
  - NP-hard in general (Saxe1979)



- Prop. The followings are equivalent for (G, p)
  - (G, p) is rigid
  - p is an isolated point in $\Re / E(d)$
  - $\exists \varepsilon > 0$ s.t. $\forall q$ with $||p - q|| < \varepsilon$, $(G, p) \sim (G, q) \Rightarrow (G, p) \equiv (G, q)$
  - For any continuous path $p_t \in \mathbb{R}^{dn}$ s.t. $p_0 = p$ and $(G, p) \sim (G, p_t)$ for $0 \le t < 1$, $(G, p) \equiv (G, p_t)$ for all $0 \le t < 1$
  - For any smooth path $p_t \in \mathbb{R}^{dn}$ s.t. $p_0 = p$ and $(G, p) \sim (G, p_t)$ for $0 \le t < 1$, $(G, p) \equiv (G, p_t)$ for all $0 \le t < 1$

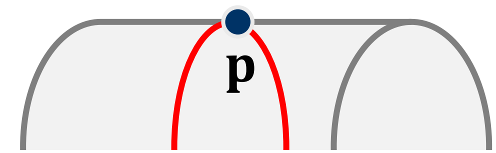


- **Rigidity map** $f_G(p): \mathbb{R}^{dn} \to \mathbb{R}^m$

  $f_G(p) := \left(\cdots, \|\mathrm{p}(v_i) - \mathrm{p}(v_j)\|^2, \cdots \right)$

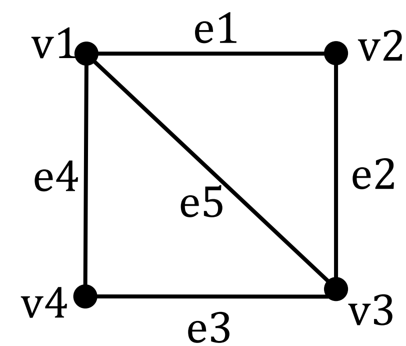
  - (G, p) is rigid $\Leftrightarrow$ $\exists$neighbor U of p s.t.
    $f_{K_n}^{-1}(f_{K_n}(p)) \cap U = f_G^{-1}(f_G(p)) \cap U$

$E(d) = f_{K_n}^{-1}(f_{K_n}(p))$



$\Re = f_G^{-1}(f_G(p))$

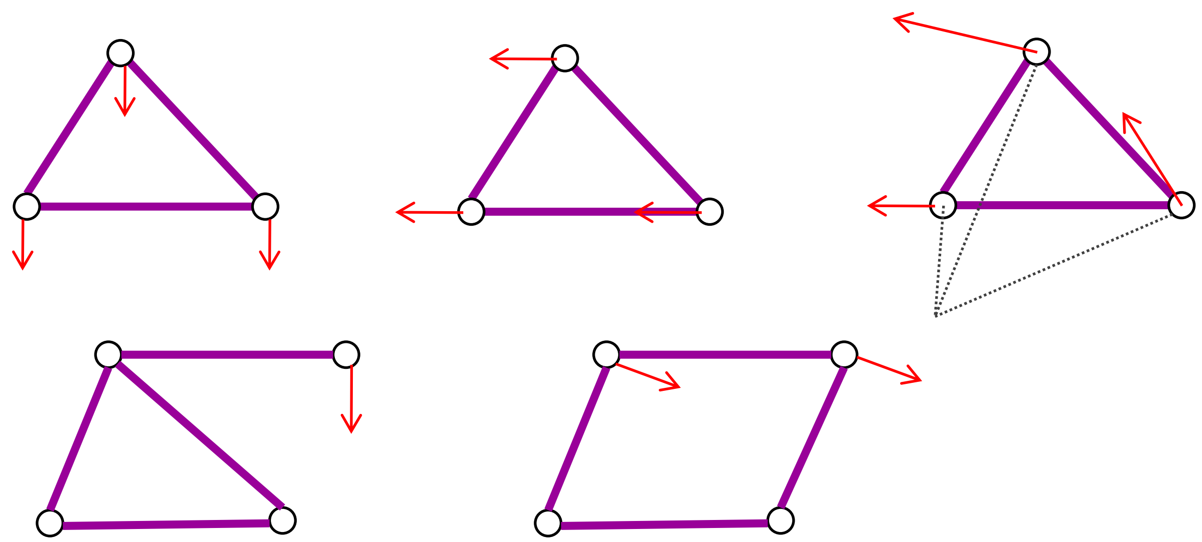
- **Rigidity matrix (剛性行列)** $R(G, \mathrm{p})$
  - The Jacobean of f_G at p
    - $m \times dn$ - matrix



