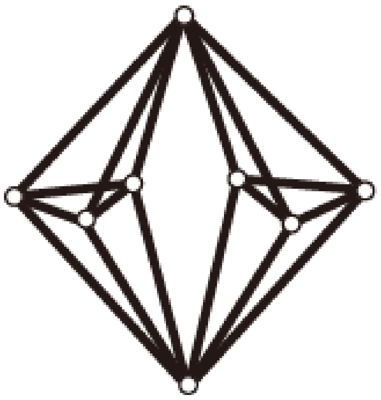
第3部　高次元一般剛性マトロイド

Motivation

- 未解決問題: グラフ上で組合せ的に定義されるマトロイドで $\mathcal{R}_3(G)$と同型なものがあるか??
- 未解決問題: $\mathcal{R}_3(G)$のランクを多項式時間で決定的に計算できるか??

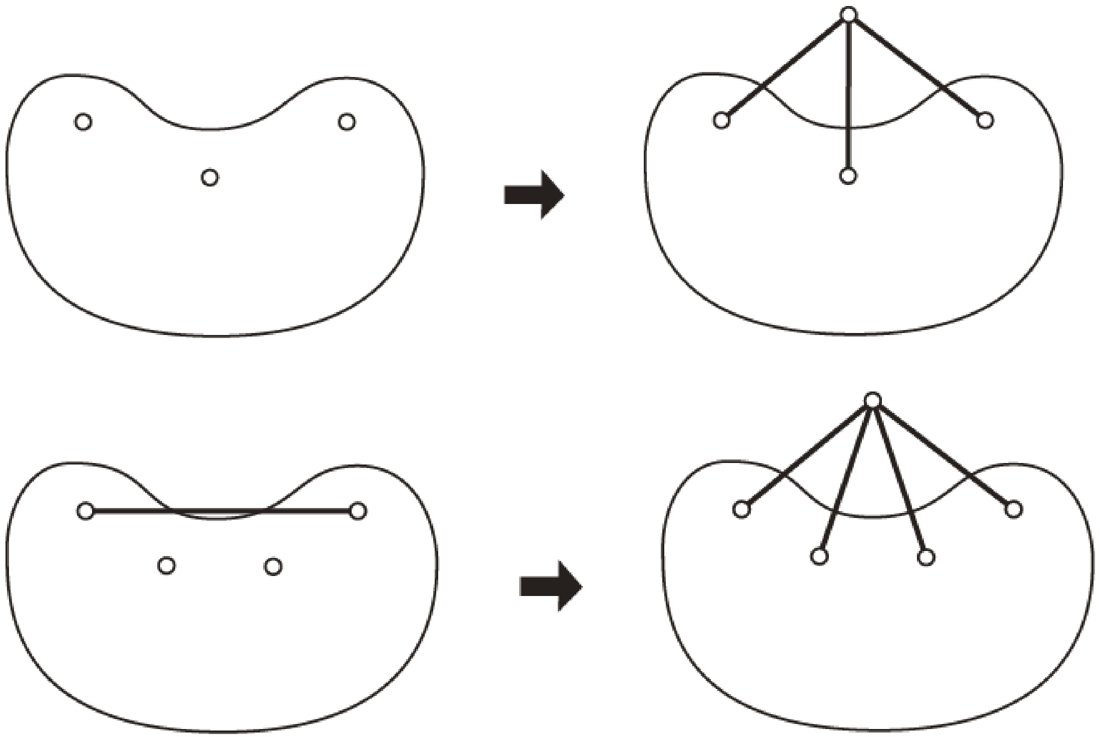
- 3-dimensional Bar-joint Frameworks
  - Inductive constructions
  - 3次元のランク関数表現予想
- 特殊構造モデル
  - より一般のモデル
  - Body-bar Frameworks (Tayの定理)
  - Body-hinge Frameworks (Tay-Whiteleyの定理)



Extensions

- 0-extension/1-extensionはd次元剛性を保持する

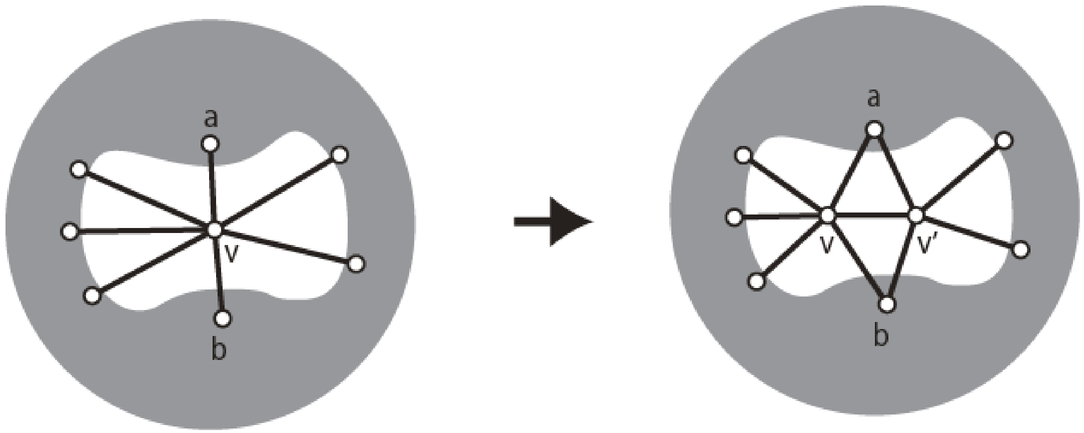
$d = 3$



- 3次元Maxwell条件($m' \leq 3n' - 6$)を満たすグラフの最小次数は5以下
  - 剛堅ならば最小次数は3以上

Glűckの定理

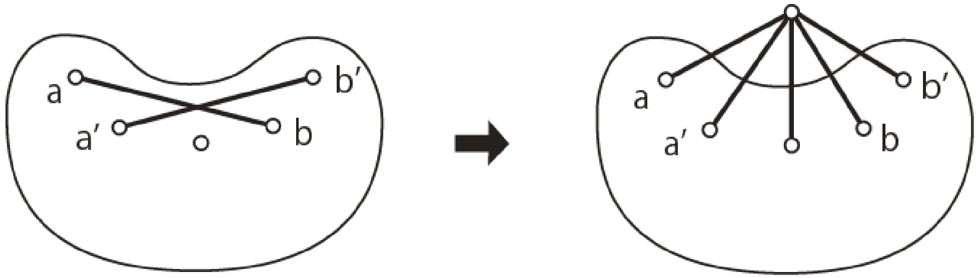
- Vertex Splitting



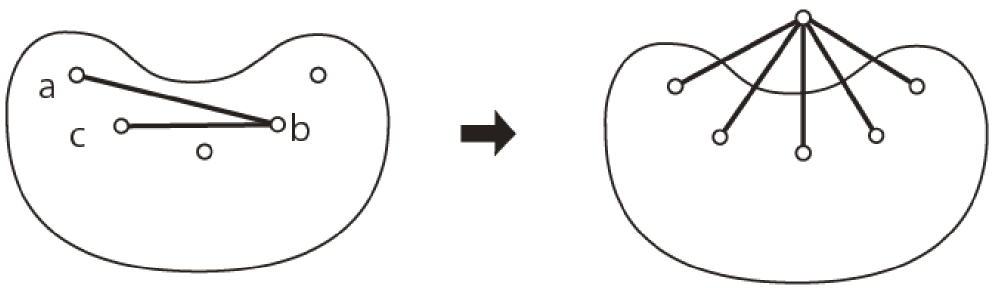
- Theorem (Whiteley1990) Vertex splitting は 3次元一般剛性を保持
- Glűckの定理(Glűck1975) 極大平面グラフは3次元極小剛堅
  - $m = 3n - 6$であるから，剛堅ならば極小剛堅
  - 極大平面グラフはvertex splittingの繰り返しによって構築可能

