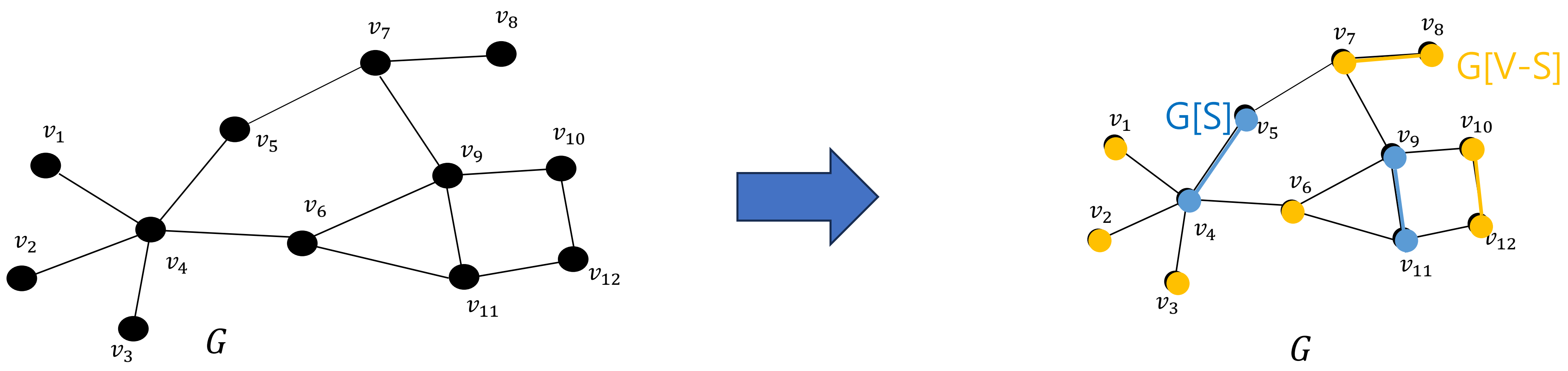
連結グラフの安全集合 (safe set) とは、次のように定義される。

Given a connected graph $G = (V, E)$, $\emptyset \neq S \subset V$ is a safe set if the following holds:

$\forall$ component C of $G[S]$, $\forall$ component D of $G[V - S]$

If there is an edge between C and D, then $|C| \geq |D|$.



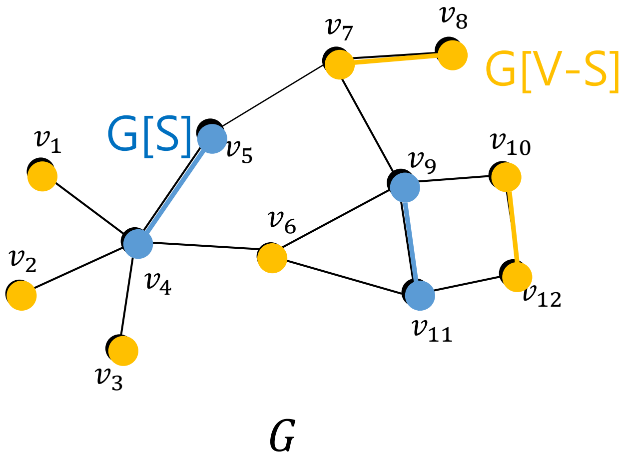
S=$\{v_4, v_5, v_9, v_{11}\}$

グラフの安全集合の最小位数をsafe numberと定義する.

[Safe number $s(G)$]

Given a connected graph $G = (V, E)$, the safe number $s(G)$ of G is the size of a safe set with the minimum cardinality.

$$s(G) = \min\{\, |S| : S \text{ is a safe set of } G \,\}$$



$s(G) \leq 4$

S=$\{v_4, v_5, v_9, v_{11}\}$

グラフの安全集合の最小位数をsafe numberと定義する．

[Safe number $s(G)$]

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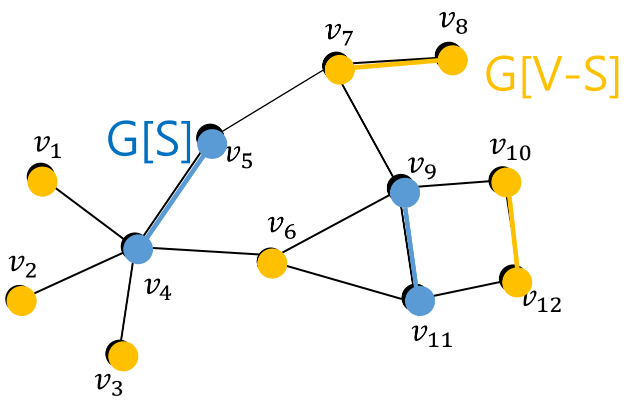
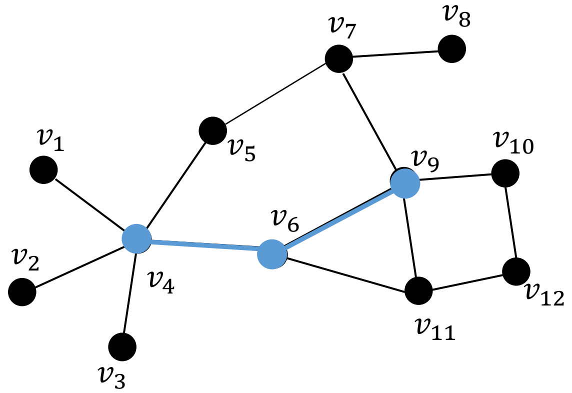


