Global rigidity of unit ball graphs

Dániel Garamvölgyi Tibor Jordán

Motivation – Sensor networks and the unit ball model

- A common application of global rigidity: localization of sensor networks.
- Sensor networks consist of many small computing units, some pairs of which can communicate with each other (and measure their distances).
- These networks are often modelled by so-called unit ball frameworks: two vertices are adjacent to each other precisely if their distance is below a given threshold (which we can take to be 1), corresponding to the sensing radius of the sensors.

Motivation – Unit ball global rigidity

- If we take this "unit ball" property into account, non-globally rigid frameworks may become localizable.
- This observation had been used before in localization algorithms, but there had been no theoretical examination of this variant of global rigidity.

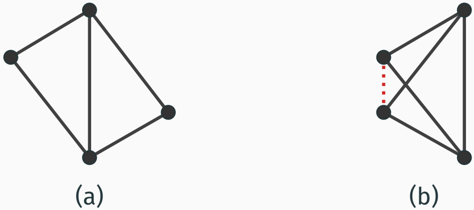

Figure 1: The framework in (a) is not globally rigid, but nonetheless it is the unique unit ball realization of the graph with the given edge lengths (up to congruences).

Definition (Equivalent and congruent frameworks)

A *(d-dimensional) framework* is a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is an embedding of the vertices of G into Euclidean space.

The frameworks (G, p) and (G, q) are *equivalent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \qquad \forall uv \in E,$$

and they are *congruent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \qquad \forall u, v \in V.$$

Definitions – Global rigidity

Definition (Rigid and globally rigid frameworks)

The framework (G, p) is *globally rigid* if every equivalent framework (G, q) is congruent to it.

The framework is *rigid* if there is some $\varepsilon > 0$ such that every equivalent framework (G, q) in the ε-neighbourhood of (G, p) is congruent to it.

- These are *generic* properties: if one generic framework is rigid (globally rigid) in a given dimension, then all of them are.

Definition (Rigid and globally rigid graphs)

A graph G is *rigid* (*globally rigid*) *in* $\mathbb{R}^d$ if every (or equivalently, if some) generic framework (G, p) is rigid (globally rigid).

Definition (Unit ball frameworks)

The framework (G, p) is *unit ball* if

$$\|p(u) - p(v)\| < 1 \Leftrightarrow uv \in E(G).$$

Definition (Unit ball graphs)

A graph G is *unit ball in* $\mathbb{R}^d$ if it has a d-dimensional unit ball realization.

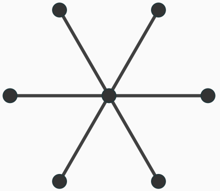

Figure 2: $K_{1,6}$ is not unit ball in $\mathbb{R}^2$.

Definitions – Unit ball graphs

- Recognizing unit ball graphs is NP-hard in any fixed dimension $d \geq 2$, and it is open whether this problem is in NP.
- Structurally, most of what is known is about forbidden induced subgraphs, e.g. $K_{1,6}$ and $K_{2,3}$ in the $d = 2$ case.
- Some problems can be solved efficiently for unit ball graphs (for $d = 2$), most notably finding a maximum clique.
- The $d = 1$ case ("unit interval" graphs) is much easier than the others – these graphs are characterized by finitely many forbidden subgraphs.

Definitions – Unit ball global rigidity

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The framework (G, p) is *globally rigid* if every equivalent framework (G, q) is congruent to it.

Definition (Unit ball globally rigid frameworks)

The unit ball framework (G, p) is *unit ball globally rigid* (or UBGR) if every equivalent unit ball framework (G, q) is congruent to it.

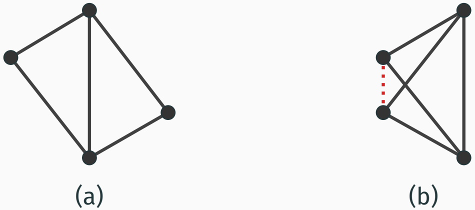

Figure 3: The framework in (a) is unit ball globally rigid.

