

# Quantum invariants and finite group actions on three-manifolds

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**Abstract.** A 3-manifold  $M$  is said to be  $p$ -periodic ( $p \geq 2$  an integer) if and only if the finite cyclic group of order  $p$  acts on  $M$  with a circle as the set of fixed points. This paper provides a criterion for periodicity of rational homology three-spheres. Namely, we give a necessary condition for a rational homology three-sphere to be periodic with a prime period. This condition is given in terms of the quantum  $SU(3)$  invariant. We also discuss similar results for the Murakami-Ohtsuki-Okada invariant.

**Key words.** Group actions, rational homology three-spheres, periodic links, quantum invariants.

**AMS Classification.** 57M27

## INTRODUCTION

The last decade has seen many new invariants introduced to low dimensional topology. In the case of three-manifolds, these invariants were first proposed by E. Witten [27] who used the Feynman path integral based on the Chern-Simons gauge theory in order to prove the existence of an invariant of oriented 3-manifolds associated with each compact Lie group. The first rigorous construction of such invariants (called quantum invariants) was by Reshetikhin and Turaev [25] in the case of the Lie group  $SU(2)$ . Various approaches to the construction of the quantum  $SU(2)$  invariant followed. Particularly, Lickorish [16] gave an elementary construction of this invariant using the skein theory associated to the Kauffman bracket [10] evaluated at primitive roots of unity of order  $4r$ . Blanchet, Habegger, Masbaum and Vogel [2] then, extended Lickorish's approach to define the invariant at roots of unity of order  $2p$ . On the lines of Lickorish's work, Ohtsuki and Yamada [21] defined skein linear theory for  $SU(3)$  using Kuperberg's skein

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relations [12] for trivalent oriented graphs. Via this skein theory, they constructed magic elements to define the quantum  $SU(3)$  invariant of 3-manifolds at primitive roots of unity of order  $6r$ . Their work was extended by Miyazawa and Okamoto [18] applying it to primitive roots of unity of order  $3r$ . A more general construction was proposed by Yokota who defined a linear skein theory for quantum  $SU(N)$  invariants [26].

It is worth mentioning that we have no idea about the power and the preciseness of the above mentioned invariants in detecting the topology of a given 3-manifold, although the large number of approaches proposed to construct these invariants. In some sense the topological meaning of the quantum invariants remains mysterious and still far from being completely understood. The main purpose of this paper is to investigate the behaviour of the quantum  $SU(3)$  invariant in the case where the manifold presents some geometric properties (symmetries), discussing whether or not the symmetry of the given manifold is reflected on its invariants. Ultimately, by this we hope to shed some light on the behaviour of the quantum invariants as well as to provide a criterion for periodicity of 3-manifolds.

Let  $M$  be a 3-manifold (all the manifolds considered in this paper are oriented compact connected and without boundary) and let  $G$  be the finite cyclic group  $\mathbb{Z}/p\mathbb{Z}$ . One of the most intriguing questions in low dimensional topology is whether the group  $G$  acts on  $M$ . Some progress in solving this problem has been made using classical algebraic and geometric tools. Nevertheless, many related questions are still open. In this paper we shall focus our interests on the case where  $G$  acts on  $M$  with a circle as the set of fixed points. A manifold with such an action is said to be  $p$ -periodic. P. Gilmer [8], studied the Witten-Reshitikhin-Turaev  $SU(2)$  invariant of periodic 3-manifolds as well as the case where the action of  $G$  has no fixed points. On the other hand, we used different techniques in [5] to give a necessary condition for a rational homology 3-sphere to be periodic, this condition is given in terms of the quantum  $SU(2)$  invariant. The present paper introduces a new criterion for periodicity of 3-manifolds using the quantum  $SU(3)$  invariant. For technical reasons we will only deal with the case of rational homology 3-spheres. We also prove similar results for the Murakami-Ohtsuki-Okada invariant (MOO for short) [19]. This invariant is known as the simplest quantum invariant.

This paper is organized as follows, in paragraph 1 we present our results. In paragraph 2 we explain the relationship between periodic links and periodic 3-manifolds. A review of skein linear theory and definition of the quantum  $SU(3)$  invariant are given in paragraph 3. The proof of our main theorem will be given in paragraph 4. Further speculations are discussed in the last two paragraphs.