G

S=$\{v_4, v_5, v_9, v_{11}\}$

$s(G) \leq 4$

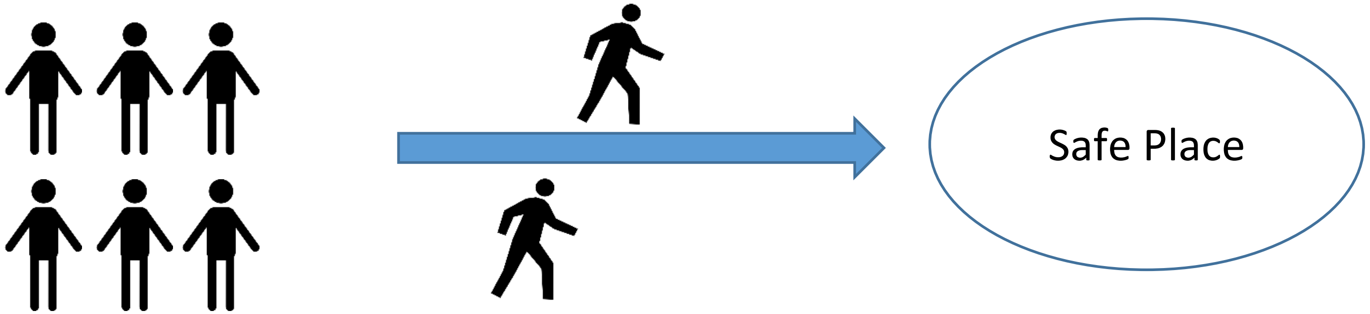
実は安全集合を取り直すと $s(G) =$ 3であることが分かる．

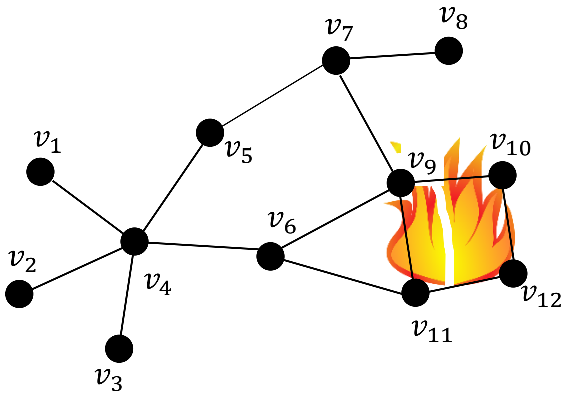


Safe Set Problem

[Safe Set Problem]

Given a connected graph $G = (V, E)$, find a safe set of G with the minimum cardinality.





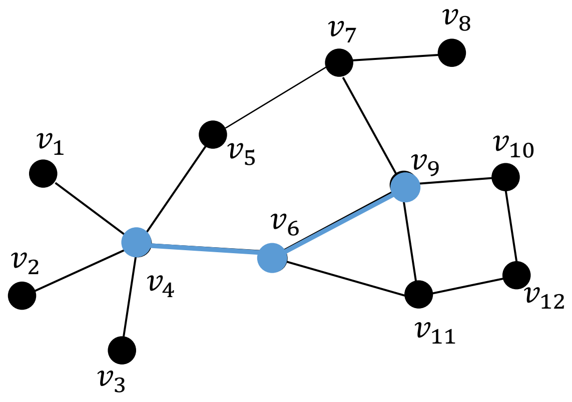
G

安全集合の連結バージョンについて

[Connected safe set]

Given a connected graph $G = (V, E)$, $\emptyset \neq S \subset V$ is a connected safe set if it is a safe set and $G[S]$ is connected.

The connected safe number $\mathrm{cs}(G)$ of G is the size of a connected safe set with the minimum cardinality.



$\mathrm{s}(G) = \mathrm{cs}(G)$

- For a path G, $\mathrm{s}(G) = \mathrm{cs}(G) = \left\lceil \frac{|V(G)|}{3} \right\rceil$.
- For a cycle G, $\mathrm{s}(G) = \mathrm{cs}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.

[Note] 定義により、s(G)≦cs(G)であり、一般にはs(G)= cs(G)は成り立たない。cs(G)の上界をs(G)を用いて表すことが出来る。※演習問題参照

安全集合の重み付き版への拡張について

[Weighted safe set]

Given a connected graph $G = (V, E)$, and a vertex weight function ω,
$\emptyset \neq S \subset V$ is a weighted safe set of (G, ω) if the following holds:

$\forall$ component C of $G[\mathrm{S}]$, $\forall$ component D of $G[\mathrm{V} - \mathrm{S}]$

If there is an edge between C and D, then $\omega(\mathrm{C}) \geq \omega(\mathrm{D})$.

