Colloquium talk at Kyoto University, May 2019

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Title: Problem of Resolution of Singularities: Past, Present, and Future

Abstract: The problem of resolution of singularities, in its simplest form, asks the following: Given a variety X defined over a field k (say, the field k is assumed to be algebraically closed, i.e., $k = \overline{k}$, for simplicity), construct a proper birational morphism $\pi : \widetilde{X} \longrightarrow X$ such that $\widetilde{X}$ is smooth over k (i.e., nonsingular).

It is the celebrated work of Hironaka in 1960's that established the existential solution to the probelm. In 1980's Bierstone-Milman and Villamayor with his collaborators refined Hironaka's main ideas and sublimated them into a constructive and explicit algorithm. At the beginning of the 21st century, Włodarczyk simplified the globalization process so that now even a beginning graduate student can understand the whole proof without hand-waving. On the other hand, no matter how wonderful the technical advances might have been, what remains at the core in characteristic zero is the inductive scheme on dimension by the notion of a (smooth) hypersurface of maximal contact.

In positive characteristic, in contrast, it is the lack of a (smooth) hypersurface of maximal contact and hence the lack of an apparent inductive scheme on dimension that restricted the known results to lower dimensions. Many methods have been known classically for resolution of singularities of a curve, a variety of dimension 1. Lipman's result provides resolution of singularities of an excellent scheme of dimension 2 in general. Cossart-Piltant removed the restriction on the characteristic of the base field from Abhyankar's result, and thus establishing resolution of singularities of a 3-fold in any positive characteristic. It may feel odd at first when one finds that embedded resolution of singularities of a 3-fold (and hence the ambient space where the 3-fold is embedded is of higher dimension) remains open.

I should mention that Hironaka has announced a proof of resolution of singularities in positive characteristic in all dimensions. To the best of my knowledge, however, nobody seems to have been able to confirm the details of the alleged proof so far.

I will start my talk demonstrating the meaning of the problem of resolution of singularities by showing some entertaining pictures. The goal of my talk is not to exhaust all the historical results mentioned above. Rather the emphasis is on explaining the inductive scheme on dimension by a smooth hypersurface of maximal contact in characteristic zero, and then on explaining our approach in positive characteristic, initiated by H. Kawanoue; how to introduce a new inductive scheme by allowing singular hypersurfaces of maximal contact. Our approach has been successful, providing a systematic proof for embedded resolution of singularities of a surface. But we still fall short of establishing embedded resolution of a 3-fold. I will discuss the major difficulty known as the Moh-hauser jumping phenomenon, and how we plan to overcome this difficulty.