組合せ最適化セミナー演習問題

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問題 1. ある単射写像 $p : V \to \mathbb{R}^2$ に対し (G, p) が剛堅となるとき，G を (2 次元) 剛実現可能グラフと呼ぶ．n 頂点の剛実現可能グラフの集合を $\mathcal{G}_n$ とするとき，$\mathsf{realization\_number}(n) = \min\{|E| \mid G = (V, E) \in \mathcal{G}_n\}$ を求めよ．例えば $\mathsf{realization\_number}(2) = 1, \mathsf{realization\_number}(3) = 3$.

問題 2. 以下の定理を証明せよ．

Theorem 1 (Tay and Whiteley 1985)**.** *グラフ $G = (V, E)$ がラーマングラフである必要十分条件は G が K_2 から 0-extension と 1-extension の繰り返して構築可能なことである．*

問題 3. グラフ $G = (V, E)$ の bicircular matroid $(E, f_{1,0})$ において，$F \subseteq E$ が独立であるための必要十分条件をグラフの用語で記述せよ．(つまり"$\forall X \subseteq F, |X| \leq |V(X)|$" 以外の条件を求めよ．) また bicircular matroid の線形表現を求めよ．

問題 4. 劣モジュラ関数 $f : 2^E \to \mathbb{R}$ とベクトル $y \in \mathbb{R}^E$ に対し

$$f^y(X) := \min\{y(X \setminus Z) + f(Z) \mid Z \subseteq X\} \quad (X \subseteq E) \tag{1}$$

と f^y を定める．$f(\emptyset) = 0$ の時，以下が成立つ事を証明せよ．

(i) f^y は劣モジュラ.

(ii) $P(f^y) = \{x \in \mathbb{R}^E \mid x \in P(f), x \leq y\}$.

問題 5. Vertex splitting が 3 次元一般剛性を保持することを証明せよ．(注意:bar-joint framework(G, p) においてジョイント配置 $p : V \to \mathbb{R}^3$ は単射でなければならない．)

問題 6 (Direction-rigidity). これまで見てきたモデルでは，グラフの各辺は 2 点間の距離制約を表現している．この距離制約に代わり，2 点間の方向制約を表現したフレームワークを考える．つまり d-dimensional bar-joint framework (G, p) に対し，その可能な (微小) 動き $\dot{p} : V \to \mathbb{R}^d$ を

$$\dot{p}(u) - \dot{p}(v) \in \{(p(u) - p(v))t \mid t \in \mathbb{R}\} \quad \forall uv \in E \tag{2}$$

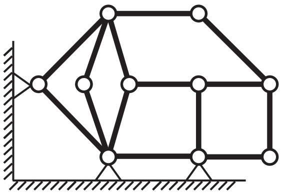
と定める．$\dot{p}$ が (K_n, p) の可能な動きの時，$\dot{p}$ は自明な動きと呼ばれ，自明な動きのみが可能な (G, p) を direction-rigid と呼ぶ.

(i) $d = 2$ の場合の自明な動きの次元を求めよ．

(ii) $d = 2$ の場合の generic direction-rigidity の特徴付けを与えよ．(ヒント：Laman の定理)

(iii) d 次元の generic direction-rigidity とグラフ的マトロイドの d 合併との関係を考察せよ (Whiteley96).

問題 7 (Pinned bar-joint frameworks). 通常，工学分野においては，下図に見られるように外部環境との接続関係を含め構造物の安定性解析を行なっている．



外部拘束数や拘束の配置によっては，Maxwell の条件を満たしていない場合においても構造物全体は剛堅となりうるため，2 次元の場合においても新たな理論の構築が必要である．

ここでは，最も典型的な外部拘束，ピン接合付きの 2 次元フレームワークを考えよう．ピン接合されている頂点集合を X と記す事で，pinned bar-joint framework を 3 つ組 (G, X, p) と定義し，

$$\langle \dot{p}(u) - \dot{p}(v), p(u) - p(v) \rangle = 0 \qquad \forall uv \in E \tag{3}$$

$$\dot{p}(v) = 0 \qquad \forall v \in X \tag{4}$$

を満たす $\dot{p} : V \to \mathbb{R}^2$ を可能な (微小) 動きと定める．非ゼロ動きが存在しないとき，(G, X, p) を剛堅と呼ぶ.

(i) 2 次元 pinned bar-joint framework の一般剛性の組合せ論的特徴付けを与えよ.

(ii) (ピン接合なしの) フレームワーク (G, p) に対し，$\min\{|X| \mid X \subseteq V; (G, X, p)$ is infinitesimally rigid$\}$ を pinning number と呼ぶ．2 次元 pinning number を求める問題が線形マトロイドのマトロイドマッチングに帰着される事を証明せよ (Lovász1980).

(ただしマトロイドマッチングが，2-ポリマトロイドの最少全域集合を求める問題と等価であることを既知とする: ポリマトロイド (E, f) が 2-ポリマトロイド $\Leftrightarrow \forall e \in E, f(e) \leq 2$; $F \subseteq E$ が最少全域集合 $\Leftrightarrow f(F) = f(E)$.)

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