Counterexample to Hilbert's fourteenth problem for the 3-dimensional additive group

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An m-dimensional linear representation of a group induces an action on the polynomial ring $\mathbf{C}[z_1, \dots, z_m]$ of m variables. This is called a *linear action* on the polynomial ring. In 1890, Hilbert[2] showed that the invariant ring was finitely generated for classical representations of the special linear groups. The following is known as his fourteenth problem:

Problem 1 Is the invariant ring $\mathbf{C}[z_1, \dots, z_m]^G$ of a linear action of an algebraic group G finitely generated?

The answer is affirmative for the additive algebraic group $\mathbf{G}_a$ (Weitzenböck [11], [9]). In 1958, Nagata[5] considered the standard unipotent linear action

$$(t_1, \dots, t_n) \in \mathbf{C}^n \curvearrowright \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n] =: S \tag{1}$$

$$\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}, \quad 1 \leq i \leq n,$$

of $\mathbf{C}^n$ on the polynomial ring S of $2n$ variables and showed that the invariant ring S^G with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. In this article, we shall prove the following:

Theorem *The invariant ring S^G of* (1) *with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension r is not finitely generated if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \leq 1. \tag{2}$$

In other words, S^G is not finitely generated if $\dim G = s \geq 3$ and if $n \geq s^2/(s-2)$. So the answer to Problem 1 is negative for $\mathbf{G}_a^3$. But the following part is still open:

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Problem 2 Is the invariant ring $\mathbf{C}[z_1, \ldots, z_m]^G$ of a linear action of the 2-dimensional additive group $G = \mathbf{G}_a \times \mathbf{G}_a$ finitely generated?

See Roberts [8] for non-linear actions.

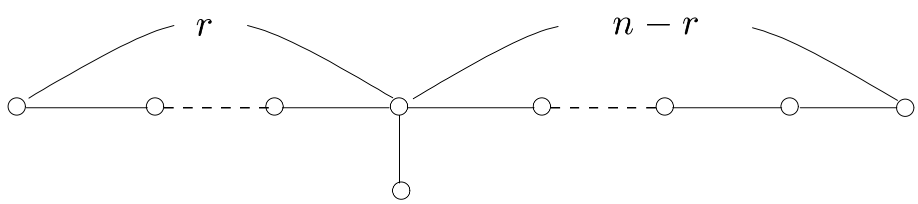
Our proof of the theorem is based on the fact that the invariant ring S^G is a certain Rees algebra (§1). In geometric term, the Rees algebra is isomorphic to the *total coordinate ring* $\mathcal{TC}(X)$ of the blow-up X of the projective space $\mathbf{P}^{r-1}$ at n points (§2). This ring $\mathcal{TC}(X)$ is graded by the Picard group $\operatorname{Pic} X \simeq \mathbf{Z}^{n+1}$ and its support is $\operatorname{Eff} X$, the semi-group of effective classes on X. Hence $\mathcal{TC}(X)$ is not finitely generated if $\operatorname{Eff} X$ is not so as semi-group (Lemma 2).

The simplest case is

$$G = \left\{ (t_1, \ldots, t_9) \,\middle|\, \sum_{i=1}^{9} t_i = \sum_{i=1}^{9} \wp(c_i) t_i = \sum_{i=1}^{9} \wp'(c_i) t_i = 0 \right\} \subset \mathbf{C}^9, \qquad (3)$$

where $\wp(z)$ is Weierstrass's $\wp$-function of an elliptic curve $C = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ and $c_1, \ldots, c_9$ are distinct points C. In this case, X is the blow-up of $\mathbf{P}^2$ at the nine points $(1 : \wp(c_i) : \wp'(c_i))$, $1 \le i \le 9$. Assume that the sum $\sum_{i=1}^{9} c_i \in C$ is zero, for simplicity. Then the nine points are the intersection of two cubics, X has an elliptic fibration $f : X \to \mathbf{P}^1$ and the nine exceptional curves are sections of f. If the difference $c_i - c_{i+1}$ is of infinite order for some $1 \le i \le 8$, then there are infinitely many exceptional curves of the first kind (cf. [6]). So S^G is not finitely generated. (Cf. Remark 1 at the end of §4.)