	$v1$	$v2$	$v3$	$v4$	$v1$	$v2$	$v3$	$v4$
$e1$	$p_{x,1} - p_{x,2}$	$p_{x,2} - p_{x,1}$	0	0	$p_{y,1} - p_{y,2}$	$p_{y,2} - p_{y,1}$	0	0
$e2$	0	$p_{x,2} - p_{x,3}$	$p_{x,3} - p_{x,2}$	0	0	$p_{y,2} - p_{y,3}$	$p_{y,3} - p_{y,2}$	0
$e3$	0	0	$p_{x,3} - p_{x,4}$	$p_{x,4} - p_{x,3}$	0	0	$p_{y,3} - p_{y,4}$	$p_{y,4} - p_{y,3}$
$e4$	$p_{x,1} - p_{x,4}$	0	0	$p_{x,4} - p_{x,1}$	$p_{y,1} - p_{y,4}$	0	0	$p_{y,4} - p_{y,1}$
$e5$	$p_{x,1} - p_{x,3}$	0	$p_{x,3} - p_{x,1}$	0	$p_{y,1} - p_{y,3}$	0	$p_{y,3} - p_{y,1}$	0

Infinitesimal motion

- infinitesimal motion (無限小移動) $\dot{p}: V \to \mathbb{R}^d$: a solution of $R(G, p)$
  - Rem. $R(G, p)$ is an $m \times dn$- matrix of a linear system in $\dot{p}$:
    - $\langle \dot{p}(u) - \dot{p}(v), p(u) - p(v) \rangle = 0 \quad \forall e = uv \in E$



- trivial motions : solutions of $R(K_n, p)$
  - i.e., $\dot{p}(v) = Sp(v) + q$ for some skew-symm. matrix S and $q \in \mathbb{R}^d$

Infinitesimal isometry

- Infinitesimal isometry
  - vector field $\boldsymbol{v}: \mathbb{R}^d \to \mathbb{R}^d$, which is obtained by taking the derivative of a smooth path in $E(d)$ at identity.

- Prop. any infinitesimal isometry $\boldsymbol{v}$ can be written by

$$\boldsymbol{v}: p \mapsto Sp + \dot{t}$$

for some skew-symmetric matrix S and $\dot{t} \in \mathbb{R}^d$

  - $\because$ Consider a smooth path $\{(A_s, t_s) \in E(d) \mid 0 \le s \le 1\}$ with $(A_0, t_0) = (I, 0)$
  - Taking the derivative at $s = 0$, we get $(\dot{A}, \dot{t})$, where $\dot{t} \in \mathbb{R}^d$ and $\dot{A} = -\dot{A}^\top$
    - $\because A_s^\top A_s = I,\ \dot{A}^\top A_0 + A_0^\top \dot{A} = \dot{A}^\top + \dot{A} = 0$

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  - vector field $\boldsymbol{v}: \mathbb{R}^d \to \mathbb{R}^d$, which is obtained by taking the derivative of a smooth path in $E(d)$ at identity.

