On primal & dual feasible base in LP

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Primal problem:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)^\top$ is an n-dim vector, and $A = (a_1, a_2, \ldots, a_n)$ is an $m \times n$ matrix with $\text{rank} A = m \leq n$ (a_i are m-dim column vectors).

Dual problem:

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \leq c \end{aligned}$$

We prove:

- if an optimal primal solution exists, there are a base $B \subseteq \{1, 2, \ldots, n\}$, an optimal primal solution x, and an optimal dual solution y such that x and y are primal and dual basic solutions to B, respectively (i.e., B is primal & dual feasible).

We use the strong duality theorem: if there exists an optimal solution x in the primal problem, then there exists an optimal solution y in the dual problem such that

$$c^\top x = b^\top y.$$

Suppose that an optimal primal solution x exists. Let $J = J_x$ be the set of indices $i \in \{1, 2, \ldots, n\}$ with $x_i > 0$. In particular, it holds

$$b = A_I x_I \quad (I : J \subseteq I \subseteq \{1, 2, \ldots, n\}), \tag{1}$$

where $A_I = (a_i : i \in I)$ and $x_I = (x_i : i \in I)$.

Take an optimal primal solution x with J minimal. Then vectors a_i ($i \in J$) are linearly independent (otherwise adding to x a coefficient vector of the linear dependency of a_i ($i \in J$) obtains another optimal solution x' with a smaller nonzero support $J_{x'}$, contradicting the minimality of J).

We are going to construct an optimal dual solution y and a base $B \subseteq \{1, 2, \ldots, n\}$ such that x and y are basic to B. Let y be an optimal dual solution; its existence is guaranteed by the strong duality theorem. By the complementary slackness condition, it holds

$$a_i^\top y = c_i \quad (i \in J := \{i : x_i > 0\}).$$

Let $K = K_y$ be the set of indices j with $a_j^\top y = c_j$. By definition, it holds

$$J \subseteq K.$$

Take an optimal dual solution y with K maximal. Let $A_K = (a_i : i \in K)$ be the $m \times |K|$ submatrix of A consisting of column vectors a_i for $i \in K$. Then it must hold

$$\text{rank} A_K = m. \tag{2}$$

Otherwise A_K has rank less than m. There is a nonzero m-dim vector w such that $w^\top A_K = 0$. Here $w^\top A \neq 0$ must hold since $\text{rank} A = m$. For some nonzero real α, $y' := y + \alpha w$ is also a dual optimum (by $b^\top y' = b^\top(y + \alpha w) = b^\top y + \alpha x_K^\top A_K^\top w = b^\top y$), and $K_{y'} \supset K$ (proper inclusion), contradicting the maximality of K.

Since a_i $(i \in J \subseteq K)$ are linearly independent and a_i $(i \in K)$ span $\mathbf{R}^m$ by (2). we can choose B with $J \subseteq B \subseteq K$ such that vectors a_i $(i \in B)$ form a base in $\mathbf{R}^m$ (linear algebra !). Namely B is a base. Then $x_B = (x_i : i \in B)(\geq 0)$ is a unique solution of $A_B x_B = b$ (see also (1)). Thus $A_B^{-1} b = x_B \geq 0$, and x is a basic solution with respect to B. In particular B is primal feasible. Also y is a unique solution of

$$a_i^\top y = c_i \quad (i \in B \subseteq K) \quad (\text{i.e., } y^\top A_B = c_B^\top).$$

This means that y is basic with respect to basis B. Namely B is dual feasible. To see that the reduced cost is actually nonnegative, substitute $y^\top = c_B^\top A_B^{-1}$ to $y^\top A_N \leq c_N^\top$, where N is the complement of B, to obtain $c_N^\top - c_B^\top A_B^{-1} A_N \geq 0$.