The proof of the theorem (§4) is similar but we replace the elliptic fibration by the symmetry of $\operatorname{Pic} X$ with respect to the Weyl group of the Dynkin diagram $T_{2,r,n-r}$ with n vertices (§3):



(4)

which was introduced by Dolgachev[1]. As is well known the inequality (2) is equivalent to the infiniteness of the Weyl group of this diagram (Lemma 4). If $G \subset \mathbf{C}^n$ is general and if (2) is satisfied, then there exist infinitely many exceptional divisors on X. Therefore, $\operatorname{Eff} X$ and hence $\mathcal{TC}(X)$ are not finitely generated (Lemma 3).

1 Invariant ring is Rees algebra

Let $G \subset \mathbf{C}^n$ be a linear subspace of codimension r and

$$\sum_{i=1}^{n} a_i^{(1)} t_i = \sum_{i=1}^{n} a_i^{(2)} t_i = \cdots = \sum_{i=1}^{n} a_i^{(r)} t_i = 0 \tag{5}$$

a system of defining equations. Since $x_1, \ldots, x_n$ are G-invariant, we obtain the induced action of G on the localization

$$S[x_1^{-1}, \ldots, x_n^{-1}] = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n] = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}].$$

Since $(t_1, \ldots, t_n) \in G$ acts by the translation $y_i/x_i \mapsto y_i/x_i + t_i$, the invariant ring $S[x_1^{-1}, \ldots, x_n^{-1}]^G$ is generated by

$$\sum_{i=1}^{n} a_i^{(1)} \frac{y_i}{x_i}, \quad \sum_{i=1}^{n} a_i^{(2)} \frac{y_i}{x_i}, \quad \ldots, \quad \sum_{i=1}^{n} a_i^{(r)} \frac{y_i}{x_i} \tag{6}$$

over the Laurent polynomial ring $\mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let

$$J^{(1)}(x,y), \quad J^{(2)}(x,y), \quad \ldots, \quad J^{(r)}(x,y) \in S^G \tag{7}$$

be the products of (6) and the monomial $\prod_{i=1}^n x_i$. Let V be the subspace and R the subring of S^G generated by them. R is a polynomial ring and V is its degree one part. The invariant ring S^G contains $R[x_1, \ldots, x_n]$ and $S[x_1^{-1}, \ldots, x_n^{-1}]^G$ coincides with $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Obviously we have

$$S^G = S[x_1^{-1}, \ldots, x_n^{-1}]^G \cap S = R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cap S. \tag{8}$$

Let V_1 be the linear subspace of V consisting of $J(x,y)$ which do not contain the monomial $y_1 \prod_{i=2}^n x_i$. Then $V_1 \subset V$ is of codimension ≤ 1. A polynomial $J(x,y) \in V$ is divisible by x_1 if and only if it belongs to V_1. Let $I_1 \subset R$ be the ideal generated by V_1. Define $V_i \subset V$ and $I_i \subset R$ for $2 \leq i \leq n$ similarly. If $F(x,y) \in R$ belongs to the b_i-th power $I_i^{b_i}$, then $F(x,y)$ is divisible by $x_i^{b_i}$ and the quotient $F(x,y)/x_i^{b_i}$ belongs to S^G. Hence S^G contains

$$R[x_1, \ldots, x_n] + \sum_{b_1, \ldots, b_n \geq 0} (I_1^{b_1} \cap \cdots \cap I_n^{b_n}) x_1^{-b_1} \cdots x_n^{-b_n} \subset R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{9}$$

as its subring. The following was proved in [5] in the case of codimension 3.

Proposition *The invariant ring S^G of the action* (1) *with respect to a subspace $G \subset \mathbf{C}^n$ coincides with the extended multi-Rees algebra* (9) *of* $(R : I_1, \ldots, I_n)$.

Proof. It suffices to show the following
claim : $f(J^{(1)}(x,y), \ldots, J^{(r)}(x,y)) \in R$ is divisible by $x_i^{b_i}$ if and only if $f(J^{(1)}, \ldots, J^{(r)})$ belongs to $I_i^{b_i}$.