Let  $r \geq 4$  be an integer and  $M$  a 3-manifold. For  $k \in \mathbb{N}^*$ , let  $\Phi_k(A)$  denote the cyclotomic polynomial of order  $k$ ,  $A$  here is an indeterminate, and  $[k]$  denote the quantum integer  $\frac{A^{3k}-A^{-3k}}{A^3-A^{-3}}$ . Here we consider the quantum  $SU(3)$  invariant defined at roots of unity of order  $3r$ . This invariant will be denoted by  $\mathcal{I}_r$  (see the definition in paragraph 3). Let  $\Lambda_r = \mathbb{Z}[A^{\pm 1}]_{/\Phi_{3r}(A)}$ , we shall explain later that the invariant  $\mathcal{I}_r$  may be considered as an element of  $\Lambda_r[\frac{1}{3r}]$ . Consequently, if  $p$  is coprime to  $3r$ , it is possible to consider  $\mathcal{I}_r$  with coefficients reduced modulo  $p$ . Throughout the rest of this paper, if  $M$  is a periodic manifold then  $\overline{M}$  denotes the quotient space. Recall here that if  $M$  is a rational homology sphere then  $\overline{M}$  is also a rational homology sphere. For  $r \geq 4$  we will denote by  $G_r$  the element of  $\Lambda_r[\frac{1}{3r}]$  given by the following formula:

$$G_r(A) = A^{-36} \frac{\left(\sum_{k=0}^{r-1} A^{6k^2}\right)^2 \left(\sum_{k=0}^{3r-1} A^{2k^2}\right)^2}{3r^2}.$$

**Theorem 1.1** *Let  $p$  be a prime and  $M$  a rational homology 3-sphere. If  $M$  is  $p$ -periodic then for all odd  $r \geq 4$  such that  $p$  and  $3r$  are coprime, the following congruence holds:*

$$\mathcal{I}_r(M) \equiv (\mathcal{I}_r(\overline{M}))^p (G_r(A))^\alpha, \text{ modulo } p, [3]^p - [3],$$

where  $\alpha$  is an integer.

The congruence given by this theorem holds in the ring  $\Lambda_r[\frac{1}{3r}]$ . In this ring we have no idea about how large is the ideal generated by  $p$  and  $[3]^p - [3]$ . However, if one considers the case  $p$  congruent to  $\pm 1$  modulo  $r$ , then we can show that working modulo  $p$  and  $[3]^p - [3]$  is equivalent to working modulo  $p$ . This is explained by the following corollary written in the case of cyclic branched coverings of the 3-sphere.

**Corollary 1.2** *Let  $p \neq 3$  be a prime and  $M$  the  $p$ -fold cyclic covering of  $S^3$  branched along a knot  $K$ . For all odd  $r \geq 4$  such that  $p \equiv \pm 1$  modulo  $r$ , the following congruence holds:*

$$\mathcal{I}_r(M) \equiv (G_r(A))^\alpha \text{ mod } p,$$

where  $\alpha$  is an integer.

**Remark 1.3** It should be mentioned here that the condition of theorem 1.1 gives no obstruction to the periodicity of a given rational homology 3-sphere. However, if we restrict the condition

to cyclic branched coverings of  $S^3$  as in corollary 2.2, the condition takes a nice form and turns to be easy to handle. Indeed, this condition is not trivial, in order to prove this we can examine the case  $r = 5$ , for example. In this case  $G_5(A) = A^{-36}$ , hence modulo  $p$ ,  $\mathcal{I}_5$  is a power of  $A$ . Thus, given a rational homology 3-sphere, we just need to verify if  $\mathcal{I}_r(M)$  is not a power of  $A$  in order to conclude that  $M$  is not the  $p$ -fold branched covering of  $S^3$  (for appropriate values of  $p$ ). This is illustrated by the following example. Let  $L(2, 1)$  be the lens space of type  $(2, 1)$ . Put  $r = 5$ , the cyclotomic polynomial of order 15 is given by  $\Phi_{15}(A) = 1 - A + A^3 - A^4 + A^5 - A^7 + A^8$ . According to the general formula for the  $SU(3)$  invariants of lens spaces obtained in [18], we have the following:

$$\mathcal{I}_5(L(2, 1)) = 1 - A - A^2 + A^3 - A^4 + A^5 - A^7.$$

The powers of  $A$  in the ring  $\Lambda_5[\frac{1}{15}]$  are given by the list below:

$$\begin{aligned} &1, A, A^2, A^3, A^4, A^5, A^6, A^7, \\ &-1 + A - A^3 + A^4 - A^5 + A^7, \\ &A^2 - A^6 - 1 - A^3 + A^7, \\ &-1 - A^5, -A - A^6, -A^2 - A^7, \\ &1 - A - A^4 + A^5 - A^7, \\ &1 - A^2 + A^3 - A^4 + A^6 - A^7. \end{aligned}$$

Obvisouly,  $\mathcal{I}_5(L(2, 1))$  does not belong to the list above. Consequently, the lens space  $L(2, 1)$  is not the  $p$ -fold cyclic covering of the three-sphere branched along a knot  $K$  for any prime  $p$  congruent to  $\pm 1$  modulo 5.