Tay-Whiteleyの構築法予想

- Vertex splittingは必ず三角形を作ってしまう.
- X-replacement

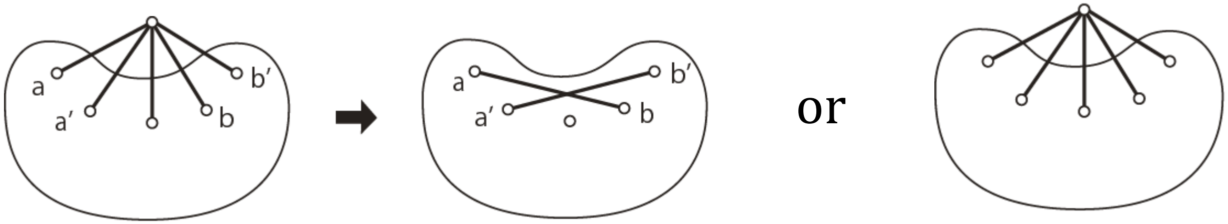


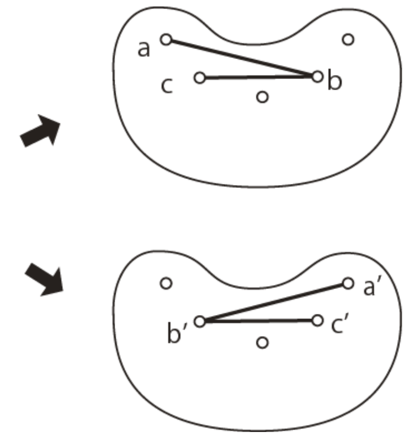
- V-replacement



Tay&Whiteley構築法予想

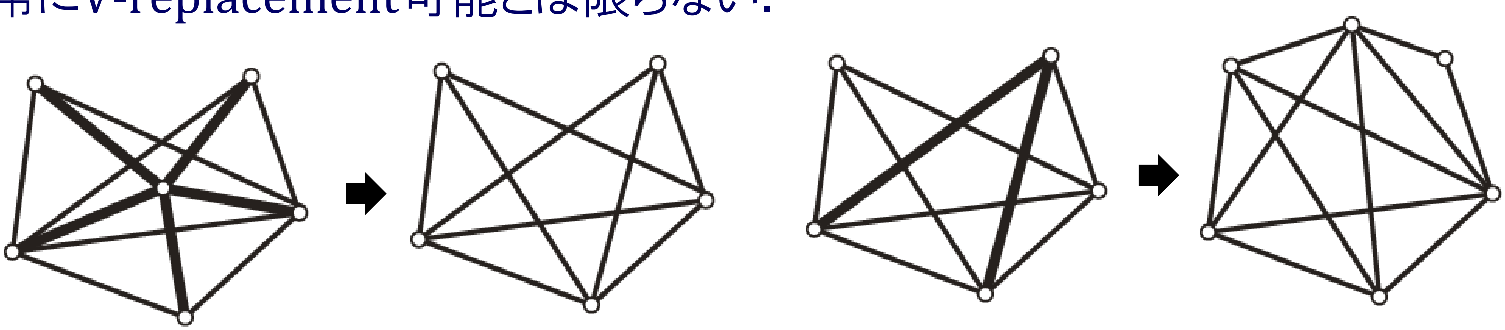
- Theorem(Tay&Whiteley1985). 次数5の点vにおいて以下の2つのいずれかが成立つ





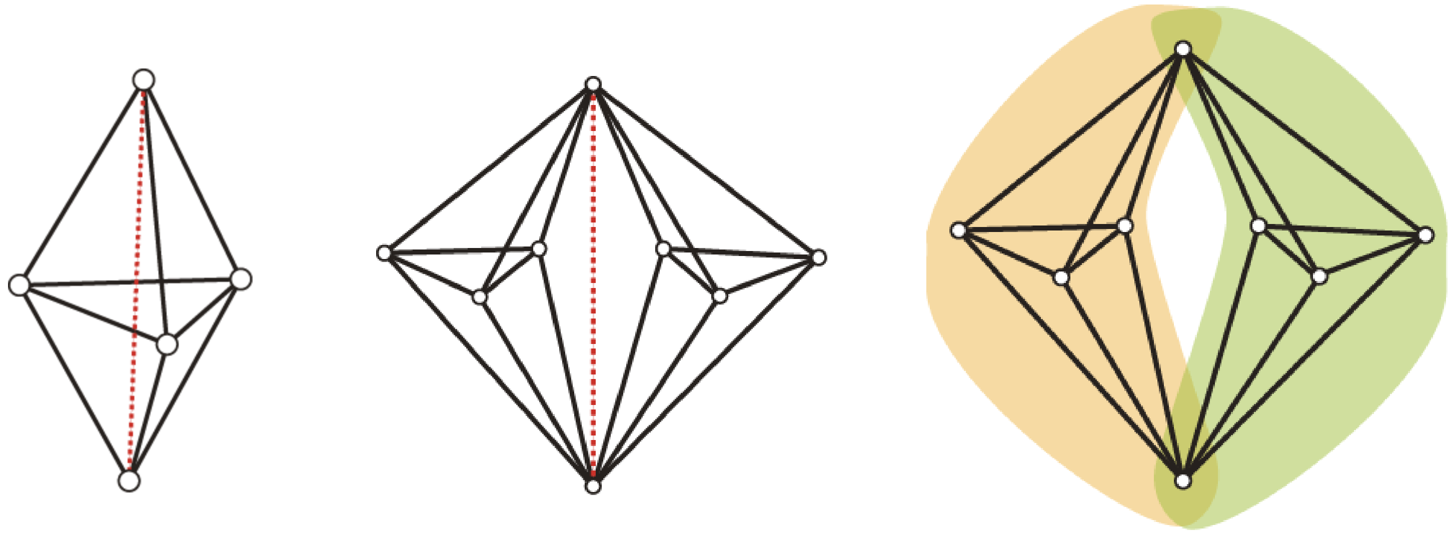
- Conjecture(Tay&Whiteley1985)
  - X-replacementは3次元一般剛性を保持
  - 上記右図のような2個の剛堅グラフが存在するならば，V-replacementは一般剛性を保持

- Remark.
  - X-replacementだけでは不十分.
  - 常にV-replacement可能とは限らない.

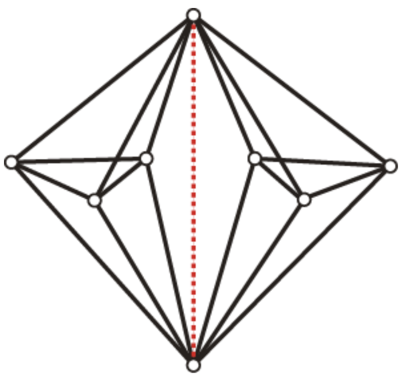
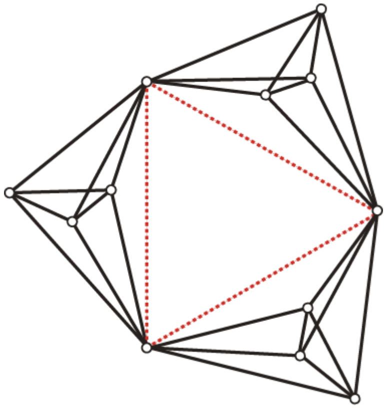
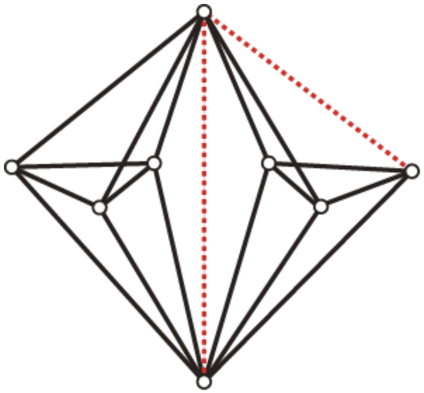


Implied Edges, Rigid Components, and Rigid Clusters

- Implied edges: $\mathrm{cl}(E) \setminus E$
- Rigid component: a maximal $X \subseteq V$ s.t. $G[X]$ is rigid
- Rigid cluster: a maximal $X \subseteq V$ s.t $\hat{G}[X]$ is a complete subgraph, where $\hat{G} = \big(V, \mathrm{cl}(E)\big)$

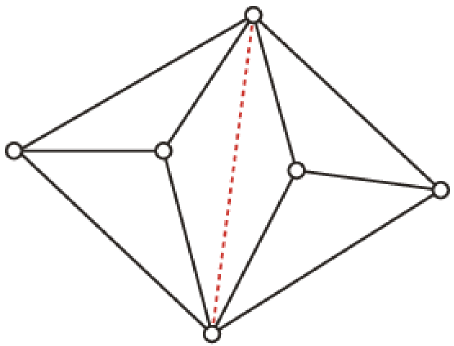
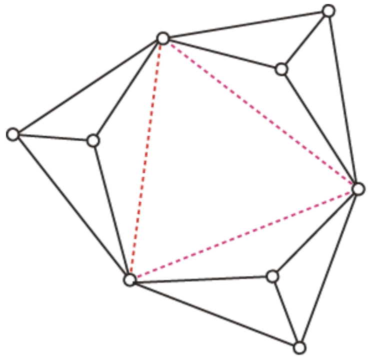
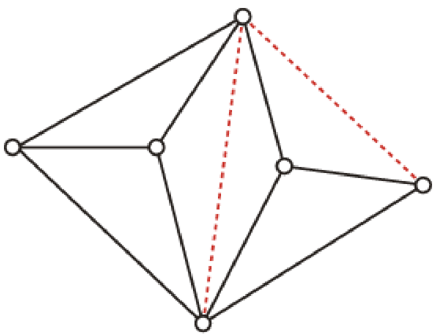


- 3次元(以上)の場合,
  - rigid componentとrigid clusterは一致しない!!