If $a_i^{(1)}, \ldots, a_i^{(r)}$ are all zero, then $J^{(1)}(x,y), \ldots, J^{(r)}(x,y)$ are all divisible by x_i. The claim is obvious, since none is divisible by x_i^2 and since $V_i = V$. So assume the contrary. By reordering (7), we may assume that $a_i^{(1)} \neq 0$. Put

$$z_1 = J^{(1)}/a_i^{(1)}, z_2 = J^{(2)} - a_i^{(2)} z_1, \ldots, z_r = J^{(r)} - a_i^{(r)} z_1.$$

Then

$$f(J^{(1)}, \ldots, J^{(r)}) = f(a^{(1)} z_1, a^{(2)} z_1 + z_2, \ldots, a^{(r)} z_1 + z_r)$$

and this belongs to the ideal $(z_2, \ldots, z_r)^{b_i}$ if and only if $f(J^{(1)}, \ldots, J^{(r)})$ belongs to $I_i^{b_i}$ by the lemma below. When regarded as polynomials of $x_1, \ldots, x_n, y_1, \ldots, y_n$, the $r-1$ polynomials $z_2, \ldots, z_r$ are divisible by x_i and only z_1 is not. Therefore, f belongs to $(z_2, \ldots, z_r)^{b_i}$ if and only if $f(J^{(1)}(x,y), \ldots, J^{(r)}(x,y))$ is divisible by $x_i^{b_i}$. □

Lemma 1 *Let I be the ideal of $\mathbf{C}[z_1, \ldots, z_r]$ generated by linear forms vanishing at*

$$(a^{(1)}, a^{(2)}, \ldots, a^{(r)}) \in \mathbf{C}^r.$$

Assume that $a^{(1)} \neq 0$. Then a polynomial $f(z_1, \ldots, z_r)$ belongs to the b-th power I^b if and only if

$$f(a^{(1)} z_1, a^{(2)} z_1 + z_2, \ldots, a^{(r)} z_1 + z_r)$$

belongs to the b-th power of the homogeneous ideal $(z_2, \ldots, z_r)$.

For small values of r, the invariant ring is very explicit.

Example 1 ($r = 1$) Assume that $G \subset \mathbf{C}^n$ is defined by $\sum_{i=1}^m t_i = 0$ for $1 \leq m \leq n$. Then S^G is generated by $x_1, \ldots, x_n$ and

$$\left(\frac{y_1}{x_1} + \cdots + \frac{y_m}{x_m}\right) \prod_{i=1}^m x_i.$$

Example 2 ($r = 2$) Assume that $G \subset \mathbf{C}^n$ is defined by $\sum_{i=1}^n t_i = \sum_{i=1}^n c_i t_i = 0$. Then $c_i J_1(x, y) - J_2(x, y)$ is divisible by x_i and the quotient $(c_i J_1(x, y) - J_2(x, y))/x_i$ belongs to S^G for every $1 \le i \le n$. S^G is generated by these invariants over $\mathbf{C}[x_1, \ldots, x_n]$ if $c_1, \ldots, c_n$ are distinct.

2 Total coordinate ring

For our purpose, it is more convenient to state the proposition in geometric term. Let $\mathbf{P}^{r-1} = \operatorname{Proj} R$ be the $(r-1)$-dimensional projective space whose homogeneous coordinates are (7). In the sequel we assume that

($\diamondsuit$) $r \ge 3$ and any two of n vectors $(a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(r)}) \in \mathbf{C}^r, 1 \le i \le n$, are linearly independent.

(The study of S^G for the action (1) is easily reduced to this case.) Then n points

$$p_i := (a_i^{(1)} : a_i^{(2)} : \ldots : a_i^{(r)}) \in \mathbf{P}^{r-1}, \quad 1 \le i \le n, \tag{10}$$

are well-defined and distinct. The ideal $I_i \subset R$ is generated by the linear forms vanishing at p_i. Let

$$\pi : X = X_G \longrightarrow \mathbf{P}^{r-1}$$

be the blow-up at these n points. The isomorphism class of X_G does not depend on the choice of the defining equation (5). The Picard group is a free abelian group of rank $n + 1$. The pull-back h of the hyperplane class H and the classes e_i, $1 \le i \le n$, of the exceptional divisors form a basis, which is called *the standard basis* of $\operatorname{Pic} X_G$ (with respect to π). The direct sum of the spaces of global sections of all line bundles (up to isomorphism)

$$\bigoplus_{a, b_1, \ldots, b_n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(ah - b_1 e_1 - \cdots - b_n e_n)) \simeq \bigoplus_{L \in \operatorname{Pic} X} H^0(X, L) \tag{11}$$

is a graded ring, which is called the *total coordinate ring* of X and denoted by $\mathcal{TC}(X)$. In our case, $\mathcal{TC}(X_G)$ is the Rees algebra (9), or more precisely, it is the $\mathbf{Z}^n$-graded ring (9) plus the extra grading of the polynomial ring R. By the proposition, we have

