RULED SURFACES WITH NON-TRIVIAL SURJECTIVE ENDOMORPHISMS

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Abstract. Let X be a non-singular ruled surface over an algebraically closed field of characteristic zero. There is a non-trivial surjective endomorphism $f\colon X \to X$ if and only if X is (1) a toric surface, (2) a relatively minimal elliptic ruled surface, or (3) a relatively minimal ruled surface of irregularity greater than one which turns to be the product of $\mathbb{P}^1$ and the base curve after a finite étale base change.

Introduction

We work over an algebraically closed field $\mathfrak{K}$ of characteristic zero. Our interest is to determine when a non-singular projective surface X has a non-trivial surjective endomorphism $f\colon X \to X$. Here an *endomorphism* simply means a morphism into itself. A *non-trivial surjective endomorphism* is a surjective endomorphism which is not an isomorphism. If $\kappa(X) \geq 0$, then the endomorphism f is étale and X is a minimal model. Moreover in the case $\kappa(X) \geq 0$, it is known (cf. [F]) that X has a non-trivial surjective endomorphism if and only if X is an abelian surface, a hyper-elliptic surface, or a minimal elliptic surface of $\kappa(X) = 1$ and $\chi(\mathcal{O}_X) = 0$. In this article, we treat the rest case: $\kappa(X) = -\infty$. This is the case X is a *ruled surface*, which is called a *birationally ruled surface* in some article. This problem is studied in several years by E. Sato and his student M. Segami. The following result is obtained by Segami [S].

Theorem 1. *Suppose that X is an irrational ruled surface with a non-trivial surjective endomorphism. Then X is relatively minimal. If further the irregularity $q(X)$ is greater than one, then the $\mathbb{P}^1$-bundle structure $X \to B$ is associated with a semi-stable vector bundle of rank two of B.*

He proved more about possible vector bundles. For the rational case, Sato posed the following:

Conjecture 2. If X is a rational surface with a non-trivial surjective endomorphism, then X is a toric variety.

1991 *Mathematics Subject Classification.* 14J26.

Key words and phrases. ruled surface, endomorphism.

A projective variety X is called a *toric variety* if there is a Zariski-open subset T such that T is a two-dimensional algebraic torus and the embedding $T \subset X$ is a torus embedding (cf. [TE]). We shall give an affirmative answer to the conjecture and characterize the irrational surfaces.

Theorem 3. *Let X be a ruled surface. It has a non-trivial surjective endomorphism if and only if X is one of following surfaces*:

(1) *a toric surface*;
(2) *a $\mathbb{P}^1$-bundle over an elliptic curve*;
(3) *a $\mathbb{P}^1$-bundle over a non-singular projective curve B of genus $g(B) > 1$ such that $X \times_B B' \simeq \mathbb{P}^1 \times B'$ for some finite étale morphism $B' \to B$.*

In the first section, we shall construct non-trivial surjective endomorphisms in the three cases above. In the case (2), we use the formula in [Mu] on the pull-back of invertible sheaves by the multiplication mapping of elliptic curve. The case (3) is reduced to the construction of equivariant endomorphisms of $\mathbb{P}^1$ with respect to a given action of a finite group. All the finite subgroups of $\mathrm{SL}(2, \mathfrak{K})$ are classified up to conjugate (cf. [K]). We shall construct endomorphisms explicitly by using some semi-invariant polynomials. In the second section, we begin with studying the set $\mathcal{S}(X)$ of irreducible curves with negative self-intersection numbers. The existence of non-trivial endomorphism f yields strong conditions. For example, $\mathcal{S}(X)$ is a finite set and there is a positive integer m such that $f^m(C) = C$ for any $C \in \mathcal{S}(X)$ (cf. Proposition 10), where f^m stands for the m-times composite $f \circ f \circ \cdots \circ f$. Thus we may assume $f(C) = C$ for any $C \in \mathcal{S}(X)$ by replacing f by f^m. The ramification formula for f also yields some condition on the dual graph of $\mathcal{S}(X)$. We then have a simplified proof of Theorem 1 in Proposition 12, and further characterize the irrational surfaces in Theorem 13. Conjecture 2 is solved affirmatively in Theorem 14.

The author thanks to Professor Y. Fujimoto for introducing him to this problem. He also thanks to Professor O. Fujino for the careful reading of the manuscript.

1. Construction of endomorphisms

Lemma 4. *A toric variety has a non-trivial surjective endomorphism.*

Proof. Let T be an algebraic torus. Let M and N, respectively, be the groups of characters and of one-parameter subgroups of T. A torus embedding $T \subset X$ is defined by a collection of rational convex polyhedral cones σ in $N \otimes \mathbb{R}$. The multiplication mapping $T \to T$ by an integer $m > 1$ induces an endomorphism of group algebras $A_\sigma := \mathfrak{K}[\sigma^\vee \cap M]$. Since X is a natural union of $\mathrm{Spec}\, A_\sigma$, the multiplication mapping extends to an endomorphism of X. □

The following statement is mentioned in [S] without proof.