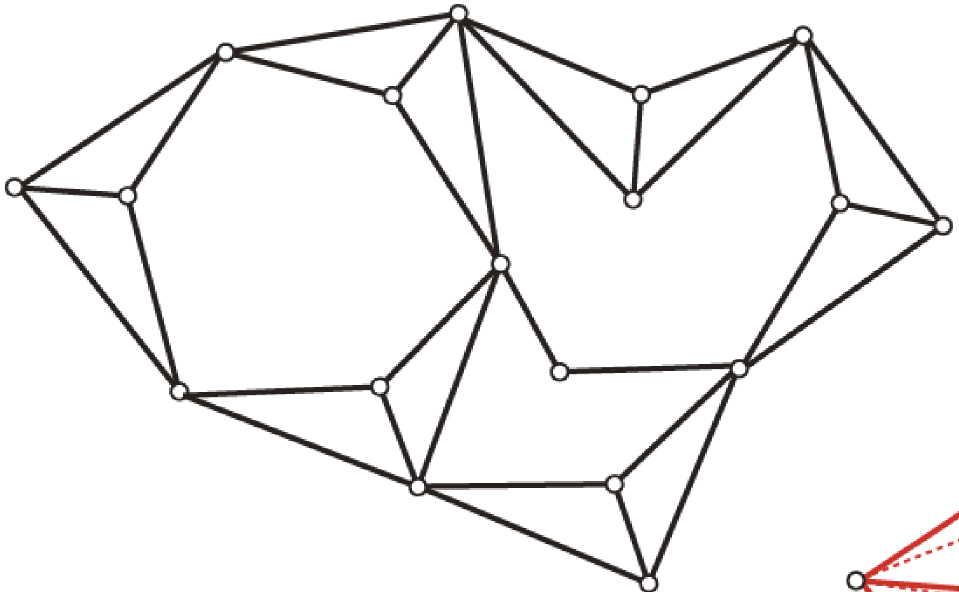




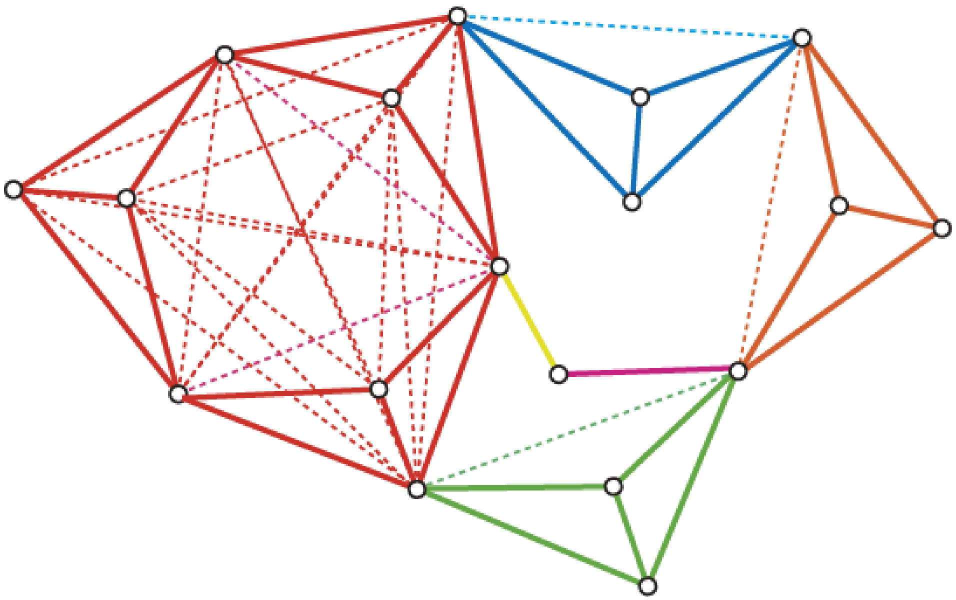
rigidな部分であっても，その部分に制限するとrigidとは限らない

- 2次元の場合
  - 2つのrigid clusterは2頂点以上共有しない
  - ⇒　rigid clusterはrigid component

$d = 2$ におけるimplied edgesとrigid clusters (rigid components)





2次元のランク関数表現

- E の cover: Vの部分集合族$\{X_1,\ldots,X_k\}$ で $E = \bigcup_i E(X_i)$
- t-thin cover $\{X_1,\ldots,X_k\}$: $|X_i \cap X_j| \le t$

- 定理(Lovász&Yemini82) 2次元一般剛性マトロイド$\mathcal{R}_2(G)$のランクは

$$\min\left\{\sum_i (2|X_i|-3) \,\middle|\, 1-\text{thin cover } \{X_1,\ldots,X_k\}\right\}$$

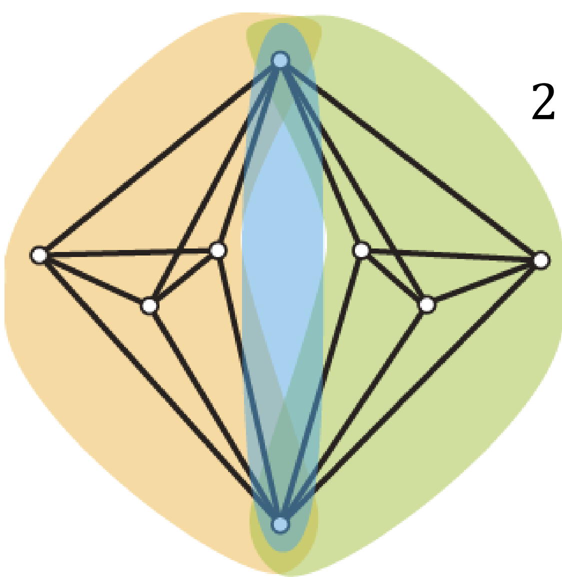
  - ∵ $\mathcal{R}_2(G)$は$f_{2,3}$で誘導されるマトロイド．そのランクは $\min\{\sum_i (2|V(F_i)|-3) \mid \{F_1,\ldots,F_k\}\text{: a partition of } V\}$
  - minimizerの各F_iは，$r(F_i) = 2|V(F_i)|-3$なので$G[F_i]$は剛堅
  - kを出来るだけ大きくとると，minimizerの各F_iはrigid clusterX_iに誘導される辺集合
  - rigid clusterは2点以上共有しないので, $\{X_1,\ldots,X_k\}$は1-thin

3次元のランク関数表現予想

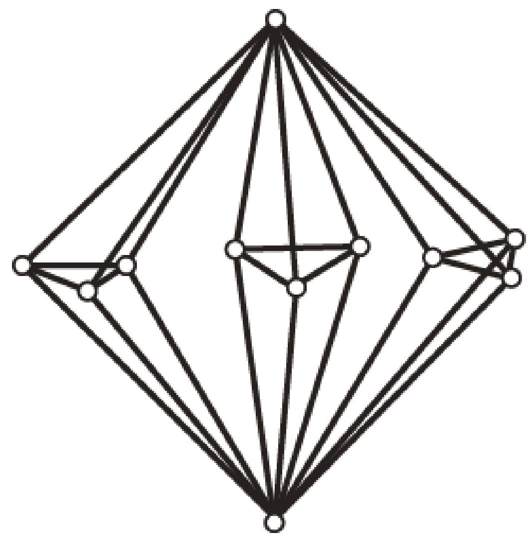
- 予想(Dress, Drieding & Haegi 1983) $\mathcal{R}_3(G)$のランクは

$$\min\left\{\sum_i f'_{3,6}(X_i) - \sum_{uv\in V\times V} (d_{\mathcal{X}}(uv) - 1) \;\middle|\; 2 - \text{thin cover } \mathcal{X} = \{X_1, \dots, X_k\}\right\}$$