Corollary *Under the condition of ($\diamondsuit$), the invariant ring S^G of the action (1) with respect to $G \subset \mathbf{C}^n$ is the total coordinate ring $\mathcal{TC}(X_G)$ of the blow-up X_G.*

Let $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ be an integral domain graded by a free abelian group Λ. The subset $\{\lambda \mid A_\lambda \neq 0\}$ of Λ is a semi-group. This is called the *support* of A and denoted by $\operatorname{Supp} A$.

Lemma 2 *If* $\operatorname{Supp} A$ *is not finitely generated as semi-group, neither is* A *as a ring over* A_0.

Proof. Assume that A is finitely generated. Then finite nonzero homogeneous elements $a_i \in A_{\lambda_i}$, $1 \leq i \leq N$, generate A and $\lambda_1, \ldots, \lambda_N$ generate $\operatorname{Supp} A$. □

For example, the support of $\mathcal{TC}(X)$ as $\mathbf{Z}^{n+1}$-graded ring is the semi-group

$$\operatorname{Eff} X := \{L \in \operatorname{Pic} X \mid H^0(X, L) \neq 0\},$$

of linear equivalence classes of effective divisors on X. If $\operatorname{Eff} X$ is not finitely generated as semi-group, neither is $\mathcal{TC}(X)$. The following is basic for our analysis of $\operatorname{Eff} X$.

Lemma 3 *Let* $\pi : X \longrightarrow Y$ *be the blowing up of a projective variety* Y *at a point. Then the linear equivalence class of the exceptional divisor* E *of* π *belongs to any system of generators of the effective semi-group* $\operatorname{Eff} X$.

Proof. Assume that E is linearly equivalent to the sum $D_1 + D_2$ of two effective divisors. Let H be the pull-back of an ample divisor on Y. Then the intersection number $(E.H^{m-1})$, $m = \dim X$, is zero. Hence so are $(D_1.H^{m-1})$ and $(D_2.H^{m-1})$. Therefore, both $\operatorname{Supp} D_1$ and $\operatorname{Supp} D_2$ are contained in E and either D_1 or D_2 is zero. □

If X and X' are isomorphic in codimension one, then the Picard groups are the same and $\operatorname{Eff} X = \operatorname{Eff} X'$. So we call $D \subset X$ a (-1)-*divisor* if there is a birational map $f : X \cdots \to X'$ and a morphism $\pi : X' \to Y$ such that f is an isomorphism in codimension one, π is the blowing up of a projective variety Y at a smooth point and D is the strict transform of the exceptional divisor of π. By the lemma, the class of a (-1)-divisor is contained in any system of generators of $\operatorname{Eff} X$. Hence $\operatorname{Eff} X$ is not finitely generated if X has infinitely many classes of (-1)-divisors.

3 Root systems and elliptic curves

Let Λ be the lattice of rank $n+1$ with orthogonal basis $h, e_1, \ldots, e_n$. In view of the standard Cremona transformation (see the next section especially the formula (16)), we set $(h^2) = r-2$ and $(e_i^2) = -1$ for $1 \le i \le n$. For $\lambda = ah - \sum_{i=1}^n b_i e_i \in \Lambda$, we denote its coefficient a in h by $\deg \lambda$. We put $\kappa = rh - \sum (r-2) \sum_{i=1}^n e_i$, which corresponds to the anti-canonial class of the blow-up of $\mathbf{P}^{r-1}$ at points. The orthogonal complement of κ together with its basis

$$e_1 - e_2, \quad e_2 - e_3, \quad \ldots, \quad e_{n-1} - e_n \quad \text{and} \quad h - \sum_{i=1}^r e_i \tag{12}$$

becomes a root system. The Dynkin Diagram is (4), that is, $T_{2,r,n-r}$ with three-legs of length $2, r$ and $n-r$. For a subset $I \subset [n] := \{1, 2, \ldots, n\}$ of cardinality r, $\alpha_I = h - \sum_{i \in I} e_i$ is a root. The reflection R_I with respect to α_I is as follows:

$$\begin{cases} h & \mapsto & h + (r-2)\alpha_I & = (r-1)h - (r-2)\sum_{i \in I} e_i \\ e_i & \mapsto & e_i + \alpha_I & \text{for } i \in I \\ e_j & \mapsto & e_j & \text{for } j \notin I \end{cases} \tag{13}$$