$\omega(v)$: positive real number

成分Cの重みの総和

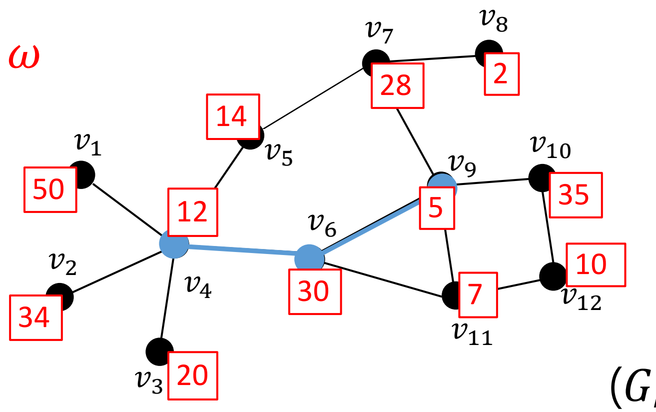
この場合，安全集合は成分の重み和を考える

[Weighted safe set]

Given a connected graph $G = (V, E)$, and a vertex weight function ω, $\emptyset \neq S \subset V$ is a weighted safe set of (G, ω) if the following holds:

∀ component C of $G[S]$, ∀ component D of $G[V - S]$

If there is an edge between C and D, then $\omega(C) \geq \omega(D)$.



(G, ω)上で weighted safe number $\mathrm{ws}(G, \omega)$ という概念も導入可能である。

同様に、weighted connected safe number $\mathrm{wcs}(G, \omega)$ も考えられる。

[Note]

- 各重みが同じ値をとる場合は通常の安全集合の概念と等価である。
- 重み付き安全集合においても、連結な安全集合を同様に定義できる。

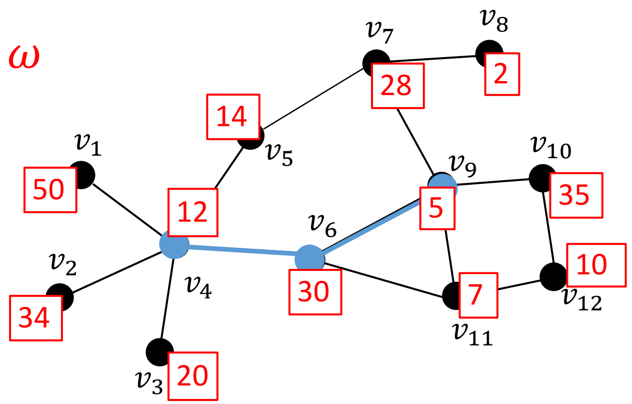
安全集合の重み付きグラフ版について

[Weighted safe set]

Given a connected graph $G = (V, E)$, and a vertex weight function ω, $\emptyset \neq S \subset V$ is a weighted safe set of (G, ω) if the following holds:

$\forall$ component C of $G[\mathrm{S}]$, $\forall$ component D of $G[\mathrm{V} - \mathrm{S}]$

If there is an edge between C and D, then $\omega(\mathrm{C}) \geq \omega(\mathrm{D})$.



連結グラフ G とvertex weight function ω に対して、(G, ω) の weighted safe number $\mathrm{ws}(G, \omega)$ を、最小重み和をもつ重み付き安全集合Sの重み和ω(S)と定義する。

Computational Results

Theorem (Fujita, MacGillivary, Sakuma, 2016)

The Safe Set Problem is NP-hard. ⇨ Reduction from the independence number of a graph

Theorem (Bapat, Fujita, Manoussakis, Matsui, Sakuma, Tuza, 2017)

The Weighted Safe Set Problem is NP-hard (even if we restrict the domain into the stars.)

Theorem (Agueda, Cohen, Fujita, Legay, Manoussakis, Matsui, Montero, Naserasr, Ono, Otachi, Sakuma, Tuza, Xu, 2018)

For an n-vertex tree, a safe set of the minimum size can be found in time $O(n^5)$.

Theorem (Agueda, Cohen, Fujita, Legay, Manoussakis, Matsui, Montero, Naserasr, Ono, Otachi, Sakuma, Tuza, Xu, 2018)

Let k be a fixed constant. For an n-vertex graph of treewidth at most k, a (connected) safe set of minimum size can be found in time $O(n^{5k+8})$.

Some other results:
Agueda et al., (2017), Ehard and D. Rautenbach (2020).