  - 2-thin cover $\mathcal{X} = \{X_1, \dots, X_k\} : |X_i \cap X_j| \leq 2$
  - $f'_{3,6}(X) = \begin{cases} 1, & |X| = 2 \\ 3|X| - 6, & otherwise \end{cases} \quad (X \subseteq \mathrm{V})$
  - $d_{\mathcal{X}}(uv) = |\{\, i \mid \{u, v\} \subseteq X_i \,\}|$



$2(3 \cdot 5 - 6) - 1 = 17$



$3(3 \cdot 5 - 6) - 2 = 25$

Jackson & Jordán (2005)によって反証

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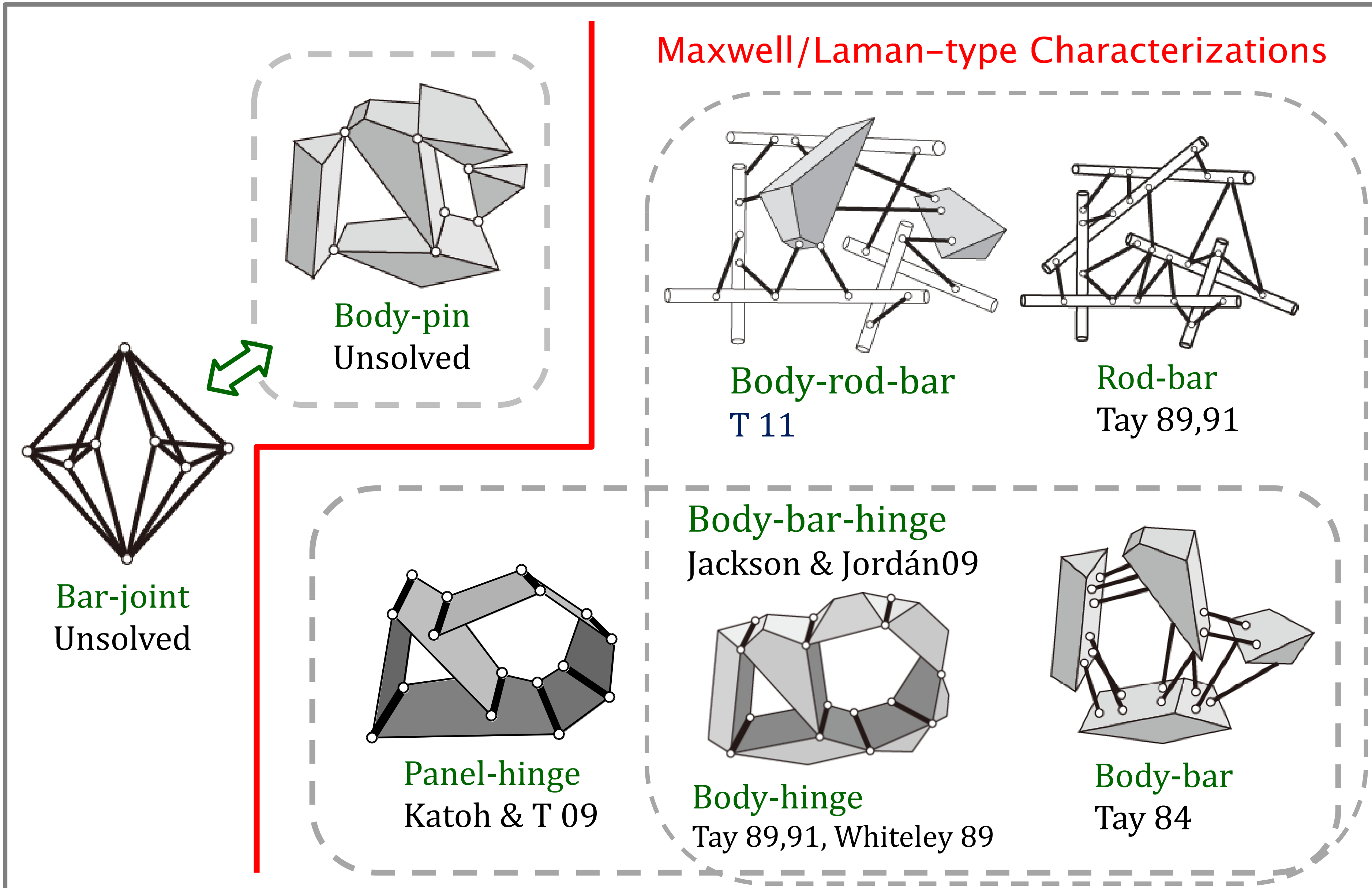
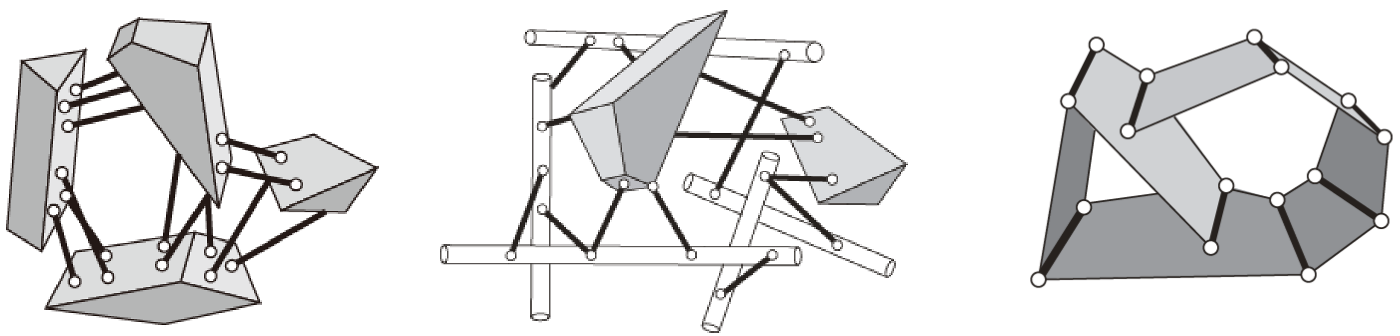
Special Structural Models

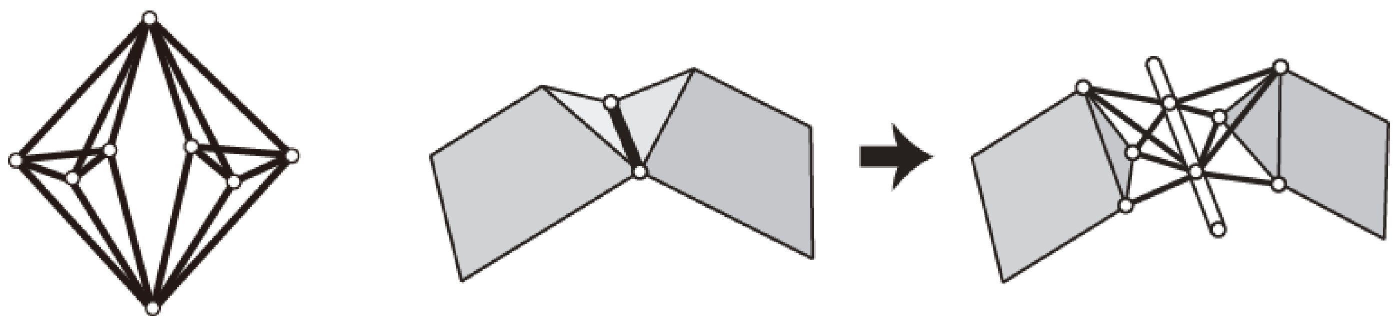


- Special solvable cases for $\mathcal{R}_3(G)$:
  - [Gluck 75] Triangulated sphere
  - [Whiteley 84] Complete bipartite graphs
  - [Nevo 04] K_5-minor free graphs
  - [Jackson and Jordán 05] Sparse graphs ($|F| \leqq 5/2|V(F)|-7/2$)
  - [Katoh and T 11] The squares of graphs (molecular graphs)

Frameworks: More generally

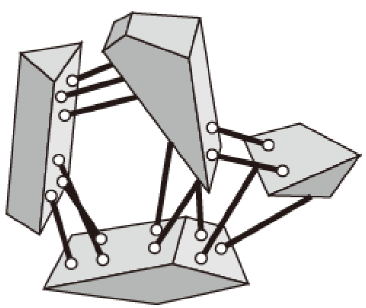
- Every discrete structure consists of objects linked by bars
  - Objects
    - body (3-d-body)
    - panel (2-d-body)
    - rod (1-d-body)
    - joint (0-d-body)