Proposition 5. *A relatively minimal elliptic ruled surface has a non-trivial endomorphism.*

Proof. Let $\pi\colon X = \mathbb{P}_B(\mathcal{E}) \to B$ be the ruling of a relatively minimal elliptic ruled surface associated with a locally free sheaf $\mathcal{E}$ of rank two over an elliptic curve B. We may assume that $\mathcal{E}$ is one of the following sheaves:

(1) $\mathcal{E} = \mathcal{O}_B \oplus \mathcal{L}$ for an invertible sheaf $\mathcal{L}$;

(2) There is a non-trivial extension

$$0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B \to 0;$$

(3) There exist a point $b \in B$ and a non-trivial extension

$$0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B([b]) \to 0.$$

Here, $\mathcal{O}_B([b])$ stands for the invertible sheaf associated with the prime divisor $[b]$ consisting of b. We shall construct endomorphisms in each cases.

Case (1). We want to construct an endomorphism $\nu\colon B \to B$ such that

$$(*_m) \qquad \nu^*\mathcal{L} \simeq \mathcal{L}^{\otimes m}$$

for some integer m. If the ν exists, then the natural embedding

$$\mathcal{O}_B \oplus \mathcal{L}^{\otimes m} \hookrightarrow \mathrm{Sym}^m(\mathcal{O}_B \oplus \mathcal{L}) = \mathcal{O}_B \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes m}$$

induces a homomorphism $\nu^*\mathcal{E} \to \mathrm{Sym}^m(\mathcal{E})$. This defines a morphism

$$X = \mathbb{P}_B(\mathcal{E}) \to X \times_{B,\nu} B = \mathbb{P}_B(\nu^*\mathcal{E})$$

over B and an endomorphism $X \to X$. Let us fix a point $0 \in B$ and let us give B a unique abelian group structure whose zero is 0. We seek a positive integer n and a point $c \in B$ such that the composite $\nu = \mu_n \circ T_c$ of the translation morphism $T_c\colon y \mapsto y + c$ and the multiplication mapping $\mu_n\colon B \to B$ by n, satisfies the condition $(*_m)$ for some m. There is an invertible sheaf $\mathcal{L}_0$ of degree zero such that

$$\mathcal{L} \simeq \mathcal{O}_B([0])^{\otimes d} \otimes \mathcal{L}_0$$

for $d = \deg \mathcal{L}$. We have the following isomorphisms (cf. [Mu]):

$$\mu_n^*\mathcal{L}_0 \simeq \mathcal{L}_0^{\otimes n}, \quad \text{and} \quad \mu_n^*\mathcal{O}_B([0]) \simeq \mathcal{O}_B([0])^{\otimes n^2}.$$

Since T_c does not change $\mathcal{L}_0$, we have

$$T_c^*\mu_n^*\mathcal{L} \simeq \mathcal{O}_B([-c])^{\otimes n^2 d} \otimes \mathcal{L}_0^{\otimes n}.$$

The condition $(*_m)$ for $\nu = \mu_n \circ T_c$ is satisfied if

$$\mathcal{O}_B([-c])^{\otimes n^2 d} \otimes \mathcal{O}_B([0])^{\otimes(-n^2 d)} \simeq \mathcal{L}_0^{\otimes(n^2-n)}.$$

For any invertible sheaf $\mathcal{M}$ of degree zero, there is a point c such that

$$T_c^*\mathcal{O}_B([0]) \otimes \mathcal{O}_B([0])^{\otimes(-1)} \simeq \mathcal{O}_B([-c]-[0]) \simeq \mathcal{M}.$$

Since the group $\mathrm{Pic}^0(B)$ of invertible sheaves of degree zero is divisible, we can find an expected point c for any positive integer n.

Case (2). Let μ_m be the multiplication mapping above. Then the induced exact sequence of (2) by μ_m^* is not split. Thus $\mu^*\mathcal{E} \simeq \mathcal{E}$.

Case (3). A stable vector bundle of rank two on B is isomorphic to the $\mathcal{E}$ twisted by an invertible sheaf for a point b. The pull-back $\mu_m^*\mathcal{E}$ for an odd integer m is still a semi-stable vector bundle of odd degree. Thus $\mu_m^*\mathcal{E}$ is stable. Hence

$$T_c^*\mu_m^*\mathcal{E} \simeq \mathcal{E} \otimes \mathcal{N}$$

for a point $c \in B$ and for an invertible sheaf $\mathcal{N}$. The isomorphism induces $X \simeq X \times_{B,\nu} B$ for $\nu = \mu_m \circ T_c$. □

Lemma 6. *Let G be a finite group acting on $\mathbb{P}^1$. Then there exists an equivariant non-trivial surjective endomorphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$; it satisfies the condition: $f(g \cdot z) = g \cdot f(z)$ for any $z \in \mathbb{P}^1$ and $g \in G$.*