Let W be the Weyl group of (12). By definition, W leaves κ invariant, that is, $rw(h) - (r-2)\sum_{i=1}^n w(e_i) = \kappa$ for every $w \in W$. In particular, we have

$$r \deg w(h) - (r-2) \sum_{i=1}^n \deg w(e_i) = r. \tag{14}$$

Lemma 4 *If the inequality* (2) *holds, then the* W*-orbit of* e_n *is infinite.*

Proof. The assumption implies $r \ge 3$. Let w be an element of the Weyl group. There exists a subset $I \subset [n]$ of cardinality r such that

$$\sum_{i \in I} \deg w(e_i) \le \frac{r}{n} \sum_{i=1}^n \deg w(e_i).$$

By (14) we have

$$\deg w(\alpha_I) = \deg w(h) - \sum_{i \in I} \deg w(e_i) \ge \deg w(h) - \frac{r^2}{n(r-2)}(\deg w(h) - 1),$$

which is positive by (2). Therefore, $\deg w(R_I(h)) - \deg w(h) = (r-2)\deg w(\alpha_I)$ is also positive. It follows that the degree is increased by a suitable reflection R_I. Hence, the orbit $W \cdot h$ is infinite. So is $W \cdot e_n$ by the equality (14). □

The Weyl group of $T_{p,q,r}$ is infinite if and only if $1/p + 1/q + 1/r \leq 1$ ([3] Chap. 4). The lemma also follows from this.

Let C be an elliptic curve and Λ_C the $(n+1)$-dimensional variety $\operatorname{Pic}^r C \times C^n$. This is canonically isomorphic to $\operatorname{Pic}^r C \times (\operatorname{Pic}^1 C)^n$. So the factor permutation of C^n and the automorphism

$$(D; c_1, \ldots, c_n) \mapsto (D'; c'_1, \ldots, c'_n),$$

$$\begin{cases} D' &= (r-1) - (r-2)\sum_{i=1}^r c_i \\ c'_i &= D - c_1 - \cdots - \check{c}_i - \cdots - c_r \quad \text{for } 1 \leq i \leq r \\ c'_j &= c_j \quad \text{for } r+1 \leq j \leq n \end{cases}$$

define the action of the Weyl group W on the variety Λ_C. For a real root $\alpha = ah - \sum_{i=1}^n b_i e_i \in \Delta^{re}$ ([3] Chap. 5), the reflection R_α interchanges

$$f_\alpha : \Lambda_C \longrightarrow \operatorname{Pic}^0 C, \quad (D; c_1, \ldots, c_n) \mapsto aD - \sum_{i=1}^n b_i c_i.$$

with $-f_\alpha$. We denote the fiber $f_\alpha^{-1}(0)$ by $\mathcal{D}(\alpha)$.

Example 3 $\mathcal{D}(e_i - e_j)$, $i \neq j$, is the diagonal $\{c_i = c_j\}$. $\mathcal{D}(h - \sum_{i=1}^r e_i)$ consists of $(D; c_1, \ldots, c_n)$ such that $\sum_{i=1}^r c_i \in |D|$.

The Weyl group W acts on the complement of all these fibers:

$$\Lambda_C - \bigcup_{\alpha \in \Delta^{re}} \mathcal{D}(\alpha). \tag{15}$$

4 Standard Cremona transformation

The map

$$\Psi : \mathbf{P}^{r-1} \cdots \to \mathbf{P}^{r-1}, \quad (x_1 : x_2 : \cdots : x_r) \mapsto (\frac{1}{x_1} : \frac{1}{x_2} : \cdots : \frac{1}{x_r}), \quad r \geq 3,$$