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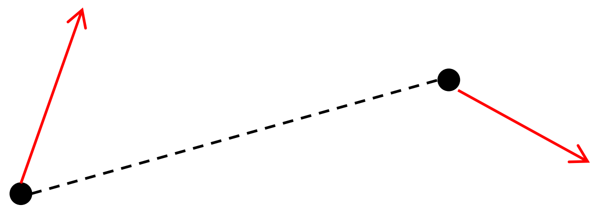
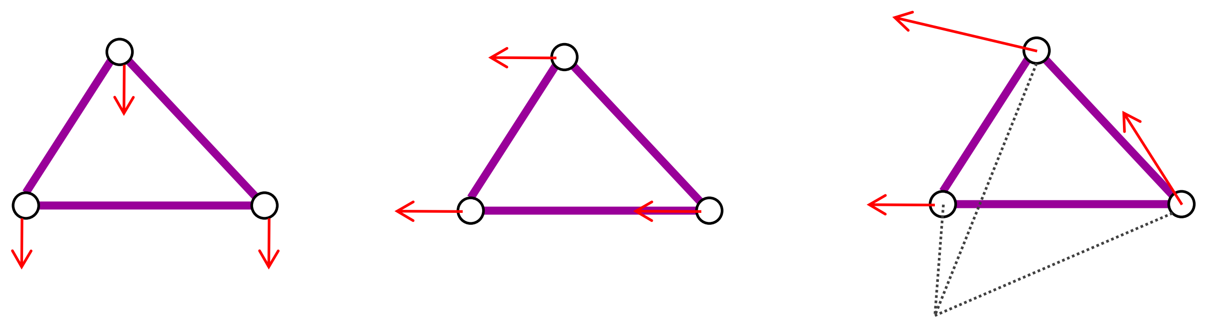
for some skew-symmetric matrix S and $\dot{t} \in \mathbb{R}^d$

- Coro. The set of infinitesimal isometries forms a $\binom{d+1}{2}$-dimensional linear space
- Coro. $\forall p, q \in \mathbb{R}^d, \langle \boldsymbol{v}(p) - \boldsymbol{v}(q), p - q \rangle = 0$

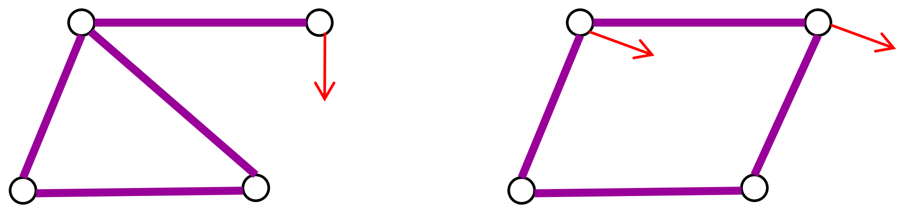


- infinitesimal motion $\dot{p}: V \to \mathbb{R}^d$: a solution of $R(G,p)$
- trivial motions of (G,p): solutions of $R(K_n, p)$
  - an infinitesimal isometry, restricted to p
  - i.e., $\dot{p}(v) = Sp(v) + q$ for some skew-symm. matrix S and $q \in \mathbb{R}^d$
  - Prop. dim {trivial motions} $= \binom{d+1}{2}$ (if p affinely spans $\mathbb{R}^d$)

trivial:



nontrivial:

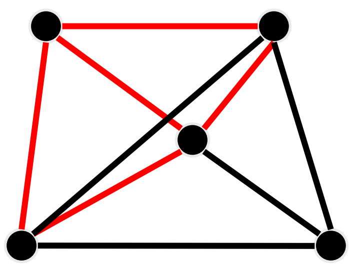


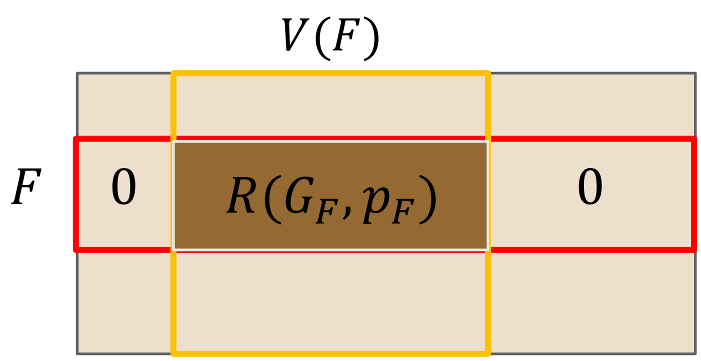
Infinitesimal Rigidity (無限小剛性)

- (G, p) is infinitesimally rigid (無限小剛堅) $\overset{\text{def}}{\Leftrightarrow}$ every motion is trivial
- $\dim \{\text{trivial motions}\} = \binom{d+1}{2}$
- $\Rightarrow \dim \ker R(G, p) \geq \binom{d+1}{2}$
- Prop. (G, p) is infinitesimally rigid $\Leftrightarrow$ $\text{rank}\, R(G, p) = d|V| - \binom{d+1}{2}$

Maxwell's rule (for infinitesimal rigidity)