Proof. We may assume that the action of G is faithful; $G \subset \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}(2, \mathfrak{K})$. Let V be the two-dimensional vector space $H^0(\mathbb{P}^1, \mathcal{O}(1))$ and let us fix a basis $\{x, y\}$ of V, which defines a homogeneous coordinate. Then $\mathbb{P}^1 = \mathbb{P}(V)$ and g^* induces an automorphism of V up to scalar. Thus there is a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{G} \to G \to 1,$$

such that V is a right $\widetilde{G}$-module and that the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as (-1). An element $\widetilde{g} \in \widetilde{G}$ acts on V as

$$\begin{pmatrix} x^{\widetilde{g}} \\ y^{\widetilde{g}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}(2, \mathfrak{K})$. The corresponding automorphism $g \in G$ is written in terms of the in-homogeneous coordinate $z = x/y$ as:

$$z \longmapsto \frac{az+b}{cz+d}.$$

It is well-known that for a suitable in-homogeneous coordinate $z \in \mathbb{P}^1$, G and the action of G are described in one of the following ways (cf. [K]):

(1) G is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m. The action of the generator 1 is:

$$z \mapsto \varepsilon_m z.$$

(2) G is a dihedral group D_n of order $2n$. The action of two generators is written as:

$$z \mapsto \varepsilon_n z, \quad \text{and} \quad z \mapsto z^{-1}.$$

(3) G is the tetrahedral group, which is isomorphic to the alternating group $\mathfrak{A}_4$. The action is given by:

$$z \mapsto -z, \quad \text{and} \quad z \mapsto \frac{z+\sqrt{-1}}{z-\sqrt{-1}}.$$

(4) G is the octahedral group, which is isomorphic to the symmetric group $\mathfrak{S}_4$. The action is given by:

$$z \mapsto \sqrt{-1}z, \quad \text{and} \quad z \mapsto \frac{z+\sqrt{-1}}{z-\sqrt{-1}}.$$

(5) G is the icosahedral group, which is isomorphic to the alternating group $\mathfrak{A}_5$. The action is given by:

$$z \mapsto \rho z, \quad \text{and} \quad z \mapsto -\frac{(\rho-\rho^{-1})z-(\rho^2-\rho^{-2})}{(\rho^2-\rho^{-2})z+(\rho-\rho^{-1})}.$$

Here, ε_m is the primitive m-th root of 1 defined as follows: The field $\mathfrak{K}$ contains the field $\overline{\mathbb{Q}}$ of algebraic numbers. We fix an inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$ to the field of complex numbers. Let $\varepsilon_m \in \mathfrak{K}$ correspond to $\exp(2\pi\sqrt{-1}/m)$. As special cases, we set $\sqrt{-1} := \varepsilon_4$ and $\rho := \varepsilon_5$.

In the cases (1) and (2), the endomorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$f(z) = z^{m+1}, \quad \text{and} \quad f(z) = -z^{-(2n-1)}$$

are G-equivariant, respectively. For the rest cases, we shall construct a $\widetilde{G}$-linear injection

$$V \otimes L \hookrightarrow \mathrm{Sym}^d(V) = H^0(\mathbb{P}^1, \mathcal{O}(d))$$

for a one-dimensional representation space L of $\widetilde{G}$ and for an integer $d > 1$. If the linear sub-system of $|\mathcal{O}(d)|$ defined by the subspace $V \otimes L \subset \mathrm{Sym}^d(V)$ is base-point free, then it induces a G-equivariant endomorphism of $\mathbb{P}^1$. Suppose that $F(x,y) \in \mathbb{C}[x,y]$ be a non-zero homogeneous polynomial of degree $d+1$ such that $F(x,y) \in \mathrm{Sym}^{d+1}(V)$ is semi-invariant under $\widetilde{G}$;

$$F(ax+by, cx+dy) = \delta(\widetilde{g})F(x,y)$$

for an one-dimensional character δ of $\widetilde{G}$. Let L be the one-dimensional representation space associated with δ. Thus F induces a $\widetilde{G}$-linear injection $L \to \mathrm{Sym}^{d+1}(V)$. We have the decomposition

$$V \otimes \mathrm{Sym}^{d+1}(V) \simeq \mathrm{Sym}^{d+2}(V) \oplus \mathrm{Sym}^d(V)$$

as $\mathrm{SL}(V)$-modules. The projection $V \otimes \mathrm{Sym}^{d+1}(V) \to \mathrm{Sym}^d(V)$ is given by:

$$(\alpha x + \beta y) \otimes H(x,y) \longmapsto \beta \frac{\partial H}{\partial x}(x,y) - \alpha \frac{\partial H}{\partial y}(x,y),$$

for α, $\beta \in \mathfrak{K}$ and for $H(x,y) \in \mathrm{Sym}^{d+1}(V)$. Thus the composite

$$\phi_F \colon V \otimes L \to V \otimes \mathrm{Sym}^{d+1}(V) \to \mathrm{Sym}^d(V)$$

is $\widetilde{G}$-linear.