⋮

安全集合とgraph integrityの関係

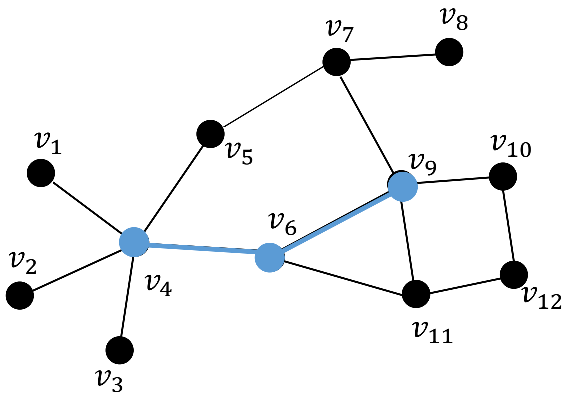
【参考文献】 C.A. Barefoot, R. Entringer and H.C. Swart, *Vulnerability in graphs–a comparative survey*, J. Combin. Math. Combin. Comput. 1 (1987)

The integrity of a graph $G = (V, E)$ is defined as

$$I(G) = \min\{|S| + t(G[V-S])|\ S \subset V\},$$

where $t(G[V-S])$ means the order of the largest components of $G[V-S]$.

$I(G) \leq 3+3=6$



$3 \leq I(G)$

$$s(G) \leq cs(G) \leq I(G) \leq 2s(G) \leq 2cs(G)$$

Theorem (Fujita and Furuya, 2018)

Let G be a connected graph.

- If G is not a star, then $2\sqrt{s(G)-2}+1 \leq I(G)$
- $2\sqrt{cs(G)-1} \leq I(G)$

【参考文献】S.Fujita, M.Furuya: *Safe number and integrity of graphs*. Discrete Appl. Math. 247 (2018)

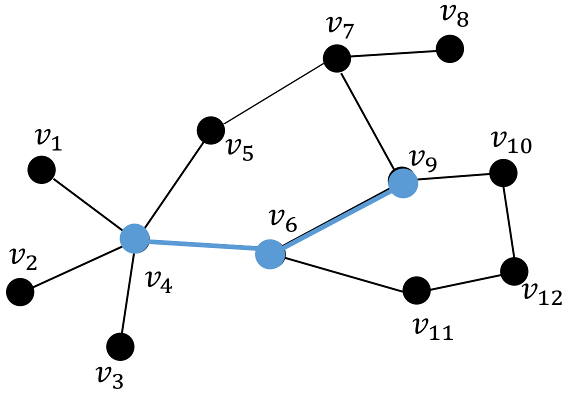
[Note] 上の論文では、不等式の上界で等号が成立するグラフの構成も与えている。

重み付き版安全集合の「安定性」について

[Recall: Weighted safe set]
Given a connected graph $G=(V,E)$, and a vertex weight function ω, $\emptyset \neq S \subset V$ is a weighted safe set of (G,ω) if the following holds:

$\forall$ component C of $G[\mathrm{S}]$, $\forall$ component D of $G[\mathrm{V}-\mathrm{S}]$

If there is an edge between C and D, then $\omega(\mathrm{C}) \geq \omega(\mathrm{D})$.



$\mathrm{s}(G)=\mathrm{cs}(G)=3.$

連結グラフGが $\mathrm{s}(G)=\mathrm{cs}(G)$を満たすことは、最小安全集合が常に連結な構造を有していることを意味する。応用面から、そのような性質をもつグラフの特徴付けを考える問題は面白そうである。

ここで重み付きグラフ上での安全集合において、

$\mathrm{s}(G,\omega) \leq \mathrm{cs}(G,\omega)$

が任意の重み関数ωで成り立つことに注意する。

Research Question

ここでネットワーク安定性の観点から、次のような性質を満たすグラフを特徴付けたい。

Q.　次を満たす連結グラフGはどんなグラフか？

∀ vertex weight function ω: V(G)→**R**, $\mathrm{ws}(G,\omega) = \mathrm{wcs}(G,\omega)$ ―――――― (★)

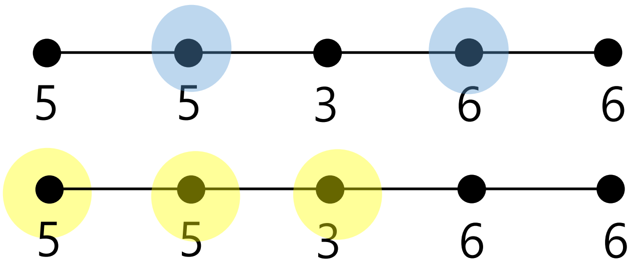
$\mathcal{G}^{CS}$ として、（★）の条件を満たすグラフクラスと定義する。