- If (G, p) is infinitesimally minimally rigid, then
  - $|E| = d|V| - \binom{d+1}{2}$
  - $\forall F \subseteq E \; with \; |V(F)| \geq d, \; |F| \leq d|V(F)| - \binom{d+1}{2}$
- Proof
  - infinitesimal rigidity $\Rightarrow$ rank $R(G,p) = d|V| - \binom{d+1}{2}$
  - minimality $\Rightarrow$ row independence
  - $\forall F \subseteq E$, consider the sub-framework (G_F, p_F) induced by F

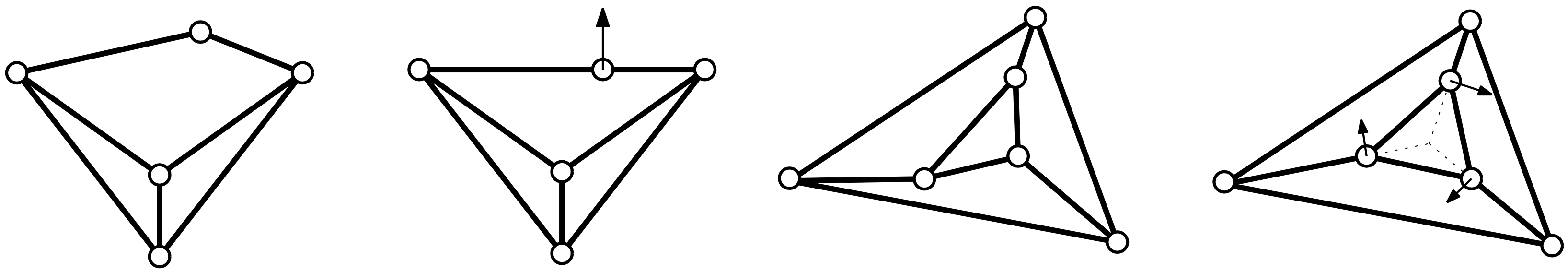




$$|F| + \dim \ker R(G_F, p_F) = d|V(F)|$$

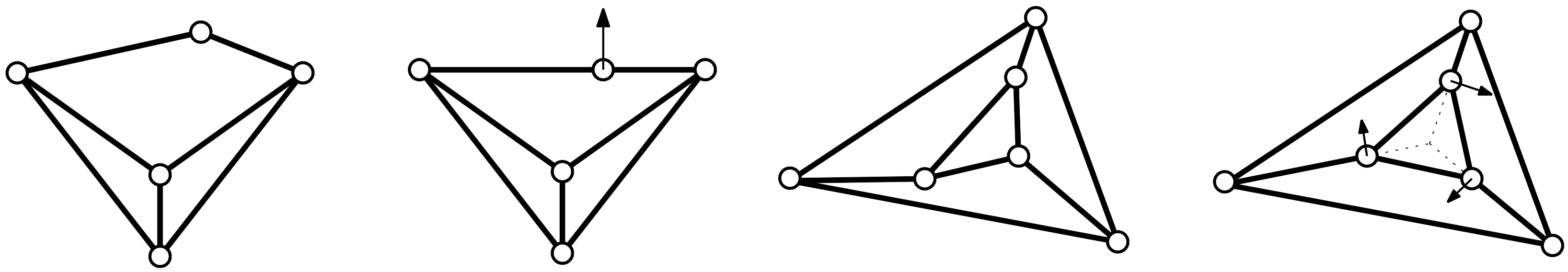
Infinitesimal Rigidity v.s. Rigidity

- Prop. (Asimow and Roth 78) Infinitesimal rigidity ⇒ Rigidity

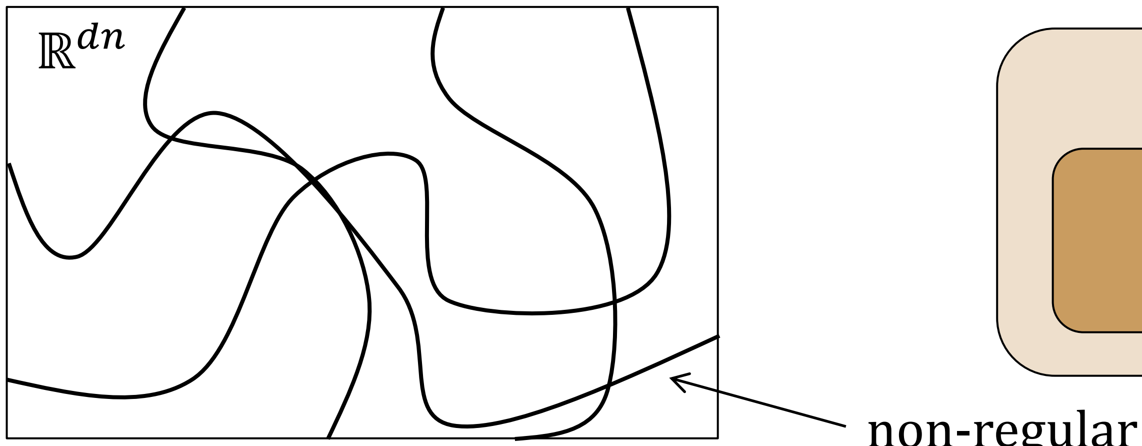


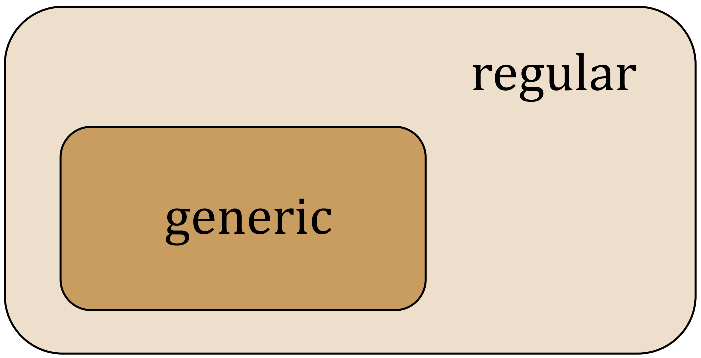
- Remarks
  - (i) Infinitesimal rigidity ⇍ Rigidity
  - (ii) Infinitesimal rigidity depends on joint configurations

- (ii) Infinitesimal rigidity depends on joint configurations