Cases (3) and (4). We know the following semi-invariant polynomial (cf. [K]):

$$F(x,y) = xy(x^4 - y^4).$$

Thus ϕ_F is given by

$$x \mapsto -x(x^4 - 5y^4), \quad \text{and} \quad y \mapsto y(5x^4 - y^4).$$

There are no common roots in the two polynomials above. Hence we have an equivariant endomorphism

$$f(z) = -\frac{z(z^4 - 5)}{5z^4 - 1}.$$

Case (5). We know the following invariant polynomial (cf. [K]):

$$F(x,y) = xy(x^{10} + 11x^5y^5 - y^{10}).$$

Thus ϕ_F is given by

$$x \mapsto -x(x^{10} + 66x^5y^5 - 11y^{10}), \quad \text{and} \quad y \mapsto y(11x^{10} + 66x^5y^5 - y^{10}).$$

There are no common roots in the two polynomials above. Hence we have an equivariant endomorphism

$$f(z) = -\frac{z(z^{10} + 66z^5 - 11)}{11z^{10} + 66z^5 - 1}. \qquad \square$$

Theorem 7. *Let $\pi \colon X \to B$ be a relatively minimal ruled surface over a non-singular curve B of genus $g(B) > 1$. Then the following conditions are equivalent*:

1. *The relative anti-canonical divisor $-K_{X/B}$ is semi-ample*;
2. *There exist at least three distinct irreducible curves C satisfying $C^2 = 0$ and $\pi(C) = B$*;
3. *There exist a finite étale covering $\tau \colon B' \to B$ and an isomorphism $X \times_B B' \simeq \mathbb{P}^1 \times B'$.*

If the mutually equivalent conditions are satisfied, then X has a non-trivial surjective endomorphism.

Proof. (1) $\implies$ (2). Since $(-K_{X/B})^2 = 0$, then the linear systems $|-mK_{X/B}|$ define a fibration $h\colon X \to C$ onto a non-singular curve C. The fibers of π dominate C. Hence $C \simeq \mathbb{P}^1$. Let D be a general fiber of h. Then $D^2 = 0$ and $\pi(D) = B$.

(2) $\implies$ (3). If there is a section C_0 of π with $C_0^2 < 0$, then any other irreducible curve C with $\pi(C) = B$ is linearly equivalent to $aC_0 + \pi^*E$ for some $a > 0$ and a divisor E of $\mathbb{P}^1$. Since $0 \le C_0 \cdot C = aC_0^2 + \deg E$, we have $\deg E > 0$ and

$$C^2 = a^2C_0^2 + 2a \deg E > 0.$$

Hence, there is no section C_0 with $C_0^2 < 0$. Therefore π is associated with a semi-stable vector bundle of rank two on B. By [Mi, 3.1], $-K_{X/B}$ and any effective divisors of X are nef. Let C_i for $i = 1, 2, 3$ be the three irreducible curves with $C_i^2 = 0$ and $\pi(C_i) = B$. There exist rational numbers $a_i > 0$ and $\mathbb{Q}$-divisors E_i of $\mathbb{P}^1$ such that C_i is numerically equivalent to $-a_iK_{X/B} + \pi^*E_i$. We have $\deg E_i = 0$ from $C_i^2 = 0$. Thus $C_i \cdot C_j = C_i \cdot K_{X/B} = 0$ for any i, j. In particular, $C_i \to B$ is an étale morphism, since $(K_{X/B} + C_i) \cdot C_i = 0$. There is a finite étale morphism $\tau\colon B' \to B$ such that any component of $C_i \times_B B'$ is a section of $X \times_B B' \to B'$. Thus we may assume that C_i are sections of π. These are mutually disjoint. There exist divisors L_2 and L_3 of B such that $C_2 \sim C_1 + \pi^*L_2$ and that $C_3 \sim C_1 + \pi^*L_3$. Since $C_1 \cap C_2 = C_1 \cap C_3 = \emptyset$, we infer that $L_2 \sim L_3$. Thus $C_2 \sim C_3$. Therefore $X \simeq \mathbb{P}^1 \times B$.

(3) $\implies$ (1). We may assume that τ is a Galois covering. Let $\mu\colon X' := X \times_B B' \to X$ be the induced étale morphism. Then $\mu^*(-K_{X/B}) = p_1^*(-K_{\mathbb{P}^1})$ for the first projection $p_1\colon X' \to \mathbb{P}^1$. The action of the Galois group G on $X' \simeq \mathbb{P}^1 \times B'$ is given by:

$$(z, b) \longmapsto (gz, gb)$$

for $g \in G$, for a suitable action of G on $\mathbb{P}^1$. This is because the morphism $B' \to \mathrm{Aut}(\mathbb{P}^1)$ induced by g is constant. We may assume that G acts faithfully on $\mathbb{P}^1$; $G \subset \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathfrak{K})$. There exist two G-invariant effective divisors E_1 and E_2 of $\mathbb{P}^1$ such that $E_1 \sim E_2$ and $E_1 \cap E_2 = \emptyset$. Then $p_1^*E_1$ and $p_1^*E_2$ define a base-point free sub-linear system of $|-mK_{X/B}|$ for $m = \deg E_1$. Hence $-K_{X/B}$ is semi-ample.