(Ex) 任意の完全グラフは $\mathcal{G}^{CS}$に属する。

【演習問題】

G が Δ(G)=|V(G)|-1を満たすならば G は $\mathcal{G}^{CS}$に属することを示せ。

(Ex) 以下の例から、５頂点からなるパスは $\mathcal{G}^{CS}$には属さないことが分かる。



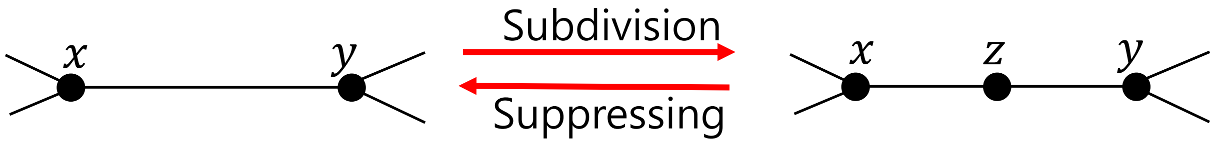
$\mathrm{ws}(G,\omega) = 5 + 6 = 11$

$\mathrm{wcs}(G,\omega) = 5 + 5 + 3 = 13$

$\mathscr{g}^{CS}$ について分かっていること

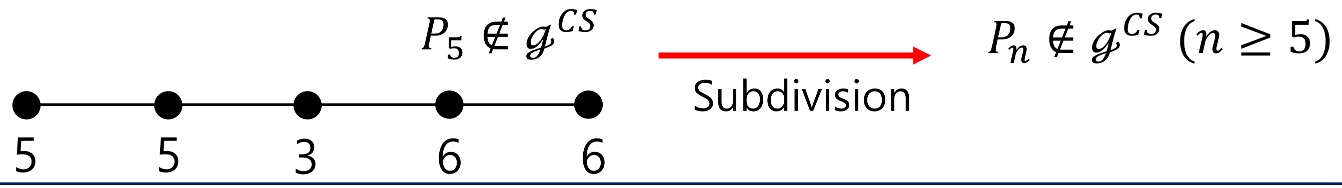
$\mathscr{g}^{CS}$ を下記の性質を満たす連結グラフの集合族とする。

$\forall$ vertex weight function ω, $\mathrm{ws}(G,\omega) = \mathrm{wcs}(G,\omega)$.



Theorem (Fujita, Jensen, Park, Sakuma, 2019)

The family $\mathscr{g}^{CS}$ is closed under suppressing .



Proposition (Fujita, Jensen, Park, Sakuma, 2019)

Let G be a path with n vertices.

$n \leq 4$ if and only if G belongs to $\mathscr{g}^{CS}$.

重み付き木については、いろいろ分かっている

Theorem (Fujita, Park, Sakuma, 2021)

Let G be a tree.

$\mathrm{diam(G)} \leq 3$ if and only if G belongs to $\mathcal{G}^{CS}$.

(Fujita, MacGillivary, Sakuma, 2016)

- polynomial-time ((O(n)) algorithm for the connected safe number of a tree

(Agueda, Cohen, Fujita, Legay, Manoussakis, Matsui, Montero, Naserasr, Ono, Otachi, Sakuma, Tuza, Xu, 2018)

- polynomial-time ((O(n^5)) algorithm for the safe number of a tree

(Bapat, Fujita, Manoussakis, Matsui, Sakuma, Tuza, 2017)

- polynomial-time ((O(n^3)) algorithm for the safe number of a weighted path
- O($n \log n$) time 2-approximation algorithm for the connected safe number of a weighted tree
  (and so ∃ 4-approximation algorithm for the safe number of a weighted tree)

(Ehard and D. Rautenbach, 2020)

- $\forall\epsilon$, O($\frac{1}{\epsilon^4} n^{O(\frac{1}{\epsilon})}$) time $(1+\epsilon)$ -approximation algorithm for the connected safe number of a weighted tree

$\mathcal{g}^{CS}$について分かったこと、いろいろ

Theorem (Fujita, Jensen, Park, Sakuma, 2019)

Any cycle belongs to $\mathcal{g}^{CS}$.

Corollary (Fujita, Park, Sakuma, 2021)

Let G be a tree.

$\mathrm{diam}(G) \leq 3$ if and only if G belongs to $\mathcal{g}^{CS}$.

$S(1,6)$ $S(4,5)$

Theorem (Fujita, Park, Sakuma, 2021)

For a connected chordal graph G, the following are equivalent:

(i) G belongs to $\mathcal{g}^{CS}$

(ii) $\mathrm{diam}(G) \leq 3$.

(iii) G has a dominating clique.

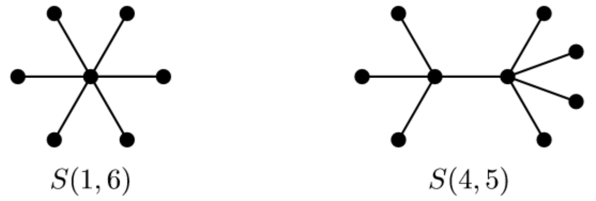
g^{CS}に属する二部グラフの特徴付け

Theorem (Fujita, Park, Sakuma, 2021)

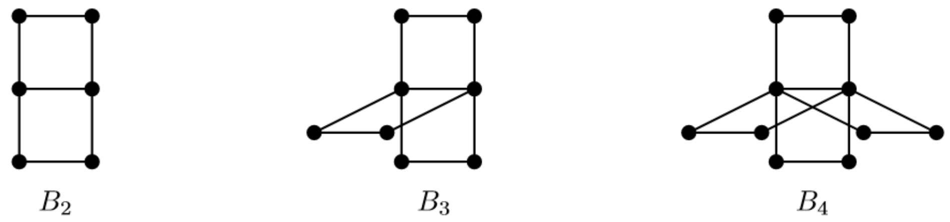
For a connected nontrivial bipartite graph G, G belongs to g^{CS} if and only if G is one of the following:

(i) an even cycle or $K_{3,3} - e$

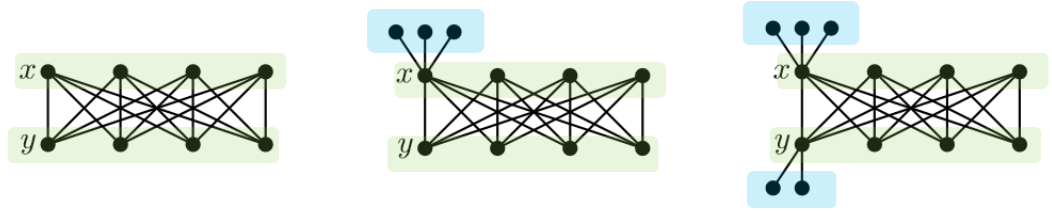
(ii) a double star $S_{m,n}$.



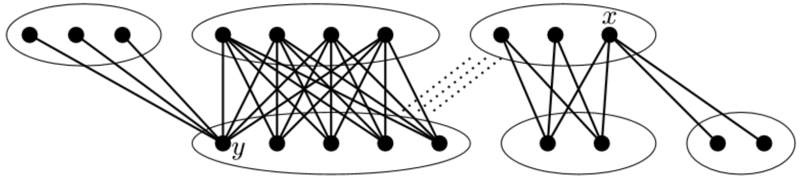
(iii) a book graph B_n



(iv) $K_{n,n} + R_{p,q}$ $(n \geq 3)$



(v) $K_{n,n+1} + K_{m+1,m} + R_{p,q}$ $(m, n \geq 2)$ whose adjacency between two big parts form a complete bipartite graph or a double star at x and y.



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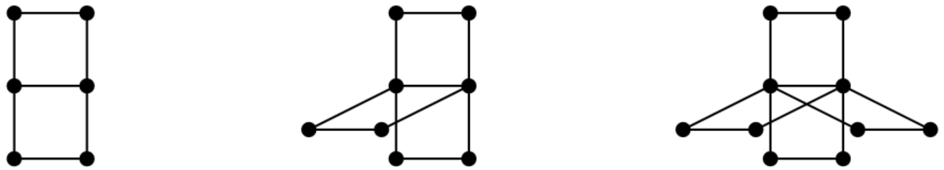
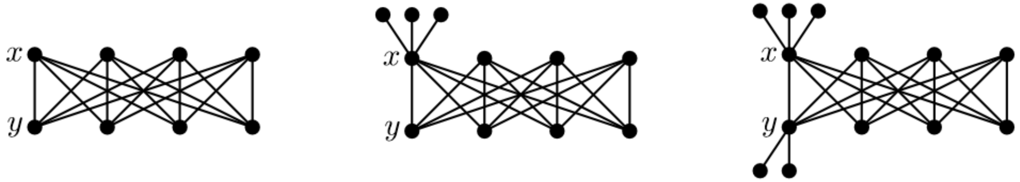


B_2 B_3 B_4

(iv) $K_{n,n} + R_{p,q}$ $(n \geq 3)$



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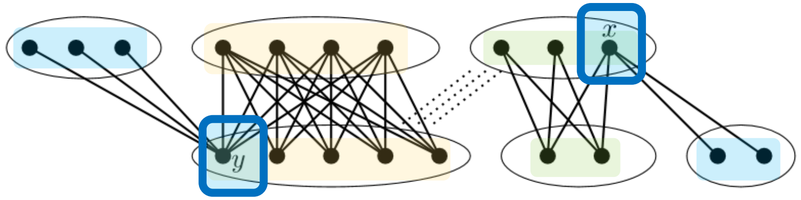

$\mathcal{g}^{CS}$に属する二部グラフの特徴付け

Theorem (Fujita, Park, Sakuma, 2021)

For a connected nontrivial bipartite graph G, G belongs to $\mathcal{g}^{CS}$ if and only if G is one of the following:

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$S(1,6)$ $S(4,5)$

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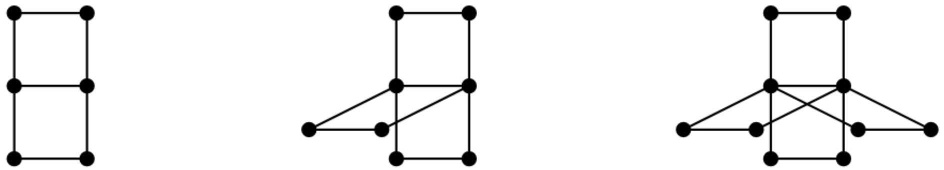