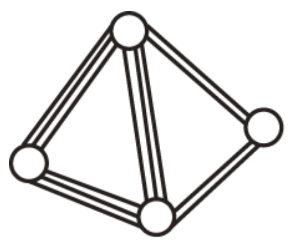


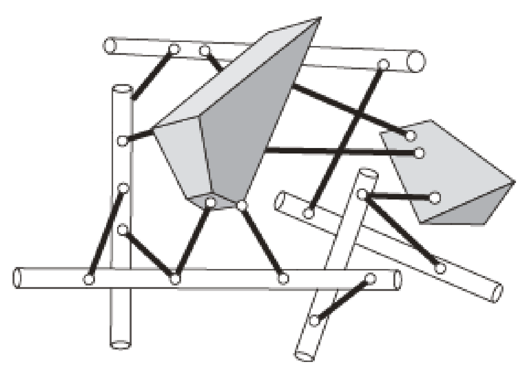


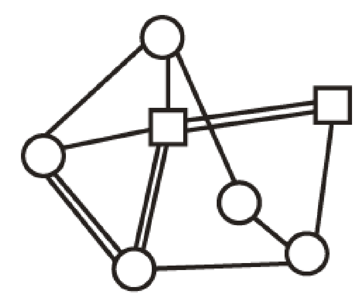
Maxwell's condition: More generally

- framework $(G, \mathbf{p}, \mathbf{q})$
  - $G = (V, E)$: graph
    - vertex ⇔ object
    - edge ⇔ bar
  - $\mathbf{p}$: an object-configuration (a mapping on V)
  - $\mathbf{q}$: a bar-configuration; $e \in E \mapsto$ a line-segment in $\mathbb{R}^3$









Maxwell's condition: More generally

- The degree of freedoms of each object in $\mathbb{R}^3$
  - Objects --- degree of freedoms
    - 3-d-body 6 d.o.f.
    - panel (2-d-body) 6 d.o.f.
    - rod (1-d-body) 5 d.o.f.
    - joint (0-d-body) 3 d.o.f.
- Let $\mathrm{dof}: V \to \mathbb{Z}$,

$$\mathrm{dof}(v) = \begin{cases} 6 & (\text{if } v \text{ corresponds to a body or a panel}) \\ 5 & (\text{if } v \text{ corresponds to a rod}) \\ 3 & (\text{if } v \text{ corresponds to a joint}) \end{cases}$$

- Let $f_{\boldsymbol{dof}}: 2^E \to \mathbb{Z}$,

$$f_{\boldsymbol{dof}}(F) := \sum_{v \in V(F)} \mathrm{dof}(v) - 6 \qquad (F \subseteq E)$$

- Maxwell's condition. If a framework $(G, \mathbf{p}, \mathbf{q})$ is minimally rigid,
  - $|E| = f_{dof}(E)$
  - $\forall F \subseteq E$ with $f_{\boldsymbol{dof}}(F) > 0$, $|F| \le f_{\boldsymbol{dof}}(F)$

準備:外積

- d次元ベクトル空間V
- Vの要素の外積$a \wedge b$とは形式的な積で以下を満たすもの
  - $a \wedge b = -b \wedge a$
  - $(a + b) \wedge c = a \wedge c + b \wedge c$
- $\Lambda^2 V$:　$a \wedge b\ (a, b \in V)$によって張られるベクトル空間
- 基底$e_1, \dots, e_d$に対し,
  - $\{e_i \wedge e_j \mid 1 \le i < j \le d\}$は線形独立
  - $\Lambda^2 V = span\{e_i \wedge e_j \mid 1 \le i < j \le d\}$
  - $dim\, \Lambda^2 V = \binom{d}{2}$
- $V = \mathbb{R}^d$, 標準基底をとって座標化:　$a = (a_1, \dots, a_d), b = (b_1, \dots, b_d)$
  - $a \wedge b = \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \dots, \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \dots, \begin{vmatrix} a_{d-1} & a_d \\ b_{d-1} & b_d \end{vmatrix} \right) \in \mathbb{R}^{\binom{d}{2}}$

準備:外積

- d次元ベクトル空間V
- Vの要素のk階外積$a_1 \wedge a_2 \wedge \cdots \wedge a_k$とは形式的な積で以下を満たすもの
  - $a_1 \wedge \cdots \wedge a_i \wedge a_{i+1} \wedge \cdots \wedge a_k = -a_1 \wedge \cdots \wedge a_{i+1} \wedge a_i \wedge \cdots \wedge a_k$
  - $a_1 \wedge \cdots \wedge (a_i + a_i{}') \wedge \cdots \wedge a_k$
    $= a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_k + a_1 \wedge \cdots \wedge a_i{}' \wedge \cdots \wedge a_k$
- $\wedge^k V$: $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ $(a_i \in V)$によって張られるベクトル空間
- 基底$e_1, \ldots, e_d$に対し,
  - $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge a_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq d\}$は線形独立
  - $\wedge^k V = span\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge a_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq d\}$
  - $dim \wedge^k V = \binom{d}{k}$
- $V = \mathbb{R}^d$, 標準基底をとって座標化: $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ は各ベクトルを行とする$k \times d$行列の$k \times k$小行列式を並べてできる$\binom{d}{k}$次元ベクトル

準備：プリュッカー座標

- グラスマン多様体$Gr(k,\mathbb{R}^d)$: $\mathbb{R}^d$内のk次元線形部分空間の集合
- プリュッカー埋込み: $p^*: Gr\left(k,\mathbb{R}^d\right) \to \mathbb{P}\left(\wedge^k \mathbb{R}^d\right)$

$$X \mapsto [v_1 \wedge v_2 \wedge \cdots \wedge v_k]$$

($\{v_1, .. v_k\}$はXの基)

- $p^*(X)$： Xのプリュッカー座標
- 例： $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3)$

  - $p^*(span\{p,q\}) = [p \wedge q] = \left[\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} : \begin{vmatrix} p_1 & p_3 \\ q_1 & q_3 \end{vmatrix} : \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}\right] \in \mathbb{P}^2$

準備:プリュッカー座標

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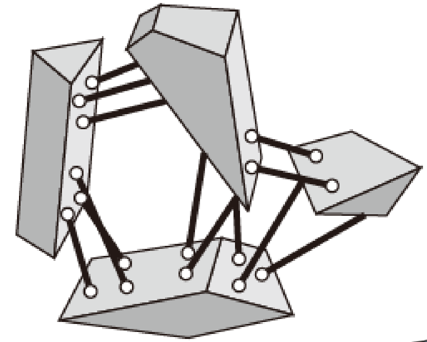
($\{v_1, .. v_k\}$はXの基)

- $p^*(X)$: Xのプリュッカー座標

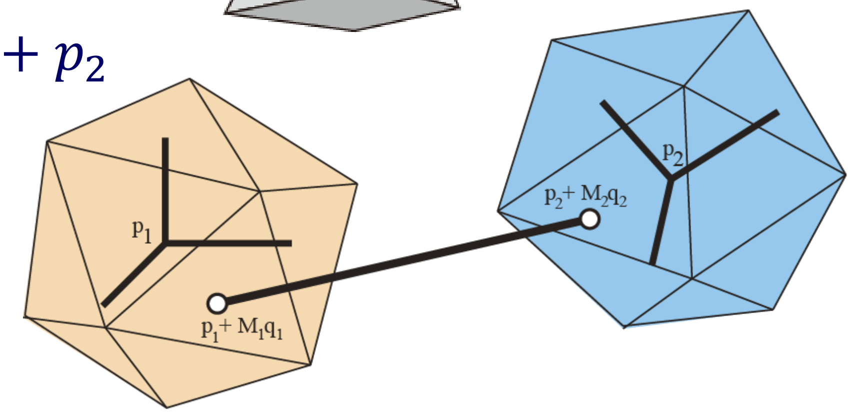
- p^*は単射

- $\to$ $Gr(k,\mathbb{R}^d)$を$\mathbb{P}^{\binom{d}{k}-1}$内の部分空間とみなすとき, $Gr(k,\mathbb{R}^d)$は$\mathbb{P}^{\binom{d}{k}-1}$内の2次の射影多様体
  - $Gr(2,\mathbb{R}^4) = \{[p_1:\ldots:p_6] \in \mathbb{P}^5 \mid p_1p_4 - p_2p_3 + p_3p_6 = 0\}$