We have a G-equivariant surjective endomorphism $\nu\colon \mathbb{P}^1 \to \mathbb{P}^1$ by Lemma 6. Thus $\nu \times \mathrm{id}$ is a G-equivariant non-trivial surjective endomorphism of $X' = \mathbb{P}^1 \times B'$. This descends to an endomorphism of X. □

2. Curves with negative self-intersection numbers

Let X be a non-singular ruled surface. Let $\mathrm{N}(X)$ denote the real vector space $\mathrm{NS}(X) \otimes \mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$. The intersection numbers $C_1 \cdot C_2$ of curves C_1 and C_2 define a natural intersection pairing on $\mathrm{N}(X)$. In this section, we assume that there exists a non-trivial surjective endomorphism $f\colon X \to X$. Then the pull-back $f^*\colon \mathrm{N}(X) \to \mathrm{N}(X)$ and the push-down $f_*\colon \mathrm{N}(X) \to \mathrm{N}(X)$ are both isomorphic and the composite $f_* \circ f^*$ is the multiplication map by $\deg f$. We note the projection formula: $f^*C \cdot D = C \cdot f_*D$ for C, $D \in \mathrm{N}(X)$.

Lemma 8. *Let C be an irreducible curve with $C^2 < 0$ and let $C_1 = f(C)$ be the image of C by f. Then there exist positive integers a and b such that $f^*C_1 = bC$ and $f_*C = aC_1$. In particular,* $\deg f = ab$ *and* $C_1^2 = (b/a)C^2 < 0$.

Proof. We have $f_*C = aC_1$ for the mapping degree a of $C \to C_1$. If C' is another irreducible curve with $f(C') = C_1$, then $f_*C' = \alpha f_*C$ in $\mathrm{N}(X)$ for some positive rational number α. Since f_* is an isomorphism, $C' = \alpha C$ in $\mathrm{N}(X)$. Thus $C' = C$, since $C^2 < 0$. Therefore $f^*C_1 = bC$ for a positive integer b. □

Let us consider the following sets of irreducible curves:

$$\mathcal{S}(X) := \{C \,|\, C^2 < 0\}, \quad \text{and} \quad \mathcal{S}_0(X) := \{C \,|\, C^2 < 0, \text{ and } C \subset \operatorname{Supp} R\},$$

where R stands for the ramification divisor of f; it is defined by the ramification formula

$$K_X \sim f^*K_X + R.$$

The map $f\colon \mathcal{S}(X) \to \mathcal{S}(X)$ given by $C \mapsto f(C)$ is bijective by Lemma 8.

Lemma 9. *If $C \in \mathcal{S}(X)$, then $f^m(C) \in \mathcal{S}_0(X)$ for a positive integer m.*

Proof. Let $C_1 = f(C)$ and let a and b be the same numbers as Lemma 8. The condition $C \subset \operatorname{Supp} R$ is equivalent to $b \geq 2$. If $b = 1$, then $|C_1^2| = (\deg f)^{-1}|C^2| < |C^2|$. Thus $f^m(C) \subset \operatorname{Supp} R$ for some m. □

Proposition 10. *The set $\mathcal{S}(X)$ is finite and there is a positive integer m such that $f^m(C) = C$ for any $C \in \mathcal{S}(X)$.*

Proof. For any curve $C \in \mathcal{S}_0(X)$, there exist infinitely many positive integers m such that $f^m(C) \in \mathcal{S}_0(X)$ by Lemma 8. If $f^m(C) = f^n(C)$ for some $0 < m < n$, then $f^m(C) = f^m(f^{n-m}(C))$. Thus $C = f^{n-m}(C)$ by the injectivity of $f\colon \mathcal{S}(X) \to \mathcal{S}(X)$. Let m_C be the smallest positive integer m such that $f^m(C) = C$. We put

$$m_0 := \prod_{C \in \mathcal{S}_0(X)} m_C.$$

Then $f^{m_0}(C) = C$ for any $C \in \mathcal{S}_0(X)$. If $C' \in \mathcal{S}(X) \setminus \mathcal{S}_0(X)$, then $f^{m'}(C') \in \mathcal{S}_0(X)$ for some $m' > 0$. Hence $f^{m_0+m'}(C') = f^{m'}(C')$ and thus $f^{m_0}(C') = C'$ by the injectivity. Since we can choose $m' < m_0$, we have $f^{m_0-m'}(f^{m'}(C')) = C'$. Hence $\mathcal{S}(X) = \bigcup_{m>0} f^m(\mathcal{S}_0(X))$. Therefore f^{m_0} is identical on $\mathcal{S}(X)$ and $\mathcal{S}(X)$ is a finite set. □