B_2 B_3 B_4

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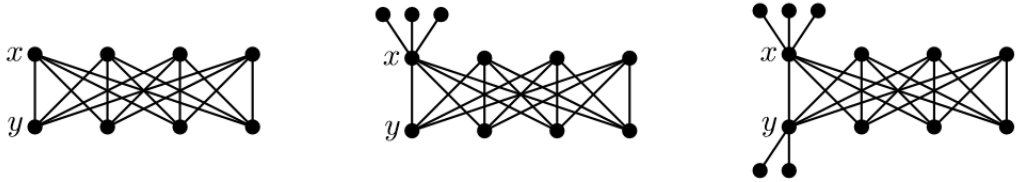
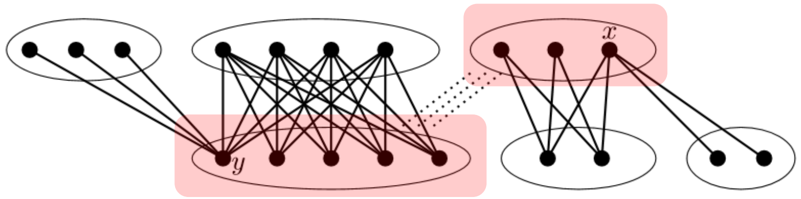
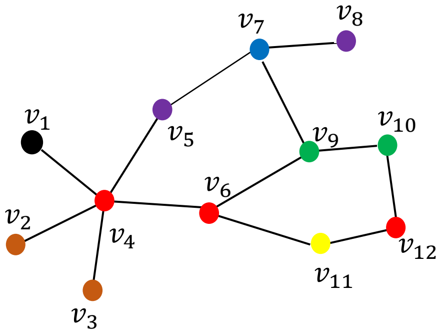


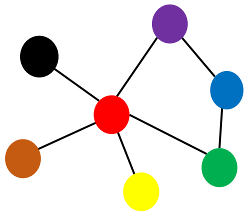
(v) $K_{n,n+1} + K_{m+1,m} + R_{p,q}$ $(m, n \geq 2)$ whose adjacency between two big parts form a complete bipartite graph or a double star at x and y.

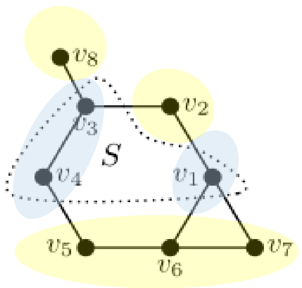


縮約操作との関係について

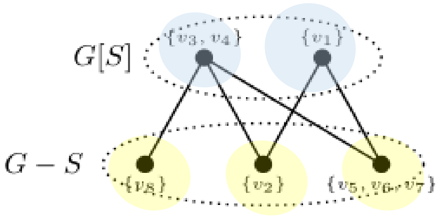
A graph G is contractible to H or (H-contractible) if H can be obtained from a partition of $V(G)$ by contracting each part to a vertex.







A graph G and $S = \{v_1, v_3, v_4\}$



$\beta(G, S)$

For a graph G not in $\mathcal{G}^{CS}$ $\beta(G,S)$ is not in $\mathcal{G}^{CS}$ for a minimum safe set S of (G,ω).

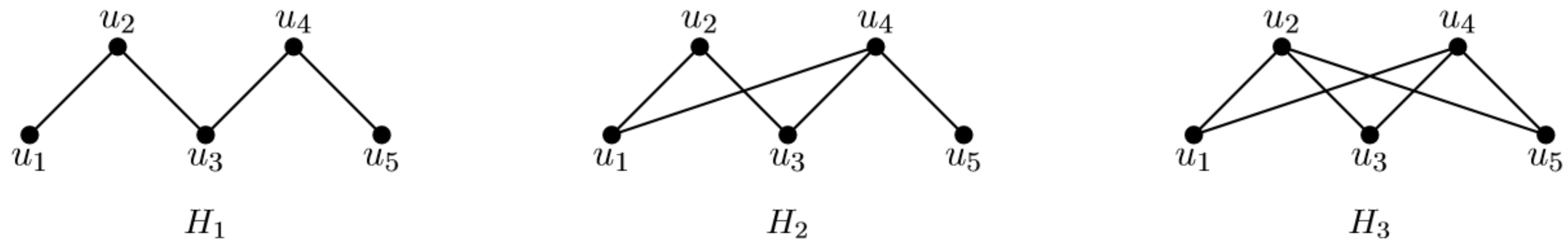
Proposition (Fujita, Park, Sakuma, 2021)

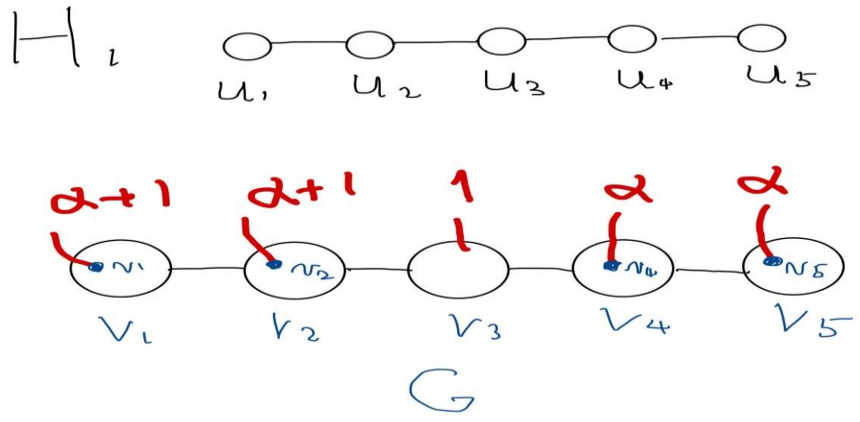
For a connected graph G, if $\beta(G,S)$ is in $\mathcal{G}^{CS}$ for any S, then G is in $\mathcal{G}^{CS}$.