**Proof of Corollary 1.2.** If  $M$  is the  $p$ -fold cyclic covering of  $S^3$  branched along a knot  $K$ , then  $M$  is  $p$ -periodic and its quotient is  $S^3$ . Recall that the invariant  $\mathcal{I}_r$  is normalized in such a way that  $\mathcal{I}_r(S^3) = 1$ , for all  $r \geq 4$ . If  $p \neq 3$  and congruent to  $\pm 1$  modulo  $r$  then  $p$  and  $3r$  are coprime. Thus, we have modulo  $p$ :

$$\begin{aligned} [3]^p - [3] &\equiv (1 + A^6 + A^{-6})^p - (1 + A^6 + A^{-6}) \\ &\equiv A^{6p} + A^{-6p} - A^6 - A^{-6} \equiv 0. \end{aligned}$$

The last equality holds because if we are working modulo  $\Phi_{3r}(A)$  then  $A^{3r} = 1$ . Hence if  $p \equiv 1$  modulo  $r$  then  $3r$  divides  $6(p - 1)$ . Therefore  $A^{6(p-1)} = 1$ , that is  $A^{6p} = A^6$  and  $A^{-6p} = A^{-6}$ , same argument for  $p \equiv -1$  modulo  $r$ .

Let  $M$  be a 3-manifold. It is well known from Lickorish's work [15] that  $M$  may be obtained from the 3-sphere  $S^3$  by surgery along a framed link  $L$  in  $S^3$ . Such a link is called a surgery presentation of  $M$ . Kirby [11] introduced elementary moves which we call here  $M1$  and  $M2$  for framed links and proved that two surgery presentations of the same manifold are related by a finite sequence of moves  $M1$  and  $M2$ . This section deals with surgery presentation of periodic manifolds. We here recall briefly some definitions and results, (see [24] for details).

**Definition 2.1** *Let  $p \geq 2$  an integer, a link  $L$  of  $S^3$  is said to be  $p$ -periodic if and only if there exists an orientation preserving auto-diffeomorphism of  $S^3$  such that:*

- 1-  $\text{Fix}(h)$  is homeomorphic to the circle  $S^1$ ,
- 2- the link  $L$  is disjoint from  $\text{Fix}(h)$ ,
- 3-  $h$  is of order  $p$ ,
- 4-  $h(L) = L$ .

If  $L$  is periodic we will denote by  $\bar{L}$  the quotient link.

If we consider  $S^3 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1\}$ , then by the positive solution of the Smith conjecture [3] we know that  $h$  is topologically conjugate to the standard rotation given by:

$$\begin{aligned} \varphi_p : S^3 &\longrightarrow S^3 \\ (z_1, z_2) &\longmapsto (e^{\frac{2i\pi}{p}} z_1, z_2). \end{aligned}$$

The set of fixed points for this diffeomorphism is the circle  $\Delta = \{(0, z_2) \in \mathbb{C}^2; |z_2|^2 = 1\}$ . If  $L$  is a  $p$ -periodic link we denote by  $lk(L, \Delta)$  the linking number of  $L$  and the circle  $\Delta$ . If we identify  $S^3$  with  $\mathbb{R}^3 \cup \infty$ ,  $\Delta$  may be seen as the standard  $z$ -axis. For example, the trefoil knot  $3_1$  is 3-periodic with linking number 2. Przytycki and Sokolov [24] introduced the notion of strongly periodic links as follows.

**Definition 2.2** *Let  $p \geq 2$  be an integer, a  $p$ -periodic link  $L$  is said to be strongly periodic if and only if the linking number of each component of  $L$  with the axis  $\Delta$  is congruent to zero modulo  $p$ .*

Now let us define periodic manifolds:

**Definition 2.3** *Let  $p \geq 2$  be an integer, a 3-manifold  $M$  is said to be  $p$ -periodic if and only if there exists an orientation preserving auto-diffeomorphism of  $M$ , such that:*

- 1-  $\text{Fix}(h)$  is homeomorphic to the circle  $S^1$ ,

2-  $h$  is of order  $p$ .

The 3-sphere  $S^3$  is  $p$ -periodic for all  $p \geq 2$ . The acting diffeomorphism is nothing but the above defined rotation  $\varphi_p$ . Another concrete example is the Brieskorn manifold  $\Sigma(2, 3, 7)$ , which has periods 2, 3 and 7. One can easily explicit the corresponding diffeomorphisms by considering  $\Sigma(2, 3, 7)$  as the intersection of the complex surface  $\{(z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^3 + |z_3|^7 = 0\}$  with an appropriate sphere of dimension 5 (see [17]).

The relationship between periodic links and periodic manifolds was first noticed by Goldsmith in [9]. Indeed, she only considered cyclic branched covers of  $S^3$  branched along a knot  $K$ . In this case, she gave a simple algorithm describing how from the knot  $K$  one can construct a periodic surgery presentation of the covering space. Recently, a much more general result was obtained by Przytycki and Sokolov who showed that any periodic 3-manifold can be obtained from the 3-sphere by surgery along a strongly periodic link. To be more precise, they proved the following theorem.