- p is regular $\overset{\text{def}}{\Longleftrightarrow}$ $p \in \cap_{G \subseteq K_n} \operatorname{argmax}_q \ \operatorname{rank} R(G, q)$
  - Infinitesimal rigidity does not depend on p, if p is restricted to regular configurations
  - The set of regular configurations is a dense open subset of $\mathbb{R}^{d|V|}$, which contains the set of generic configurations





Generic Rigidity (一般剛性)

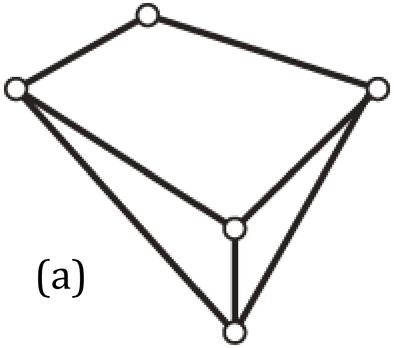
- (ii) Infinitesimal rigidity $\not\Leftarrow$ Rigidity

- Prop. (Asimow & Roth 78)

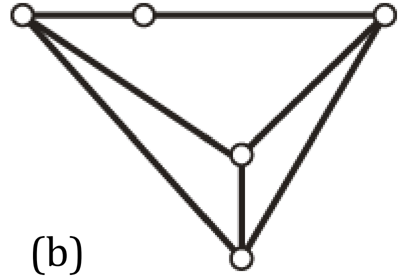
  If p is regular, then (G,p) is rigid iff rank $R(G,p) = d|V| - \binom{d+1}{2}$

- Generic rigidity (一般剛性) : rigidity on generic joint configurations p
  - A property of a graph
- We can say "(G,p) is generically rigid (一般剛堅)" or simply "G is rigid (剛堅)".