We may assume that $f(C) = C$ for $C \in \mathcal{S}(X)$ by replacing f by f^{m_0}. Then we have $a = b$ in Lemma 8 for $C \in \mathcal{S}(X)$, since $(\deg f)C_1^2 = b^2C^2$. Therefore, $\deg f = a^2$ and $\mathrm{mult}_C R = a - 1$ for any curve $C \in \mathcal{S}(X)$. In particular, $\mathcal{S}(X) = \mathcal{S}_0(X)$ for the f. We define

$$\Delta := R - (a-1) \sum_{C \in \mathcal{S}(X)} C.$$

Then Δ is a nef and effective divisor. We have the ramification formula

$$K_X \sim f^*K_X + \Delta + (a-1) \sum_{C \in \mathcal{S}(X)} C. \tag{2.1}$$

Let C be a curve in $\mathcal{S}(X)$. The ramification divisor R_C for $f|_C \colon C \to C$ is calculated as:

$$R_C = (R + C - f^*C)|_C = \Delta|_C + (a-1) \sum_{C \neq C_\lambda \in \mathcal{S}(X)} C_\lambda|_C.$$

Hence we have the following relation of intersection numbers with C:

$$(a-1)(K_X \cdot C + C^2) + \Delta \cdot C + (a-1) \sum_{C \neq C_\lambda \in \mathcal{S}(X)} C_\lambda \cdot C = 0. \tag{2.2}$$

Lemma 11. *Let C be a curve in $\mathcal{S}(X)$. Then the following three properties hold:*

(1) *The arithmetic genus $p_a(C)$ is at most one.*
(2) *If $p_a(C) = 1$, then C is a connected component of* $\mathrm{Supp}\, R$.
(3) *C intersects at most two other irreducible curves in $\mathcal{S}(X)$. The intersection is locally transversal.*

If a connected component of $\mathcal{S}(X)$ is not irreducible, then it is a chain or a cycle of non-singular rational curves. Curves in the component are apart from $\mathrm{Supp}\, \Delta$ *except for edge curves of chain.*

Proof. (1) and (2) follow from the inequality

$$2p_a(C) - 2 + \sum_{C \neq C_\lambda \in \mathcal{S}(X)} C_\lambda \cdot C \leq 0$$

induced from (2.2).

(3). If C intersects another $C' \in \mathcal{S}(X)$, then C and C' are non-singular rational curves and

$$\sum_{C \neq C_\lambda \in \mathcal{S}(X)} C_\lambda \cdot C \leq 2.$$

Suppose that $C \cap C'$ consists of one point P and $C \cdot C' = 2$. Then $C \cup C'$ is a connected component of $\mathrm{Supp}\, R$ and $R_C = (a-1)C'|_C = 2(a-1)P$. This is a contradiction since $f|_C$ is unramified over the affine line $C \setminus \{P\}$. Therefore, if $C \cdot C' = 2$, then C and C'

intersects transversely at two distinct points. If C intersects two other irreducible curves C_1 and C_2 in $\mathcal{S}(X)$, then the intersection points $C \cap C_1$ and $C \cap C_2$ are distinct, by the same reason. The rest assertion is derived from these properties. □

We call an exceptional curve of the first kind by a (-1)-curve for short. Let C be a (-1)-curve and let $X \to X_1$ be the contraction of C. Then f descends to X_1. Any curve in $\mathcal{S}(X_1)$ is the image of a curve in $\mathcal{S}(X)$. Thus f also stabilizes $\mathcal{S}(X_1)$. Let us choose a successive blow-downs

$$\mu \colon X \to X_1 \to X_2 \to \cdots \to X_l,$$

of (-1)-curves. Then f descends to the final X_l and it stabilizes $\mathcal{S}(X_l)$. We assume that X_l is relatively minimal.

Proposition 12. (1) *If X is an irrational surface, then X is isomorphic to the total space of the $\mathbb{P}^1$-bundle over a non-singular irrational curve.*

(2) *If the irregularity $q(X)$ is greater than one, then the $\mathbb{P}^1$-bundle is associated with a semi-stable vector bundle of rank two.*

(3) *If X is rational, then any curve in $\mathcal{S}(X)$ is a non-singular rational curve.*

Proof. (1) and (2). We use some argument of [S]. Let $\pi \colon X \to B$ be the ruling induced from the Albanese map. Then there is a unique endomorphism $f_B \colon B \to B$ such that $f_B \circ \pi = \pi \circ f$. Suppose that π is not a $\mathbb{P}^1$-bundle. Then an irreducible component C of any singular fiber is contained in $\mathcal{S}(X)$. Since $f^{-1}C = C$, the endomorphism f_B fixes the point $b := \pi(C)$. Thus f_B is an isomorphism since B is irrational. We infer that $f^*C = aC = C$ from $f^*\pi^*(b) = \pi^*(b)$. This contradicts to $a > 1$. Thus X is relatively minimal and π is a $\mathbb{P}^1$-bundle. Suppose that $q(X) > 1$. Then the induced morphism f_B is an isomorphism. If π is not associated with a semi-stable vector bundle of B, then there is a section C with $C \in \mathcal{S}(X)$. We know that the mapping degree of $f|_C \colon C \to C$ is a. Thus the mapping degree of the composite

$$C \subset X \xrightarrow{f} X \xrightarrow{\pi} B$$

is also a. This is a contradiction.