**Theorem 2.4** ([24]). *Let  $p$  be a prime. A 3-manifold  $M$  is  $p$ -periodic if and only if  $M$  is obtained from  $S^3$  by surgery along a strongly periodic link of period  $p$ .*

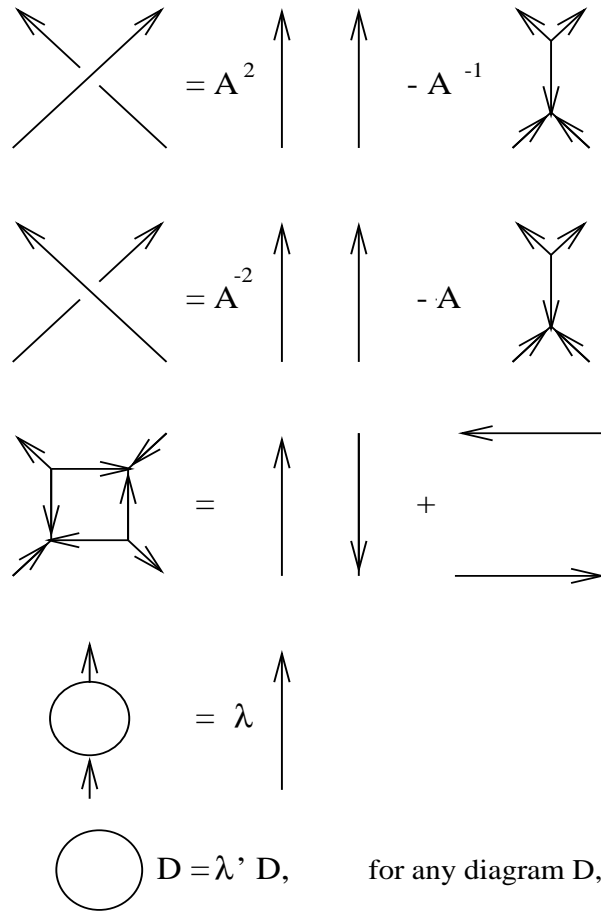
### 3- THE QUANTUM $SU(3)$ INVARIANT

#### **Linear skein theory for $SU(3)$**

This section is to review the construction of the quantum  $SU(3)$  invariant of 3-manifolds via linear skein theory. This skein theory was first introduced by Kuperberg [12] in order to construct an invariant of links in the three-sphere. Ohtsuki and Yamada proved in a joint work [21] that Kuperberg's invariant can be used to define the  $SU(3)$  invariant for 3-manifolds, details about the material exposed in this section can be found in [12], [18] and [21].

Let  $F$  be an oriented surface. The graphs considered in the sequel are oriented trivalent graphs possibly with loops with no vertices. Moreover, we assume that at each vertex the three edges are oriented in the same way, all inner or all outer. A diagram in  $F$  is an oriented trivalent graph immersed in  $F$  such that the singular points are only transverse double points, to each of which over and under crossing information is associated. Equivalently, locally a diagram is a trivalent graph or the diagram of an oriented link. As usual diagrams are considered here up

to isotopy. Throughout the rest of this paper  $\emptyset$  denotes the empty diagram and  $\bigcirc$  denotes the trivial knot. The following skein relations were introduced by Kuperberg [12].



the empty diagram  $\emptyset = 1$ .

FIGURE 1.

Where  $\lambda = \frac{A^6 - A^{-6}}{A^3 - A^{-3}}$  and  $\lambda' = \frac{A^9 - A^{-9}}{A^3 - A^{-3}}$ .

**Remark 3.1** These relations allowed Kuperberg to define an invariant of oriented links which we denote here by  $J$ . In fact, this invariant is nothing but a specialization of the Homfly polynomial [20]. By setting  $q = A^6$ , this invariant can be defined only by the skein relations:

- (i)  $J(\emptyset) = 1$
- (ii)  $J(\bigcirc \cup L) = (1 + q + q^{-1})J(L)$ ,
- (iii)  $q^{3/2}J(L_+) - q^{-3/2}J(L_-) = (q^{1/2} - q^{-1/2})J(L_0)$

where  $L_+$ ,  $L_-$  and  $L_0$  are three oriented links which are identical except near one crossing where they look like in the following figure:

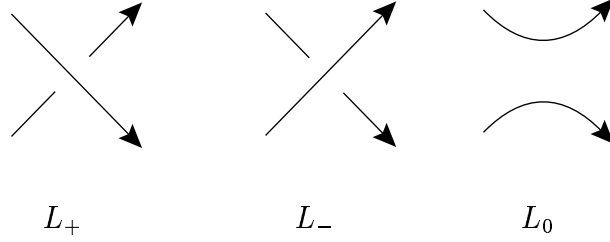


FIGURE 2

**Definition 3.2** Let  $F$  be an oriented surface. The skein module  $\mathcal{S}(F)$  is defined as the quotient  $\Lambda_r$ -module of formal linear sums of diagrams in  $F$  with coefficient in  $\Lambda_r$ , divided by the relations of Figure 1.

Let  $I$  be the unit interval  $[0, 1]$  and  $S^1 \times I$  the annulus. There is a natural product that we can define on  $\mathcal{S}(S^1 \times I)$ . Moreover, if we consider the two diagrams  $x$  and  $y$  as in figure 3, we have the following proposition.



