ADDENDUM TO "CURVES AND SYMMETRIC SPACES, II"

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Let C be a curve of genus 9 and asuume that C has no g^1_5. We denote by $M_C(3, K)$ the set of isomorphism classes of rank 3 stable vector bundles E on C with canonical determinant, *i.e.*, $\bigwedge^3 E \simeq K_C$. In this note we prove the following, which was announced in [14, Proposition 2]:

Theorem 1. *The maximum $\eta_3(C)$ of the number $h^0(E)$ of linearly independent global sections of E, when E runs over $M_C(3, K)$, is equal to 6. Moreover, $E_{max} \in M_C(3, K)$ with $h^0(E_{max}) = 6$ is unique (up to isomorphism).*

Let E be a stable rank 3 vector bundle on C of canonical determinant.

Lemma 2. (1) $h^0(\xi) \leq 1$ *for every line subbundle ξ of E.*

(2) $h^0(F) \leq 3$ *for every rank two subbundle F of E. Moreover, if $h^0(F) = 3$, then*

$$\lambda_2 : \bigwedge^2 H^0(F) \longrightarrow H^0(\bigwedge^2 F)$$

is injective and $\deg F \geq 8$.

Proof. We have $\deg \xi \leq 5$ by stability and have (1) by non-pentagonality. By stability we have $\deg F \leq 10$ also. By non-pentagonality (or non-tetragonality more precisely), C has no g^2_6. By Serre duality, C has no g^4_{10}, either. Hence we have $h^0(\det F) \leq 4$. Assume that $h^0(F) = 4$. Then, by Proposition 3.2, F contains a line subbundle η with $h^0(\eta) \geq 2$, which contradicts (1). Hence we have the first half of (2). Assume that $h^0(F) = 3$. By (1), $\lambda_2(s_1 \wedge s_2)$ is nonzero for every pair of linearly independent global sections s_1 and $s_2 \in H^0(\bigwedge^2 E)$. Therefore,

$$\bigwedge^2 H^0(F) \longrightarrow H^0(\bigwedge^2 F) \subset H^0(\bigwedge^2 E)$$

is injective and we have $h^0(\det F) \geq 3$. By non-pentagonality C has no g^2_7. Hence, we have $\deg F \geq 8$. □

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Proposition 3. (1) $h^0(E) \leq 6$.

(2) *If* $h^0(E) = 6$*, then* E *has a rank two subbundle* F *with* $h^0(F) = 3$.

(3) *If* $h^0(E) = 6$*, then* $|E|$ *is free and semi-irreducible.*

Proof. We may assume that $h^0(E) \geq 6$. By Proposition 3.2, there exists a 3-dimensional subspace $W \subset H^0(E)$ such that the evaluation homomorphism $ev_W : W \otimes \mathcal{O}_C \longrightarrow E$ is not injective. Let G be the saturation of the image of ev_W. By the preceding lemma, F is of rank two and $h^0(G) = 3$, which shows (2).

Let F be an arbitrary rank two subbundle of E with $h^0(F) = 3$ and set $\beta = E/F$. By (2) of the preceding lemma, we have $\deg \beta = 16 - \deg F \leq 8$. Since C has no g^3_8, we have

$$h^0(E) \leq h^0(F) + h^0(\beta) \leq 3 + 3 = 6.$$

This and (2) show (1). Since $h^0(E) \geq 6$, we have $h^0(\beta) = 3$. Hence β is a g^2_8. Since $\alpha := \bigwedge^2 F$ is isomorphic to $K_C\beta^{-1}$, α is a g^2_8, too. Therefore,

$$\bigwedge^2 H^0(F) \longrightarrow H^0(\bigwedge^2 F)$$

is an isomorphism, by (2) of the preceding lemma. Since $|\bigwedge^2 F|$ is free, so is $|F|$. This shows the semi-irreducibility of E by Proposition 3.5. Since β is also free and since

$$0 \longrightarrow H^0(F) \longrightarrow H^0(E) \longrightarrow H^0(\beta) \longrightarrow 0$$

is exact, $|E|$ is free, too. This shows (3). □

Proof of Theorem 1. (1) of the proposition implies $\eta_3(C) \leq 6$. Since the vector bundle E_{max} constructed in Section 5 is stable by its semi-irreducibility and (1) of Proposition 3.5, we have the first assertion.

If E is stable and if $h^0(E) = 6$, then $|E|$ is semi-irreducible by (3) of the proposition. Hence the second assertion follows from Proposition 5.8. □

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