(3). If $p_a(C) = 1$ for a curve $C \in \mathcal{S}(X)$, then $\mu \colon X \to X_l$ is an isomorphism along C by Lemma 11. Thus $p_a(C_l) = 1$ and $C_l^2 < 0$ for the image $C_l := \mu(C)$. We may assume that X_l is isomorphic to the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated with $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)$ for $e > 0$. Then C_l should be the minimal section of the $\mathbb{P}^1$-bundle, since this is the unique curve in X with negative self-intersection number. Thus $p_a(C) = 0$. □

Theorem 13. *Let $\pi \colon X \to B$ be a $\mathbb{P}^1$-bundle over a non-singular curve B of genus $g(B) > 1$. Then the following two conditions are equivalent*:

(1) *X has a non-trivial surjective endomorphism*;
(2) *There is a finite étale morphism $B' \to B$ such that $X \times_B B' \simeq \mathbb{P}^1 \times B'$ over B'.*

Proof. (2) $\Longrightarrow$ (1) is proved in Theorem 7. We shall show (1) $\Longrightarrow$ (2). Let $f\colon X \to X$ be a non-trivial surjective endomorphism. We may assume that f_B is identical by replacing f by f^m for some m. Hence $\pi \circ f = \pi$. The ramification divisor R for f is not zero, since f is not étale along fibers of π. The $\mathbb{P}^1$-bundle π is associated with a semi-stable vector bundle of rank two by Proposition 12. Hence the divisor $-K_{X/B}$ and any effective divisors are nef by [Mi, 3.1]. In particular, $R^2 = f^*(-K_{X/B}) \cdot (-K_{X/B}) = 0$ and $\Delta_i \cdot \Delta_j = 0$ for any irreducible components Δ_i and Δ_j of R. We see that $\Delta_j \to B$ is étale, since $(K_{X/B} + \Delta_j) \cdot \Delta_j = 0$. Let $B' \to B$ be any finite étale morphism. Then f induces an endomorphism f' of $X' = X \times_B B'$. Here the ramification divisor R' of f' is the pull-back of R. Hence we may assume from the beginning that every irreducible component Δ_j of R is a section of π. Then R has at least two irreducible components; otherwise, f is unramified over $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\text{one point}\}$ on fibers of π. Therefore, π is associated with a vector bundle $\mathcal{E}$ of rank two over B such that $\mathcal{E} \simeq \mathcal{O}_B \oplus \mathcal{L}$ for an invertible sheaf $\mathcal{L}$ with $\deg \mathcal{L} = 0$.

Let $\mathcal{O}_X(1)$ be the tautological line bundle associated with $\mathcal{E}$. We have an isomorphism $f^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d) \otimes \pi^*\mathcal{M}$ for an invertible sheaf $\mathcal{M}$ of B and for $d := \deg f > 1$. Note that $\deg \mathcal{M} = 0$, since $\mathcal{O}_X(1) \cdot \mathcal{O}_X(1) = \deg \mathcal{E} = 0$. Thus we have an injection

$$\phi\colon \mathcal{E} \simeq \pi_*\mathcal{O}_X(1) \hookrightarrow \pi_* f^*\mathcal{O}_X(1) = \mathrm{Sym}^d(\mathcal{E}) \otimes \mathcal{M}.$$

Here, $\phi(\mathcal{E})$ is a direct summand, since $\mathcal{O}_X$ is a direct summand of $f_*\mathcal{O}_X$. Let ϕ_j be the composite of ϕ and the projection to $\mathcal{L}^{\otimes j} \otimes \mathcal{M}$ induced from

$$\mathrm{Sym}^d(\mathcal{O}_B \oplus \mathcal{L}) \simeq \mathcal{O}_B \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes d} \to \mathcal{L}^{\otimes j},$$