Key lemmas

Lemma (Fujita, Park, Sakuma, 2021)

If a graph is contractible to one of the following (in some sense), then $G \notin \mathcal{g}^{CS}$.





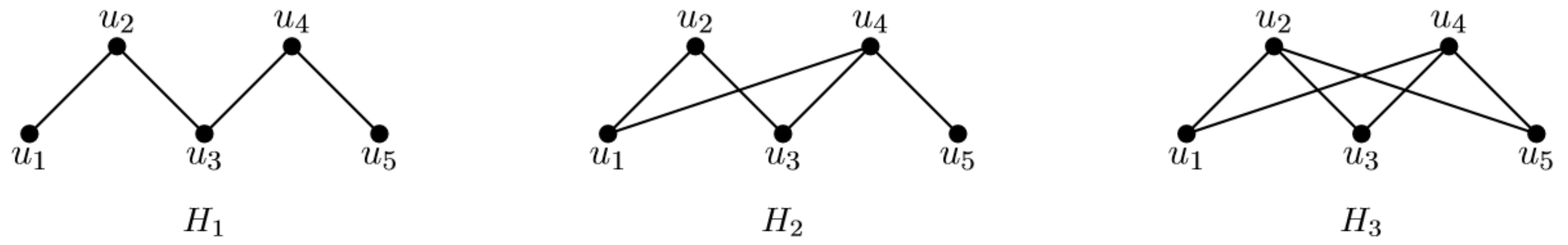
$$\omega(x) = \begin{cases} \alpha+1 & x \in \{v_1, v_2\} \\ \alpha & x \in \{v_4, v_5\} \\ \frac{1}{|V_3|} & x \in V_3 \\ 0 & 他 \end{cases}$$

$$S(G, \omega) = 2\alpha+1 < cs(G, \omega) = 2\alpha+1+\varepsilon$$

Key lemmas

Lemma (Fujita, Park, Sakuma, 2021)

If a graph is contractible to one of the following (in some sense), then $G \notin \mathcal{G}^{CS}$.



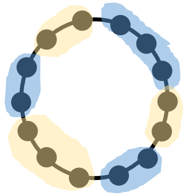
Proposition (Fujita, Park, Sakuma, 2021)

For a connected triangle-free graph G in $\mathcal{G}^{CS}$, if G is not a cycle, then $\mathrm{diam}(G) \leq 3$.

重み付きサイクル上の安全集合について

Theorem (Fujita, Jensen, Park, Sakuma, 2019)

Any cycle belongs to $\mathcal{g}^{CS}$.



For any weighted cycle (C, ω),
$\mathrm{ws}(\mathrm{C},\omega) \geq \omega((V(\mathrm{C}))/2$

上の結果から、最小安全集合はCの重み総和の過半数をとらなければならない。

Theorem (Fujita, Jensen, Park, Sakuma, 2019)

(i) G is a cycle or a complete graph.

⟺ (ii) For any weight function ω, $\mathrm{ws}(G, \omega) \geq \omega(\mathrm{V}(G))/2$.

この同値であることを示していく過程で、
subgraph component polynomial (introduced by Tittmann, Averbouch, Makowsky (2011))
という概念と深い関連があることが分かった。

Subgraph component polynomialとの関連

$k(G)$: the number of the components of G.
The subgraph component polynomial $Q = Q(G;x,y)$ of G is a polynomial s.t.

$$Q[x^i y^j] = q_{i,j}(G) = |\{ X \subset V(G) | \ |X| = i \ and \ k(G[X]) = j \}|$$

[Note]
- $q_{1,1}(G) = |V(G)|$
- $q_{2,1}(G) = |E(G)|$
- $q_{1,1}(G) = q_{n-1,1}(G)$ ならば、G は2連結である。

Theorem (Fujita, Jensen, Park, Sakuma, 2019)

Gを位数5以上のグラフとするとき、(i)–(iv)は同値である。

(i) G はサイクル、または完全グラフである。

(ii) 任意の重み付き関数 ω に対して、$ws(G,\omega) \geq \omega(V(G))/2$が成り立つ。

(iii) 任意の非隣接二点u, vに対して、 $k(G-\{u,v\}) \geq 2$が成り立つ。

(iv) $q_{1,1}(G) = q_{n-1,1}(G)$ かつ $q_{2,1}(G) = q_{n-2,1}(G)$が成り立つ。

【演習問題】

(iii) ⇒ (i) を示せ。

未解決問題

【問題】 任意の重み付き関数ωに対して、
$s(G, \omega) = cs(G, \omega)$ がつねに成り立つようなグラフGを特徴付けよ。

上の部分問題として、「連結グラフGが
$s(G) = cs(G)$ 　を満たすとき、Gはどのようなグラフか？」という問いも未解決である。

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