Body-bar Frameworks



- a body : (M, p), $M \in O(d)$, $p \in \mathbb{R}^d$
- a bar : $(q_1, q_2) \in \mathbb{R}^d \times \mathbb{R}^d$
  - connecting $M_1 q_1 + p_1$ and $M_2 q_2 + p_2$



- Bar-constraint
  - $\langle p_2 + M_2 q_2 - p_1 - M_1 q_1 , p_2 + M_2 q_2 - p_1 - M_1 q_1 \rangle = \ell^2$
  - Differentiate it,
    $\langle p_2 + M_2 q_2 - p_1 - M_1 q_1,\ \dot{p}_2 + \dot{M}_2 q_2 - \dot{p}_1 - \dot{M}_1 q_1 \rangle = 0$
  - By setting $M_i = I$ and $p_i = 0$,
    $$\langle q_2 - q_1, \dot{p}_2 + A_2 q_2 - \dot{p}_1 - A_1 q_1 \rangle = 0$$
    with skew-symmetric matrices $A_i = \dot{M}_i$
  - $\langle q_2 - q_1,\ \dot{p}_2 - \dot{p}_1 \rangle + \langle q_2 - q_1, A_2 q_2 - A_1 q_1 \rangle = 0$

- $\langle q_2 - q_1,\ \dot{p}_2 - \dot{p}_1\rangle + \langle q_2 - q_1, A_2 q_2 - A_1 q_1\rangle = 0$
- If we identify A_i with a vector $\omega_i \in \mathbb{R}^{\binom{d}{2}}$,

$$\langle q_2 - q_1, A_2 q_2 - A_1 q_1\rangle = \langle (q_2 - q_1) \wedge q_2, \omega_2\rangle - \langle (q_2 - q_1) \wedge q_1, \omega_1\rangle$$
$$= \langle q_2 \wedge q_1, \omega_2 - \omega_1\rangle$$

In general, $\langle a, Ab\rangle = \langle a \wedge b, \omega\rangle$

- $\langle q_2 - q_1,\ \dot{p}_2 - \dot{p}_1\rangle + \langle q_2 - q_1, A_2 q_2 - A_1 q_1\rangle =$
  $\langle q_2 - q_1,\ \dot{p}_2 - \dot{p}_1\rangle + \langle q_2 \wedge q_1, \omega_2 - \omega_1\rangle$
  $= \langle (q_2, 1) \wedge (q_1, 1), (\omega_2, \dot{p}_2) - (\omega_1, \dot{p}_1)\rangle = 0$

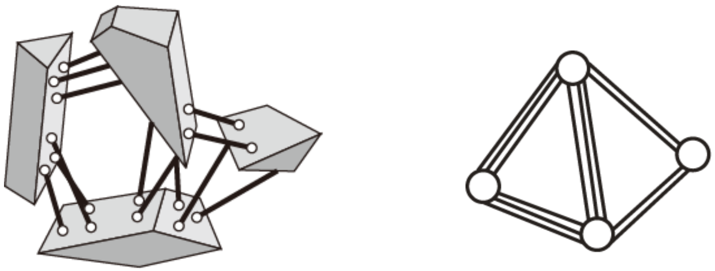
homogeneous coordinate

screw motion of the body i

Plucker coordinate of the line
(in the corresponding projective space)

Body-bar Frameworks

- d-dimensional body-bar framework: (G, b)
  - $G = (V, E)$: graph,
  - b: bar-configuration, $e \in E \mapsto [b_e] \in Gr\left(2, \mathbb{R}^{d+1}\right) \subseteq \mathbb{P}^{\binom{d+1}{2}-1}$
    - $e \in E \ \mapsto$ Plűcker coordinate of a bar (a line) in $\mathbb{P}^d$



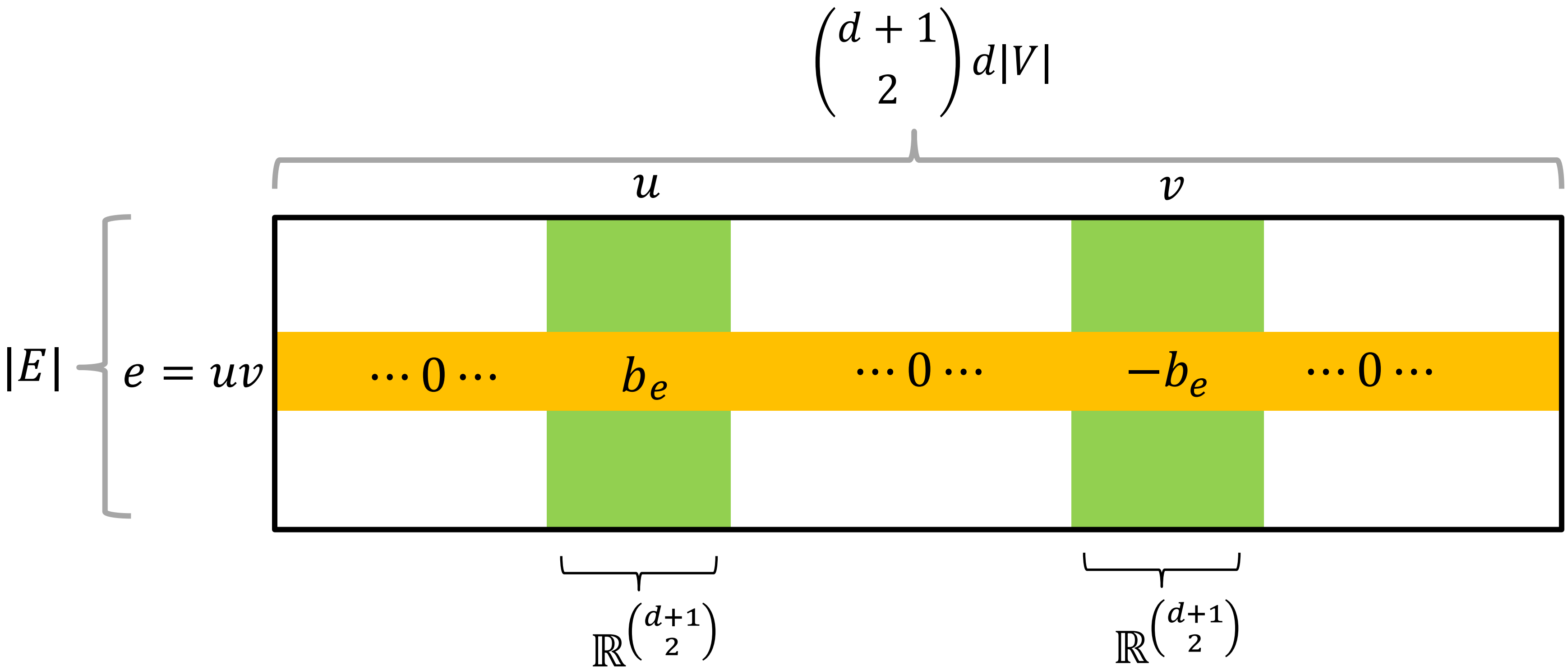
- infinitesimal motion, $m: v \in V \mapsto m_v \in \mathbb{R}^{\binom{d+1}{2}} \left(= \Lambda^{d-1}\, \mathbb{R}^{d+1}\right)$ with

$$\langle b_{uv}\, m_u - m_v \rangle = 0 \ \ \forall uv \in E$$

- trivial motion: $\forall u, v \in V,\ m_u = m_v$
- (G, b) is infinitesimally rigid $\overset{\text{def}}{\Longleftrightarrow}$ every motion is trivial