例題



(a)



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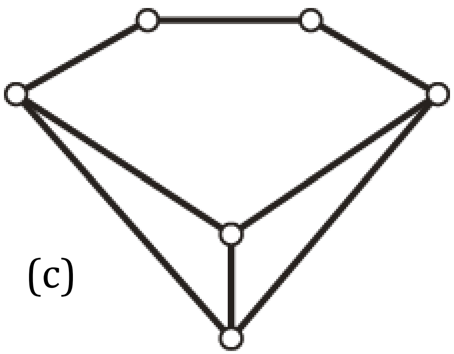

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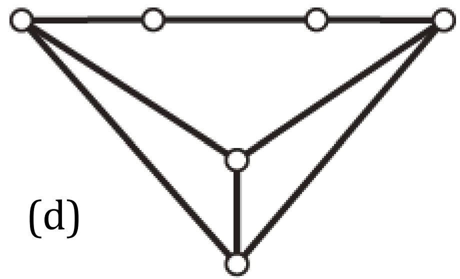
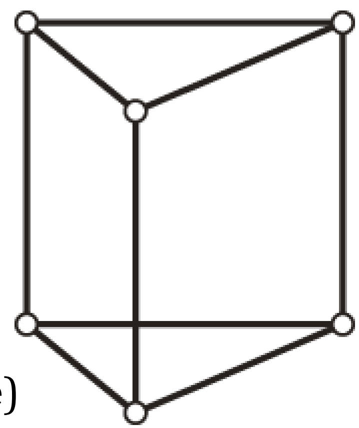
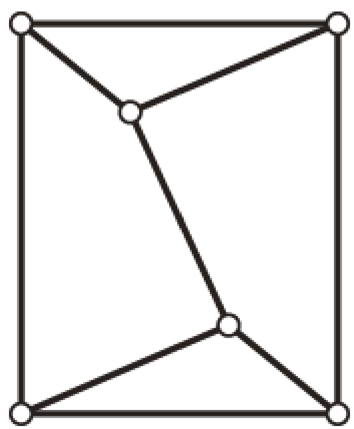


(d)

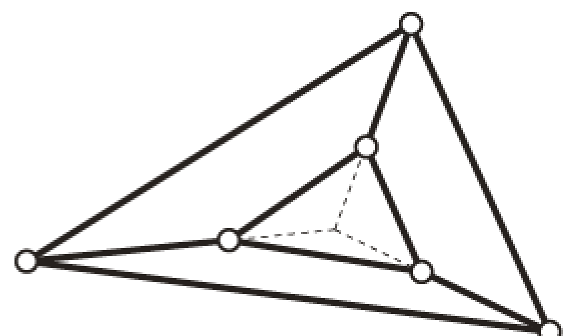
	剛堅	無限小剛堅	一般剛堅
(a)	○	○	○
(b)	○	×	○
(c)	×	×	×
(d)	○	×	×
(e)	×	×	○
(f)	○	○	○
(g)	○	×	○



(e)



(f)



(g)

第1部　Bar-joint Frameworks

- 基礎
  - 等長変換
  - 剛性
  - 無限小剛性
  - 一般剛性
- Lamanの定理
  - Henneberg構築
  - Lamanの定理

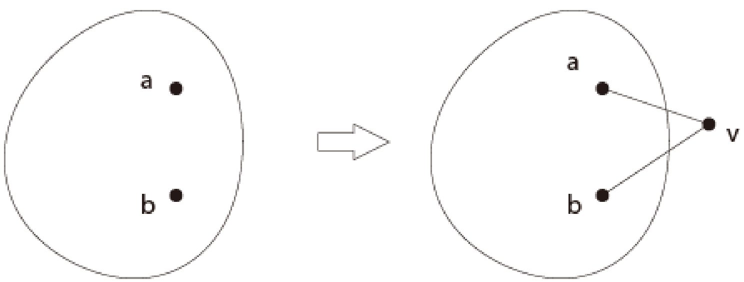
- Lamanの定理 (1970) 任意の一般的 p に対し, (G, p) が2次元極小剛堅 $\Leftrightarrow$
  - $|E| = 2|V| - 3$
  - $\emptyset \neq \forall F \subseteq E, |F| \leq 2|V(F)| - 3$