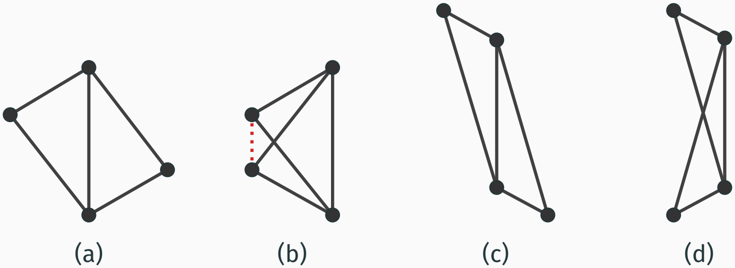

Figure 4: (a) A UBGR, and (c) a non-UBGR unit ball realization of the same graph.

Unit ball global rigidity is not a generic property!

Definition

A graph is *unit ball globally rigid* (or UBGR) in $\mathbb{R}^d$ if it has a d-dimensional generic unit ball globally rigid realization.

More observations

We have

$$\{\text{Globally rigid graphs}\} \subseteq \{\text{UBGR graphs}\} \subseteq \{\text{Rigid graphs}\}$$

within the family of d-dimensional unit ball graphs.

For non-generic frameworks, UBGR $\not\Rightarrow$ rigid!

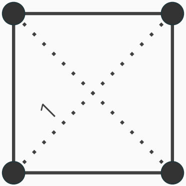

Figure 5: The square with unit diagonals is UBGR, but not rigid.

Obtaining unit ball globally rigid frameworks

Consider the following construction:

1. Start with a generic rigid unit ball framework (G, p).
2. Take one framework from each of the (finitely many) congruence classes of equivalent frameworks: $(G, p = p_1), ..., (G, p_k)$.
3. Now scale them by a factor of $0 < \alpha \leq 1$ to obtain $(G, \alpha \cdot p_1), ..., (G, \alpha \cdot p_k)$. This may result in non-neighbouring vertices with distance less than 1.
4. Decrease α until (hopefully) precisely one of $(G, \alpha \cdot p_1), ..., (G, \alpha \cdot p_k)$ is unit ball; then it is unit ball globally rigid.

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For this to work, it would be enough to show that scaling destroys the unit ball property of the frameworks one by one.

SNGR graphs

For all equivalent frameworks (G, p) and (G, q) we require:

$$\|p(u) - p(v)\| \neq \|q(u') - q(v')\| \qquad \forall uv, u'v' \notin E(G). \tag{$*$}$$

Concentrate on

$$\|p(u) - p(v)\| \neq \|q(u) - q(v)\| \qquad \forall uv \notin E(G). \tag{$**$}$$

For (G, p), $(**)$ is equivalent to the requirement that $(G + uv, p)$ is globally rigid for any $uv \notin E(G)$.

Definition (SNGR graphs)

G is SNGR (saturated non-globally rigid) in $\mathbb{R}^d$ if it is not globally rigid in $\mathbb{R}^d$, but $G + uv$ is globally rigid for any pair $u, v \in V(G)$ with $uv \notin E(G)$.

SNGR graphs

Does being SNGR imply $(*)$ for equivalent frameworks (G, p) and (G, q)?

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$$\|p(u) - p(v)\| \neq \|q(u') - q(v')\| \qquad \forall uv, u'v' \notin E(G) \qquad (*)$$

Yes.

Lemma

Let (G, p) and (G, q) be equivalent d-dimensional frameworks, where G is SNGR in $\mathbb{R}^d$ and (G, p) is generic. Then $(*)$ holds.