FIGURE 3

**Proposition 3.3** ([21]). (i)  $\mathcal{S}(\mathbb{R}^2)$  is isomorphic to  $\Lambda_r$ .

(ii)  $\mathcal{S}(S^1 \times I)$  is isomorphic to  $\Lambda_r[x, y]$  as an algebra.

**The invariant  $\mathcal{L}_r$ .**

For any pair of positive integers  $(n, m)$ , we define the element  $P_{n,m}$  as in [21]. Let  $r \geq 4$ , we define  $\omega_r$  as the element of  $\mathcal{S}(S^1 \times I)$  given by the following formula:

$$\sum_{n+m \leq r-3, n, m \geq 0} [n+1][m+1][n+m+2]P_{n,m}.$$

If  $L = l_1 \cup l_2 \dots \cup l_k$  is a  $k$ -component framed link in the three-sphere, we denote by  $L_{(i_1, i'_1), \dots, (i_k, i'_k)}$  the link obtained from  $L$  by decorating the component  $l_j$  by the element  $x^{i_j} y^{i'_j}$  for all  $1 \leq j \leq k$ . We define the multibracket  $\langle \cdot, \dots, \cdot \rangle_L$  as the multilinear form :  $\mathcal{S} \times \mathcal{S} \times \dots \times \mathcal{S} \longrightarrow \mathbf{Z}[A^{\pm 1}]$  evaluated on set of generators as follows:

$$\langle x^{i_1} y^{i'_1}, \dots, x^{i_k} y^{i'_k} \rangle_L = J(L_{(i_1, i'_1), \dots, (i_k, i'_k)}).$$



Throughout the rest of this paper if  $L$  is a framed link we denote by  $\sigma_+(L)$  (resp.  $\sigma_-(L)$ ) the number of positive (resp. negative) eigenvalues of the linking matrix of  $L$ . In addition,  $U_+$  (resp  $U_-$ ) denotes the trivial knot with framing  $+1$  (resp.  $-1$ ). Let  $M$  be a three-manifold and  $L$  a surgery presentation of  $M$ . The quantum  $SU(3)$  invariant is defined by the following formula:

$$\mathcal{I}_r(M) = \frac{\langle \omega_r, \dots, \omega_r \rangle_L}{\langle \omega_r \rangle_{U_+}^{\sigma_+(L)} \langle \omega_r \rangle_{U_-}^{\sigma_-(L)}}.$$

**Remark 3.4** In [21], the authors have noticed that  $\langle \omega_r \rangle_{U_+}$  and  $\langle \omega_r \rangle_{U_-}$  are complex conjugate. Moreover, they proved that  $|\langle \omega_r \rangle_{U_+}| = |\langle \omega_r \rangle_{U_-}| = \frac{\sqrt{3}r}{|A^6 - 1|^3}$ . Keeping this in mind, we can see that the only possible denominator for  $\mathcal{I}_r$  is a power  $3r^2$ . Consequently,  $\mathcal{I}_r$  may be seen as an element of  $\Lambda_r[\frac{1}{3r}]$ .

#### 4- PROOF OF THEOREM 1.1

In [4], we introduced a criterion for periodicity of links using the polynomial invariant  $J$ . The proof of theorem 1.1 relies heavily on this criterion. Note that the invariant  $J$  corresponds to the invariant  $P_3$  in [4] and that the variable  $q$  corresponds to  $A^6$ .

**Theorem 4.1** ([4]) *Let  $p$  be a prime and  $L$  a  $p$ -periodic link. Then we have:*

$$J(L) \equiv (J(\overline{L}))^p \text{ modulo } p, [3]^p - [3].$$

As in the case of the Kauffman bracket, the key observation in the proof of theorem 1.1 is that the congruence obtained for the invariant  $J$  may be extended to the multibracket of strongly periodic links. Indeed, we can prove the following proposition.

**Proposition 4.2** *Let  $p$  be a prime and  $L$  a strongly periodic link with period  $p$ . Then for all  $r \geq 4$  the following congruence holds:*

$$\langle \omega_r, \dots, \omega_r \rangle_L(A) = (\langle \omega_r, \dots, \omega_r \rangle_{\overline{L}}(A))^p \text{ modulo } p, [3]^p - [3].$$

Assume proposition 4.2 is proved. Let  $M$  be a  $p$ -periodic 3-manifold. According to theorem 2.4,  $M$  is obtained from the three-sphere by surgery along a strongly periodic link  $L$ . The quotient  $\overline{M}$  is obtained from  $S^3$  by surgery along the quotient link  $\overline{L}$ . The condition given by proposition 4.2 leads to the following congruence modulo  $p$  and  $[3]^p - [3]$ .

$$\mathcal{I}_r(M) \langle \omega_r \rangle_{U_+}^{\sigma_+(L)} \langle \omega_r \rangle_{U_-}^{\sigma_-(L)} \equiv \mathcal{I}_r(\overline{M})^p \langle \omega_r \rangle_{U_+}^{p\sigma_+(\overline{L})} \langle \omega_r \rangle_{U_+}^{p\sigma_-(\overline{L})}.$$