  } ラーマングラフ

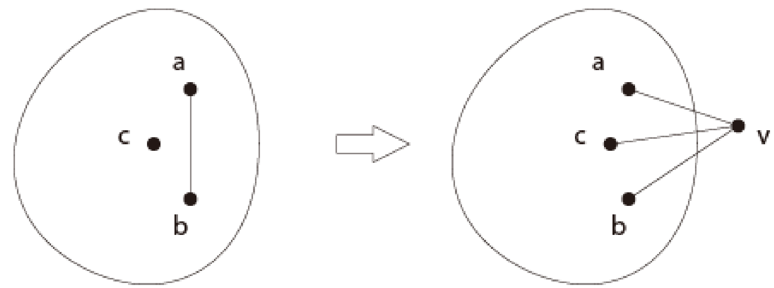
- 必要性 --- Maxwellの条件とAsimow&Rothの定理より
- 十分性 --- ラーマングラフGに対し, (G, p)が剛堅となるpを一つ見つければ良い
  - 構築法による証明
    - 小さなラーマングラフから新たに少し大きなラーマングラフを構築する操作を定義
    - 任意のラーマングラフが(K_1から)これらの操作の繰り返しで構築可能であることを証明
    - これらの操作が一般剛性を保持することを証明

Henneberg 構築

- 0-extension (Henneberg I) & 1-extension (Henneberg II)

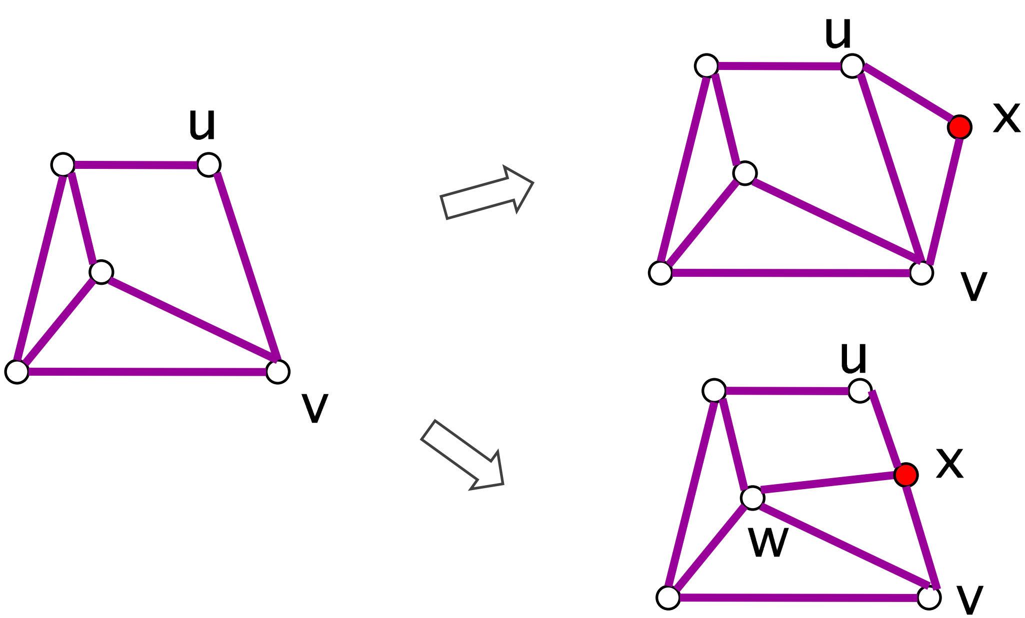


0-extension



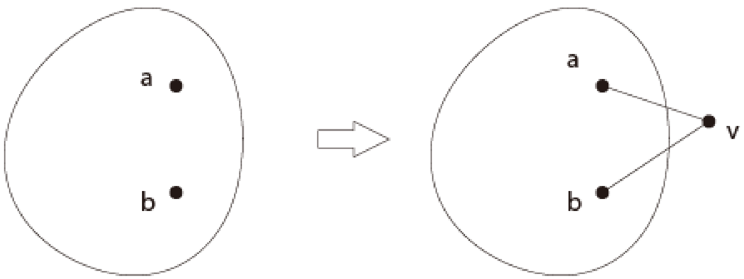
1-extension

- これらの操作はラーマンの条件を保持する

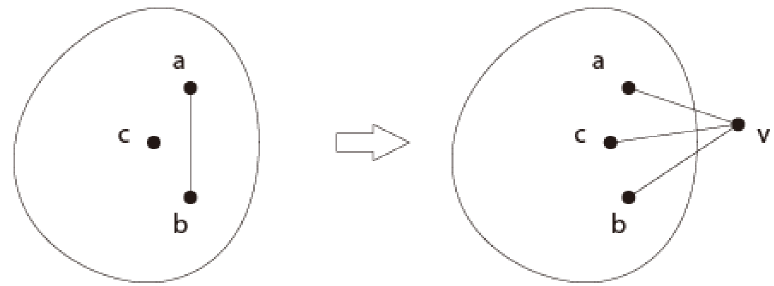


Henneberg 構築

- 0-extension (Henneberg I) & 1-extension (Henneberg II)