Rigidity Matrix in Body-bar Model

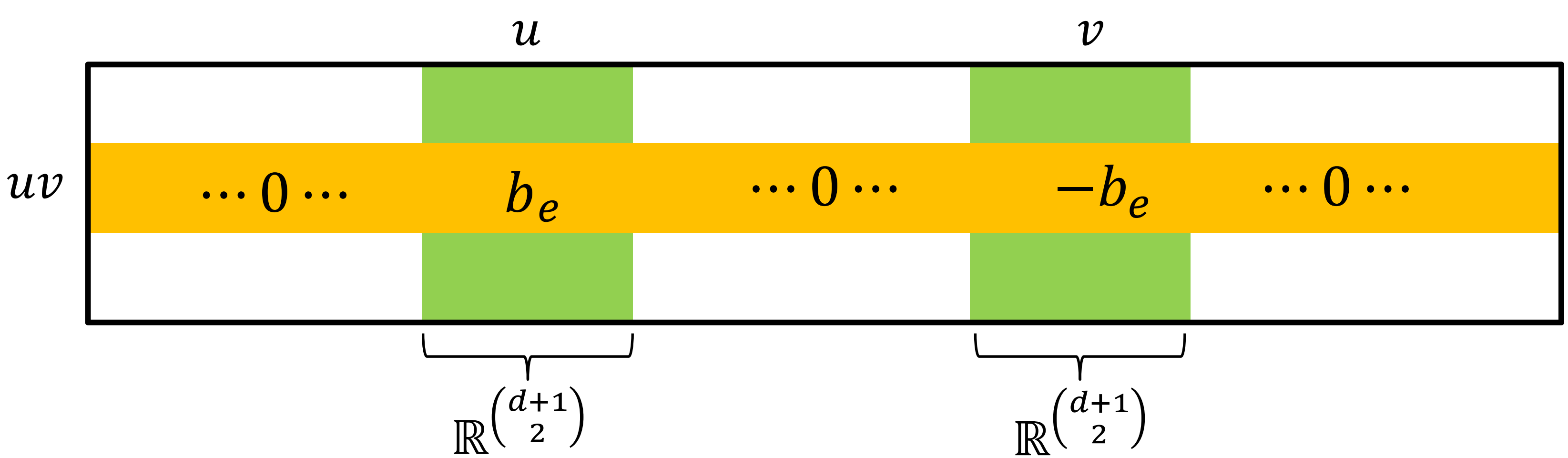
- Rigidity matrix $R(G,b)$: $|E| \times \binom{d+1}{2}|V|$-matrix



- $[b_e]$ is restricted to $Gr(2, \mathbb{R}^{d+1})$
- Generic rigidity matroid in the body-bar model
- (G,b) is infinitesimally rigid $\Leftrightarrow$ the rank is $\binom{d+1}{2}|V| - \binom{d+1}{2}$

Tay's Theorem (1984)

- **Theorem.** The generic rigidity matroid in the body-bar model is $\binom{d+1}{2}\mathcal{G}(G)$
- Namely, for generic b, the followings are equivalent:
  - rank $R(G,b) = \binom{d+1}{2}|V| - \binom{d+1}{2}$
  - $|E| = \binom{d+1}{2}|V| - \binom{d+1}{2}$ *and* $\forall F \subseteq E, |F| \leq \binom{d+1}{2}|V(F)| - \binom{d+1}{2}$
  - E can be partitioned into $\binom{d+1}{2}$ spanning trees

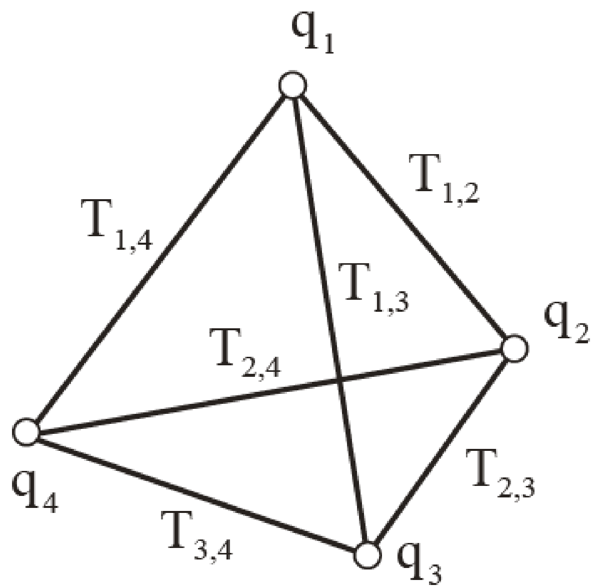


Proof of Tay's theorem by Whiteley

- (Necessity) Maxwell's condition
- (Sufficiency)
  - E can be partitioned into $\binom{d+1}{2}$ spanning trees $T_{1,2}, T_{1,3} \dots, T_{d,d+1}$
  - Take any $d+1$ points $[q_1], \dots, [q_{d+1}]$ s.t. $\{q_1, \dots, q_{d+1}\}$ is linearly independent
  - Then, $\{q_i \wedge q_j \mid 1 \le i < j \le d+1\}$ is linearly independent.
  - Define a realization $b: E \to Gr(2, \mathbb{R}^{d+1})$ by

$$b(e) = [q_i \wedge q_j] \quad \text{if} \quad e \in T_{i,j}$$

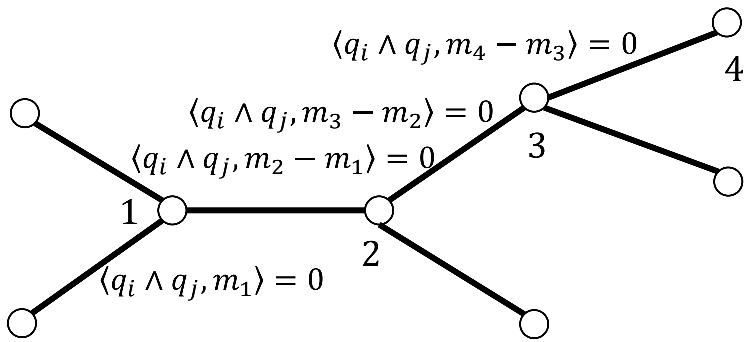
  - Then, (G, b) is infinitesimally rigid !!

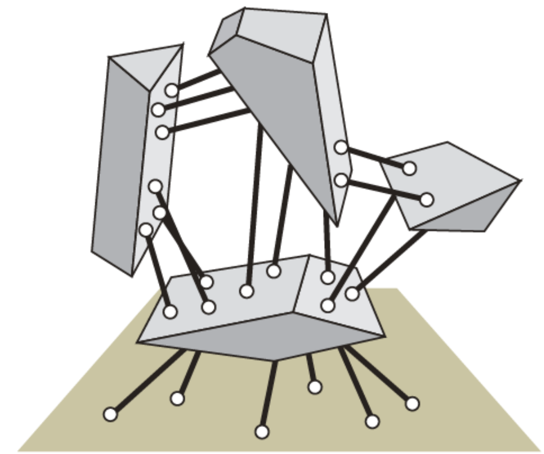


- Define a realization $b: E \to Gr(2, \mathbb{R}^{d+1})$ by

$$b(e) = [q_i \wedge q_j] \quad \text{if} \quad e \in T_{i,j}$$

- We may assume that the body of v_1 is anchored by $\binom{d+1}{2}$ bars to eliminate trivial motions.
- We choose the bars $\{[q_i \wedge q_j] \mid 1 \leq i < j \leq d+1\}$ for this purpose.
- Namely, $\forall i, j,\ \langle q_i \wedge q_j, m_{v_1} \rangle = 0$.
- Consider any motion $m: v \mapsto m_v$
- For each $v_k \in V$, $T_{i,j}$ has a unique path.

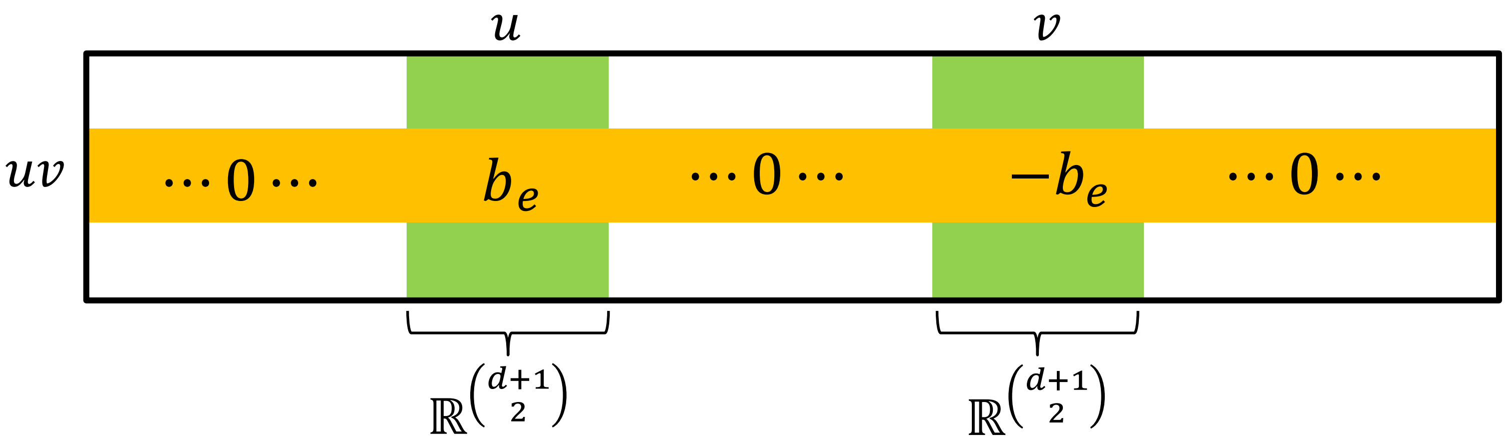




- Summing up the equations for the edges along this path, $\langle q_i \wedge q_j, m_{v_k} \rangle = 0$ for every i, j.
- Thus, $m_{v_k} = 0$. ■