From the fact that  $|\langle \omega_r \rangle_{U_+}| = |\langle \omega_r \rangle_{U_-}| = \frac{\sqrt{3r}}{|A^6-1|^3}$ , we deduce that both  $\langle \omega_r \rangle_{U_+}$  and  $\langle \omega_r \rangle_{U_-}$  are invertible in the  $\Lambda_r[\frac{1}{3r}]$ . Thus,

$$\mathcal{I}_r(M) \equiv \mathcal{I}_r(\overline{M})^p \langle \omega_r \rangle_{U_+}^{p\sigma_+(\overline{L})-\sigma_+(L)} \langle \omega_r \rangle_{U_-}^{p\sigma_-(\overline{L})-\sigma_-(L)}.$$

As the link  $L$  is strongly  $p$ -periodic, then the number of components of  $L$  is  $p$  times the number of components of  $\overline{L}$ . Assume that  $\overline{L}$  has  $n$  components. If  $M$  is a rational homology sphere then  $\overline{M}$  is also a rational homology sphere and we have the following easy facts.

$$\begin{aligned} p\sigma_+(\overline{L}) - \sigma_+(L) + p\sigma_-(\overline{L}) - \sigma_-(L) &= p(\sigma_+(\overline{L}) + \sigma_-(\overline{L})) - (\sigma_+(L) + \sigma_-(L)) \\ &= pn - pn = 0. \end{aligned}$$

Thus

$$\mathcal{I}_r(M) \equiv \mathcal{I}_r(\overline{M})^p \left( \frac{\langle \omega_r \rangle_{U_+}}{\langle \omega_r \rangle_{U_-}} \right)^{p\sigma_+(\overline{L})-\sigma_+(L)}.$$

In order to compute the value of the term  $\frac{\langle \omega_r \rangle_{U_+}}{\langle \omega_r \rangle_{U_-}}$  we are going to use the calculus developed in [18] in proving the invertibility of  $\langle \omega_r \rangle_{U_+}$  and  $\langle \omega_r \rangle_{U_-}$ . In fact, it was established that :

$$\langle \omega_r \rangle_{U_+} = \frac{-A^{-18}}{(A^3 - A^{-3})} \left( \sum_{r=0}^{r-1} A^{6k^2} \right) \left( \sum_{r=0}^{3r-1} A^{2k^2} \right).$$

Since  $\langle \omega_r \rangle_{U_+}$  and  $\langle \omega_r \rangle_{U_-}$  are complex conjugate we get :

$$\langle \omega_r \rangle_{U_-} = \frac{-A^{18}}{(A^{-3} - A^3)} \left( \sum_{r=0}^{r-1} A^{-6k^2} \right) \left( \sum_{r=0}^{3r-1} A^{-2k^2} \right).$$

Thus,

$$\frac{\langle \omega_r \rangle_{U_+}}{\langle \omega_r \rangle_{U_-}} = -A^{-36} \frac{\left( \sum_{r=0}^{r-1} A^{6k^2} \right) \left( \sum_{r=0}^{3r-1} A^{2k^2} \right)}{\left( \sum_{r=0}^{r-1} A^{-6k^2} \right) \left( \sum_{r=0}^{3r-1} A^{-2k^2} \right)}.$$

Using the fact that  $\left| \sum_{r=0}^{r-1} A^{6k^2} \right|^2 = r$  and  $\left| \sum_{r=0}^{3r-1} A^{2k^2} \right|^2 = 3r$ , we can conclude that :

$$\frac{\langle \omega_r \rangle_{U_+}}{\langle \omega_r \rangle_{U_-}} = -A^{-36} \frac{\left( \sum_{r=0}^{r-1} A^{6k^2} \right)^2 \left( \sum_{r=0}^{3r-1} A^{2k^2} \right)^2}{3r^2}.$$

Setting  $\alpha = p\sigma_+(\overline{L}) - \sigma_+(L)$  ends the proof of theorem 1.1.

**Proof of proposition 4.2.** Here we adapt the techniques developed in [5] to deal with the multibracket derived from the Kauffman bracket. We shall briefly recall the main idea of the proof and refer the reader to [5] for details. Let  $L$  be a framed link, we denote by  $\mathcal{E}_L$  the set of all possible (coming from  $\omega_r$ ) decorations of  $L$ . The element  $\omega_r$  is of the form  $\sum_{i \geq 0} Q_{i,i'}(A)x^i y^{i'}$  for some polynomials  $Q_{i,i'}(A)$ . Therefore, the multibracket is given by the following formula:

$$\langle \omega_r, \dots, \omega_r \rangle_L = \sum_{K \in \mathcal{E}_L} Q_K(A)J(K).$$

Let  $L$  be a strongly  $p$ -periodic link and  $\bar{L}$  its factor link. Assume that  $\bar{L} = l_1 \cup l_2 \cup \dots \cup l_n$ . As the linking number of each component  $l_i$  with the axis  $\Delta$  is zero modulo  $p$  then the link  $L$  is of the following form:

$$(l_1^1 \cup \dots \cup l_1^p) \cup (l_2^1 \cup \dots \cup l_2^p) \cup \dots \cup (l_n^1 \cup \dots \cup l_n^p).$$

Moreover, for all  $i$  the components  $l_i^j$  are identical and cyclically permuted by the rotation  $\varphi_p$ .