0-extension



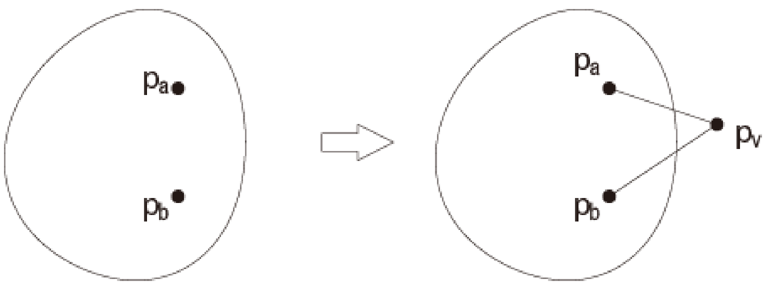
1-extension

- これらの操作はラーマンの条件を保持する

- Theorem(Tay&Whiteley 85)：
  - G がラーマングラフ ⇔ G がK_2 から0/1-extensions の繰り返しで構築可能 (演習問題)

WhiteleyによるLamanの定理の証明

- |V|に関する帰納法
  - ケース1：*G*が*G'*から0-extensionで構築される



$M(G,p) =$

	v	a	b	
va	$p_v - p_a$	$p_a - p_v$	0	0
vb	$p_v - p_b$	0	$p_b - p_v$	0
	0	$M(G', p')$		

  - $\operatorname{rank}\begin{bmatrix} p_v - p_a \\ p_v - p_b \end{bmatrix} = 2 \Leftrightarrow p_v$ is no on the line $p_a p_b$
  - If p_v is not on the line $p_a p_b$

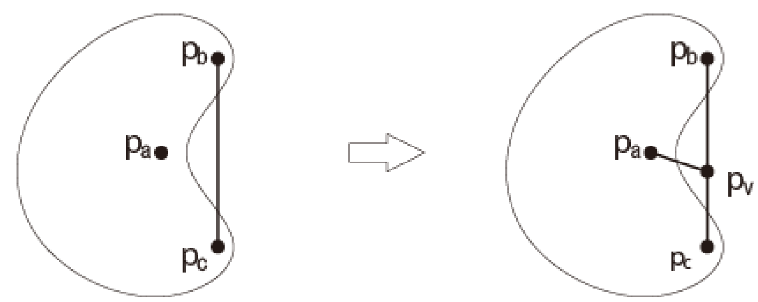
$$\operatorname{rank} R(G,p) \geq \operatorname{rank}\begin{bmatrix} p_v - p_a \\ p_v - p_b \end{bmatrix} + \operatorname{rank} R(G', p')$$

$$= 2 + 2|V| - 5 = 2|V| - 3$$

ケース2：G がG' から1-extensionで構築される

$R(G,p)=$

	v	a	b	c	
va	$p_v - p_a$	$p_a - p_v$	0	0	0
vb	$p_v - p_b$	0	$p_b - p_v$	0	0
vc	$p_v - p_c$	0	0	$p_c - p_v$	0
	0	$*$			



	v	a	b	c	
va	$p_v - p_a$	$p_a - p_v$	0	0	0
vb	$p_b - p_c$	0	$p_c - p_b$	0	0
vc	$p_b - p_c$	0	0	$p_c - p_b$	0
	0	$*$			

	v	a	b	c	
va	$p_v - p_a$	$p_a - p_v$	0	0	0
vb	$p_b - p_c$	0	$p_c - p_b$	0	0
vc	0	0	$p_b - p_c$	$p_c - p_b$	0
	0	$R(G',p')$			

$$\text{rank } R(G,p) \geq \text{rank} \begin{bmatrix} p_v - p_a \\ p_b - p_c \end{bmatrix} + \text{rank } R(G',p')$$

$$= 2 + 2|V| - 5 = 2|V| - 3 \qquad \square$$