This follows from the recent result of Gortler, Theran and Thurston: in $d \geq 2$ dimensions, the set of edge lengths of a generic globally rigid framework (G, p) on $n \geq d + 2$ vertices uniquely determines not only p (up to congruence), but G as well (up to isomorphism), among d-dimensional frameworks on n vertices.

SNGR graphs

Using the idea of scaling equivalent frameworks, outlined before, we get:

Theorem

Unit ball SNGR graphs have (generic) unit ball globally rigid relizations.

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But do such graphs exist?

They do. (At least in $\mathbb{R}^2$.)

SNGR graphs in $\mathbb{R}^2$

Theorem

SNGR graphs on at least $d+2$ vertices are rigid, and they are either $(d+1)$-connected, or can be obtained from two complete graphs (of size at least $d+1$) by gluing them along d vertices.

Theorem

Let G be a minimally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices. If G is SNGR, then every proper rigid subgraph of G is complete.

Minimally rigid graphs with the latter property are sometimes called *special* graphs.

Theorem

For $d=2$, the converse is true as well: if G is special, then it is SNGR.

Theorem

Let $G = (V, E)$ be a minimally rigid SNGR graph in $\mathbb{R}^2$ and let $u, v, w \in V$ be different vertices with $uv \in E$. Let $G' = (V', E')$ be an 1-extension of G on uv and w. Then G' is SNGR if and only if neither $\{u, w\}$ nor $\{v, w\}$ are contained in a triangle in G.

This helps us in finding infinite families of unit ball SNGR graphs in $\mathbb{R}^2$.

It also implies the following:

Corollary

Any minimally rigid SNGR graph in $\mathbb{R}^2$ on at least 5 vertices has a 1-extension that is also minimally rigid and SNGR.

An example

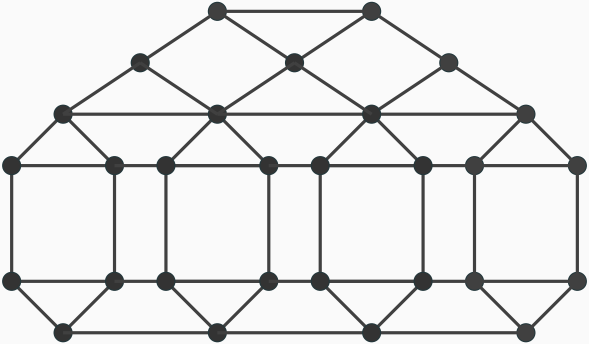

Figure 6: A minimally rigid SNGR graph that is also unit ball in $\mathbb{R}^2$. By the main Theorem, this graph has a generic unit ball globally rigid realization. Note that, being minimally rigid, this graph has fewer edges than any globally rigid graph on the same number of vertices.

Another example

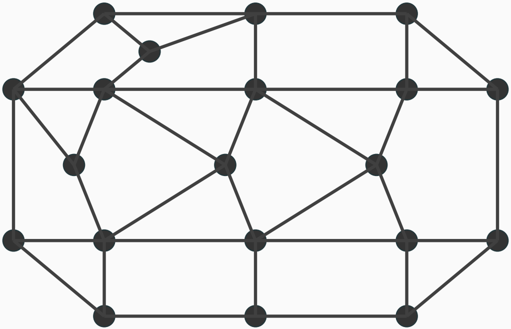

Figure 7: A different example of a unit ball SNGR graph in $\mathbb{R}^2$.

These examples both give rise to infinite families of unit ball SNGR graphs in $\mathbb{R}^2$.

Open questions

Being a previously unexamined notion, there are many open questions relating to unit ball global rigidity. Here are two:

- Are there unit ball graphs in $d \geq 2$ dimensions that are not globally rigid, but every unit ball realization of them is unit ball globally rigid ("strongly unit ball globally rigid graphs")?
- By a result of Jordán and Tanigawa, 4-connected, maximal planar graphs are SNGR in $\mathbb{R}^3$. Are there unit ball graphs (in $\mathbb{R}^3$) among these?