**Definition 4.3** Let  $L$  be a strongly periodic link with period  $p$ . A decoration of  $L$  is said to be periodic if and only if for all  $i$  the components  $l_i^j$  are decorated in the same way. Let  $\mathcal{E}'_L$  be the subset of  $\mathcal{E}_L$  made up of periodic decorations.

**Lemma 4.4** Let  $p$  be a prime and  $L$  a strongly  $p$ -periodic link then we have the following:

$$\langle \omega_r, \dots, \omega_r \rangle_L \equiv \sum_{K \in \mathcal{E}'_L} Q_K(A)J(K) \text{ modulo } p.$$

Proof of lemma 4.4. The proof is based on the fact that the cyclic group  $\mathbf{Z}/p\mathbf{Z}$  acts on  $\mathcal{E}_L$ . As  $p$  is prime, orbits of this action are made up of 1 or  $p$  elements. In the case of a  $p$ -elements orbit, these elements are identical and their contribution to the multibracket vanishes modulo  $p$ . A 1-element orbit corresponds to a periodic decoration of  $L$ . Hence, for the computation of the multibracket modulo  $p$  one may only consider elements of  $\mathcal{E}'_L$ , see [5] for details.

Now note that if  $K$  is in  $\mathcal{E}'_L$  then  $K$  is of the form :

$$K = L_{(i_1, i'_1), \dots, (i_1, i'_1), (i_2, i'_2), \dots, (i_2, i'_2), \dots, (i_k, i'_k), \dots, (i_k, i'_k)}$$

where each  $(i_j, i'_j)$  appears  $p$ -times. Thus  $K$  is a  $p$ -periodic link and its quotient link is:

$$\bar{K} = \bar{L}_{(i_1, i'_1), (i_2, i'_2), \dots, (i_k, i'_k)}.$$

Therefore for all  $K \in \mathcal{E}'_L$ , the condition given by of theorem 4.1 is satisfied :

$$J(K) \equiv (J(\bar{K}))^p \text{ modulo } p, [3]^p - [3].$$

Finally, it is easy to see that every element of  $\mathcal{E}'_L$  induces a natural decoration of  $\overline{L}$ , thus an element of  $\mathcal{E}_{\overline{L}}$ , and vice-versa. Moreover the coefficients that appear in the sum satisfy  $Q_K(A) = (Q_{\overline{K}}(A))^p$ . This ends the proof of proposition 4.2.

## 5- SIMILAR RESULTS FOR THE MOO INVARIANT

In [19], Murakami, Ohtsuki and Okada introduced an invariant of 3-manifolds so called the MOO invariant. This invariant is defined from the linking matrix of the surgery presentation of the 3-manifold. Let  $M$  be a 3-manifold obtained from  $S^3$  by surgery along an  $m$ -component framed link  $L$ . By  $\mathcal{B}_L$  we denote the linking matrix of  $L$ . Let  $A$  be a primitive root of unity of order  $N$  (resp.  $2N$ ) if  $N$  is odd (resp. even). Let  $G_N(A)$  denotes the Gaussian sum  $\sum_{k \in \mathbf{Z}/N\mathbf{Z}} A^{k^2}$ . The MOO invariant is defined as follows:

$$Z_N(M) = \left( \frac{G_N(A)}{|G_N(A)|} \right)^{\sigma(L)} |G_N(A)|^{-m} \sum_{l \in (\mathbf{Z}/N\mathbf{Z})^m} A^{t(l)\mathcal{B}_L l}.$$

Where  $\sigma(L)$  is the signature of the linking matrix  $\mathcal{B}_L$ ,  $l$  is regarded as a column vector and  $t(l)$  is its transposed row vector. As in the case of the other quantum invariants, the MOO invariant can be constructed via skein theory [7]. We shall recall here briefly this very simple construction. Let  $\Lambda'_N = \mathbf{Z}[A^{\pm 1}]_{/\Phi_N(A)}$ . We denote by  $\mathcal{L}(M)$  the free  $\Lambda'_N$ -module generated by the set of links in the 3-manifold  $M$ . Now consider the following skein relations:

$$\begin{aligned} \mathcal{R}_1 : \emptyset &= 1 \\ \mathcal{R}_2 : L_+ &= AL_0 \\ \mathcal{R}_3 : L_- &= A^{-1}L_0 \\ \mathcal{R}_4 : L \cup \bigcirc &= L \end{aligned}$$

The linking skein module of  $M$  which is denoted here as  $\mathcal{S}'(M)$  is defined as the quotient module of  $\mathcal{L}(M)$  by relations  $\mathcal{R}_i$  for  $1 \leq i \leq 4$ . We can easily prove that  $\mathcal{S}(S^3) = \Lambda'_N$  and that the linking module of the solid torus  $\mathcal{S}'(S^1 \times D^2)$  is isomorphic as an algebra to  $\Lambda'_N[x, x^{-1}]$  where  $x$  and  $x^{-1}$  are as in figure 4.