Tay's theorem (more geometrically)

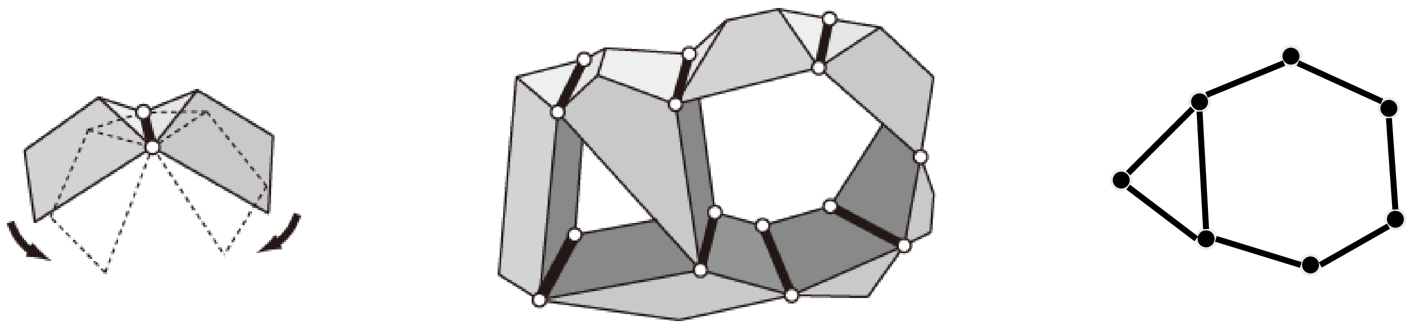
- Theorem. The generic rigidity matroid in the body-bar model is $\binom{d+1}{2}\mathcal{G}(G)$



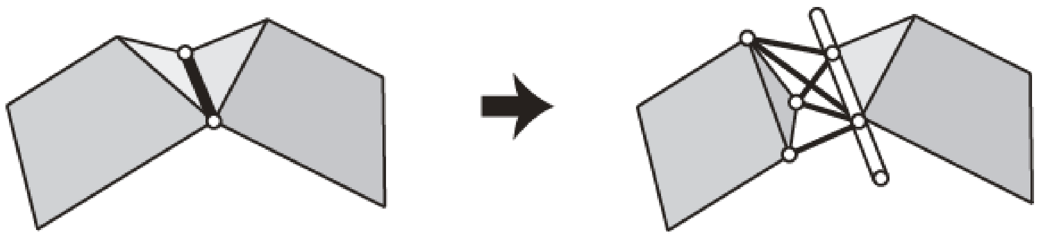
- $[b_e]$ が$Gr(2, \mathbb{R}^{d+1})$ に制限されている
- $\binom{d+1}{2}\mathcal{G}$は各$e \in E$ に対し, $W_e := A_e^1 \oplus A_e^2 \oplus \cdots \oplus A_e^{\binom{d+1}{2}}$から一般的に代表ベクトルを選んで得られる線形マトロイド
- 各代表ベクトルを$Gr(2, \mathbb{R}^{d+1})$からとってこれれば良い
- $Gr\left(2, \mathbb{R}^{d+1}\right)$は2次の多様体. 一方、非一般性の条件は線形

Body-hinge Frameworks

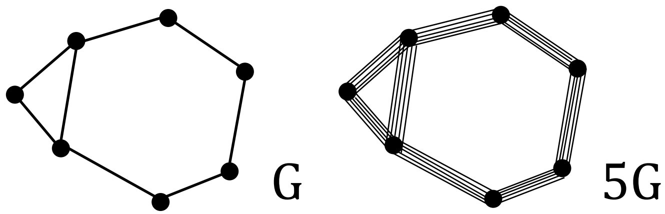
- d-dimensional body-hinge framework: (G, h)
  - $G = (V, E)$: graph,
  - h: hinge-configuration, $e \in E \mapsto [h_e] \in Gr(d-1, \mathbb{R}^{d+1}) \subseteq \mathbb{P}^{\binom{d+1}{2}-1}$
    - $e \in E \mapsto$ Plűcker coordinate of a hinge (a line) in $\mathbb{P}^d$



- Each hinge can be replaced with five bars incident to the hinge !!



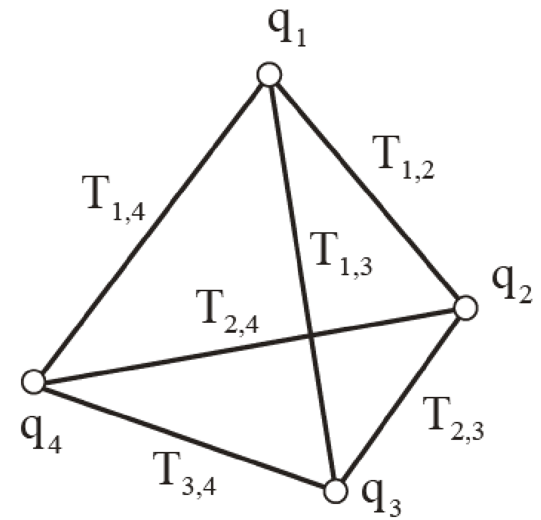
- A body-hinge framework (G, h) is equivalent to the body-bar framework $((D-1)G, b)$, where $D = \binom{d+1}{2}$ and b satisfies
  - for the $D-1$ copies $e_1, \dots, e_{D-1}$ of e, $\{b_{e_1}, \dots, b_{e_{D-1}}\}$ is a basis of the orthogonal complement of $h_e\mathbb{R}$



- The infinitesimal rigidity of (G, h) can be defined in terms of $((D-1)G, b)$
- The rigidity matrix has the size $(D-1)|E| \times D|V|$

- Tay-Whiteley's theorem (Whiteley 88, Tay 89, 91)
  - For any generic hinge-configuration h, (G, h) is infinitesimally rigid if and only if $(D-1)G$ contains D edge-disjoint spanning trees.

- (Necessity): by Maxwell's rule
- (Sufficiency):
  - $(D-1)G$ contains D edge-disjoint spanning trees $T_{i,j}$, based on which we got a rigid realization $((D-1)G, b)$
  - For each $e \in E$, there is an index i, j for which $T_{i,j}$ does not contain a copy of e
  - Define $h(e) = [q_i \wedge q_j]$
  - Notice $\langle b_{e_i}, h_e \rangle = 0$ for all copies e_i of e
  - Thus h is indeed a hinge-configuration ■



Identified Body-hinge Frameworks

- 各ヒンジが複数個の剛体を連結する事を許すモデル
- Identified body-hinge framework (G, h)
  - $G = (B, H; E)$
    - B: 剛体の集合, H:ヒンジ集合, E: 接続関係
  - $h: H \to Gr\left(d-1, \mathbb{R}^{d+1}\right)$

- 定理 (Tay 89,91, T10) 一般的hに対し, (G, h) が無限小剛堅 $\Leftrightarrow$ $\exists I \subseteq \left(\binom{d+1}{2} - 1\right) G$ s.t.
  - $|I| = \binom{d+1}{2}|B(I)| + \left(\binom{d+1}{2} - 1\right)|H(I)| - \binom{d+1}{2}$
  - $\emptyset \neq \forall X \subseteq I, |X| \leq \binom{d+1}{2}|B(X)| + \left(\binom{d+1}{2} - 1\right)|H(X)| - \binom{d+1}{2}$

- 未解決問題(Whiteley89, Jackson&Jordán09). 同じ条件で identified panel-hinge frameworkの無限小剛性が特徴付け可能か？