for $0 \le j \le d$. Then ϕ_0 and ϕ_d are surjective, since the homomorphism $\pi^*\mathcal{E} \to f^*\mathcal{O}_X(1)$ induced from ϕ is surjective. Suppose that the composite of $\mathcal{O}_B \subset \mathcal{E}$ and ϕ_0 is not zero. Then $\mathcal{O}_B \simeq \mathcal{M}$. If $\mathcal{O}_B \not\simeq \mathcal{L}^{\otimes d}$, then the composite of $\mathcal{L} \subset \mathcal{E}$ and ϕ_d is surjective. Hence $\mathcal{L} \simeq \mathcal{L}^{\otimes d}$. Suppose next that the composite of $\mathcal{O}_B \subset \mathcal{E}$ and ϕ_d is not zero. Then $\mathcal{O}_B \simeq \mathcal{L}^{\otimes d} \otimes \mathcal{M}$. If $\mathcal{L}^{\otimes d} \not\simeq \mathcal{O}_B$, then the composite of $\mathcal{L} \subset \mathcal{E}$ and ϕ_0 is surjective. Hence $\mathcal{L} \simeq \mathcal{M}$. Therefore in any case, $\mathcal{L}^{\otimes(d-1)}$, $\mathcal{L}^{\otimes d}$, or $\mathcal{L}^{\otimes(d+1)}$ is isomorphic to $\mathcal{O}_B$. Since $d > 1$, $\mathcal{L}$ is a torsion element of $\mathrm{Pic}(B)$. We have a finite étale cyclic covering $\tau\colon B' \to B$ such that $\tau^*\mathcal{L} \simeq \mathcal{O}_B$. Therefore $X \times_B B' \simeq \mathbb{P}^1 \times B'$ over B'. □

Theorem 14. *If X is a rational surface with a non-trivial surjective endomorphism, then X is a toric variety.*

Proof. We may assume that X is not relatively minimal and the X_l above is associated with $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)$ for $e > 0$. Let $p\colon X_l \to B = \mathbb{P}^1$ denote the $\mathbb{P}^1$-bundle structure and let $\pi := p \circ \mu\colon X \to X_l \to \mathbb{P}^1$ denote the composite. Irreducible components of any singular fiber of π and the proper transform C_0 of the minimal section of p belong to $\mathcal{S}(X)$. Therefore $\mathcal{S}(X)$ is connected and the number of singular fibers of π is at most two by Lemma 11. Let $F_b = \pi^*(b)$ be a singular fiber. Then F_b is a chain of non-singular rational curves. Let

$$F_b = \Gamma_{b,0} + e_{b,1}\Gamma_{b,1} + \cdots + e_{b,l_b-1}\Gamma_{b,l_b-1} + \Gamma_{b,l_b}$$

be the irreducible decomposition such that

- C_0 intersects only $\Gamma_{b,0}$ in F_b,
- $\Gamma_{b,j}$ intersects only $\Gamma_{b,j-1}$ and $\Gamma_{b,j+1}$ in F_b for $1 \le j \le l_b - 1$,
- Γ_{b,l_b} intersects only Γ_{b,l_b-1} in $F_{[b]}$,
- $e_{b,j}$ is the multiplicity of F_b along $\Gamma_{b,j}$.

One of the following two cases occurs.

Case 1. $\mathcal{S}(X)$ contains a horizontal curve C' different from C_0.

The curve C' is unique by Lemma 11; C' intersects only Γ_{b,l_b} in singular fibers F_b.

Subcase 1-1. X has two singular fibers.

The morphism $\mu\colon X \to X_l$ is considered to be a sequence of blow-ups whose centers are double points of the image of $\mathcal{S}(X)$. The image of $\mathcal{S}(X)$ in X_l consists of two fibers, the minimal section, and a section apart from the minimal section. Hence X is a toric variety.

Subcase 1-2. X has only one singular fiber F_b.

If C' intersect C_0, then the point $P := C' \cap C_0$ is apart from F_b and is fixed by f, i.e., $f^{-1}(P) = P$. Thus $\pi(P)$ is contained in the ramification locus of the induced morphism $f_B\colon B \to B$. It follows that the fiber $\pi^{-1}(\pi(P))$ is also contained in the ramification locus $\operatorname{Supp} R$ of f. This contradicts to Lemma 11. Therefore C' is apart from C_0. The morphism $\mu\colon X \to X_l$ is considered to be a sequence of blow-ups whose centers are double points of the image of $\mathcal{S}(X)$. The image of $\mathcal{S}(X)$ in X_l consists of a fiber, the minimal section, and a section apart from the minimal section. Hence X is a toric variety.

Case 2. $\mathcal{S}(X)$ contains no horizontal curve except for C_0.

Then $\mathcal{S}(X)$ is a chain. In the singular fiber F_b, there is a (-1)-curve different from Γ_{b,l_b}. Hence we have a sequence of contraction of (-1)-curves

$$\mu'\colon X \to X'_1 \to X'_2 \to \cdots \to X'_l$$

which does not contract Γ_{b,l_b}. Thus μ' is a sequence of blow-ups whose centers are double points of the image of $\mathcal{S}(X)$. If there is a section C'_0 of $X'_l \to B$ such that $(C'_0)^2 < 0$, then $C_0' = \mu'(C_0)$, since the proper transform of C_0' in X should be contained in $\mathcal{S}(X)$. Therefore, we have a section C' of $\pi\colon X \to B$ such that C' is apart from C_0 and that C' intersects Γ_{b,l_b} in each fiber F_b. Since the image $\mu'(C')$ is apart from $\mu'(C_0)$, X is a toric variety. □

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