FIGURE 4

If  $L$  is an  $m$ -component oriented link, we define the bracket of  $L$  by  $\langle L \rangle = A^{lk(L)}$ , where  $lk(L)$  is the total linking number of  $L$ . The multibracket  $\langle \cdot, \dots, \cdot \rangle_L$  is the multilinear form  $\mathcal{S}' \times \mathcal{S}' \times \dots \times \mathcal{S}' \rightarrow \Lambda'_r$  defined on the set of generators by  $\langle x^{i_1}, x^{i_2}, \dots, x^{i_m} \rangle_L = \langle L_{i_1, i_2, \dots, i_m} \rangle$ . Let  $N \geq 1$  be an integer and  $\omega_N = \sum_{k=0}^{N-1} x^k$ . The MOO invariant is obtained as follows:

$$Z_N(M) = |G_N(A)|^{-b_1(M)} \frac{\langle \omega_N, \dots, \omega_N \rangle_L}{\langle \omega_N \rangle_{U_+}^{\sigma_+(L)} \langle \omega_N \rangle_{U_-}^{\sigma_-(L)}}.$$

Where  $b_1(M)$  is the first Betti number of the manifold  $M$ . In the following we only consider the case  $N$  is odd. We are going to consider the MOO invariant as an element of  $\Lambda'_r[\frac{1}{N}]$  (this is an immediate consequence from the definition). Let  $M$  be a  $p$ -periodic rational homology sphere. We know that  $M$  can be obtained from the three-sphere by surgery along a strongly  $p$ -periodic link  $L$ . Using the easy fact that  $\langle L \rangle = \langle \bar{L} \rangle^p$  and the techniques of paragraph 4 we can prove that :

$$\langle \omega_N, \dots, \omega_N \rangle_L(A) = (\langle \omega_N, \dots, \omega_N \rangle_{\bar{L}}(A))^p \text{ modulo } p.$$

This means that for  $N$  coprime to  $p$  we have :

$$Z_N(M) = \left( \frac{\langle \omega_N \rangle_{U_+}}{\langle \omega_N \rangle_{U_-}} \right)^{p\sigma_+(L) - \sigma_-(L)} (Z_N(M))^p \text{ modulo } p.$$

It is easy to see that  $\langle \omega_N \rangle_{U_+} = \overline{\langle \omega_N \rangle_{U_-}} = G_N(A)$ . As  $N$  is odd, elementary properties of Gaussian sums [13] shows that:

$$\frac{\langle \omega_N \rangle_{U_+}}{\langle \omega_N \rangle_{U_-}} = \frac{G_N(A)}{\overline{G_N(A)}} = \pm 1.$$

Thus we get the following:

**Theorem 5.1** *Let  $p$  be a prime and  $M$  a rational homology 3-sphere. If  $M$  is  $p$ -periodic then for all odd  $N \geq 2$  coprime to  $p$ , the following congruence holds:*

$$Z_N(M) \equiv \pm (Z_N(\bar{M}))^p, \text{ modulo } p$$

where  $\alpha$  is an integer.

## 6- CONCLUDING REMARKS

**Remark 6.1** Theorem 1.1 extends in a natural way to invariants of framed links in three-manifolds (see [25] for a definition). More precisely, let  $M$  be a rational homology three-sphere

and  $K$  a link in  $M$ . Assume that the pair  $(M, K)$  is  $p$ -periodic, this means that  $M$  is  $p$ -periodic and  $K$  is invariant under this action. The techniques used to prove theorem 1.1 may be used here to prove that for all prime  $p$  and  $r \geq 4$  such that  $p$  and  $3r$  are coprime the following congruence holds in  $\Lambda_r[\frac{1}{3r}]$ :

$$\mathcal{I}_r(M, K) \equiv (\mathcal{I}_r(\overline{M}, \overline{K}))^p (G_r(A))^\alpha, \text{ modulo } p, [3]^p - [3],$$

where  $\alpha$  is an integer and  $\overline{K}$  is the quotient link.

**Remark 6.2** The key observation in the proof of our main result here is the condition given by theorem 4.1. Recently, Przytycki and Sikora [23] obtained a generalisation of this condition for the  $SU(N)$  invariants of periodic links,  $N \geq 3$  is an odd integer. On the other hand, Yokota [26] introduced a skein theory for the  $SU(N)$  quantum invariants of 3-manifolds. This indicates that the methods used in our paper may lead to a generalisation of theorem 1.1 to the  $SU(N)$  quantum invariants of periodic rational homology spheres. We plan to describe this in a forthcoming paper.

**Remark 6.3** On the lines of the ideas discussed in the present paper. We discuss in [6], the surgery formula introduced by Lescop [14] for the Casson-Walker invariant. Namely, we prove a necessary condition for the Casson invariant of periodic three-manifolds.

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