is a birational transformation of the projective space $\mathbf{P}^{r-1}$. It contracts the r coordinate hyperplanes to the r coordinate points and its square is the identity. A birational map which is projectively equivalent to Ψ is called *a standard Cremona transformation*. Let $P = \{p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_r\}$ be a pair of sets of r points of $\mathbf{P}^{r-1}$. If both P and Q span $\mathbf{P}^{r-1}$, then there exists the unique standard Cremona transformation which contracts the hyperplane H_i passing through the $r-1$ points $p_1, \dots, \check{p}_i, \dots, p_r$ to the point q_i for every $1 \le i \le r$. We denote this by $\Psi_{P,Q}$. P and Q are called its *center* and *cocenter*, respectively. $\Psi_{P,Q}$ is the rational map associated with $|(r-1)H - (r-2)\sum_{i=1}^{n} p_i|$, the linear system of hypersurfaces of degree $(r-1)$ passing through P with multiplicity $\ge r-2$. (The sum of $r-1$ of $H_1, \dots, H_r$ form a basis of the linear system.) The indeterminacy locus of $\Psi_{P,Q}$ is the union $I_P := \cup_{1 \le i < j \le r} H_i \cap H_j$ of the intersection of all pairs of the hyperplanes H_i's.

Let X_P and X_Q be the blow-up of $\mathbf{P}^{r-1}$ with center P and Q, respectively. $\Psi_{P,Q}$ induces the birational map $\tilde{\Psi}_{P,Q}$ from X_P to X_Q. The diagram

$$\begin{array}{ccc} X_P & \overset{\tilde{\Psi}_{P,Q}}{\cdots\longrightarrow} & X_Q \\ \downarrow & & \downarrow \\ \mathbf{P}^{r-1} & \underset{\Psi_{P,Q}}{\cdots\cdots\rightarrow} & \mathbf{P}^{r-1} \end{array}$$

is commutative and $\tilde{\Psi}_{P,Q}$ induces an isomorphism between the complement of the strict transform of I_P and that of I_Q. Hence $\tilde{\Psi}_{P,Q}$ is an isomorphism in codimension one. (More precisely, $\tilde{\Psi}_{P,Q} : X_P \cdots \to X_Q$ is the composite of certain flops.) In particular it induces an isomorphism $\operatorname{Pic} X_P \xrightarrow{\sim} \operatorname{Pic} X_Q$ between the Picard groups and that between the semi-groups of effective classes. Let $\{h, e_1, \dots e_r\}$ be the standard basis of $\operatorname{Pic} X_P$. Then the standard basis of $\operatorname{Pic} X_Q$ consists of

$$(r-1)h-(r-2)\sum_{i=1}^{r} e_i, \quad \text{and} \quad h-e_1-\cdots-\check{e}_i-\cdots-p_r, \quad 1 \le i \le r. \tag{16}$$

Proof of Theorem. Let C be an elliptic curve and take an $(n+1)$-tuple $(D; c_1, \dots, c_n)$ from the W-invariant open subset (15) of Λ_C. The complete linear system $|D|$ embeds C into the $(r-1)$-dimensional projective space $\mathbf{P}_D := \mathbf{P}^* H^0(C, \mathcal{O}_C(D))$. Let $p_1, \dots, p_n \in \mathbf{P}_D$ be the image of $c_1, \dots, c_n$ by the embedding Φ_D. Since $(D; c_1, \dots, c_n)$ does not belong to the divisor $\mathcal{D}(e_i - e_j) \subset \Lambda_C$ for any $1 \le i < j \le n$, the n points $p_1, \dots, p_n$ are distinct. Moreover, since it does not belongs to $\mathcal{D}(\alpha_I)$ for any $I \subset [n]$ with $|I| = r$, any r of $p_1, \dots, p_n$ spans the projective space $\mathbf{P}_D$ (Example 3).

Hence we can perform the standard Cremona transformation of $\mathbf{P}_D$ with any r of $p_1, \ldots, p_n$ as center. Put $(D'; c'_1, \ldots, c'_n) = R_I(D; c_1, \ldots, c_n)$ and $p'_i = \Phi_{D'}(c'_i)$ for $1 \leq i \leq n$. Then we have the commutative diagram:

$$\begin{array}{ccccc} & C & = & C & \\ \Phi_D & \downarrow & & \downarrow & \Phi_{D'} \\ & \mathbf{P}_D & \underset{\Psi_I}{\cdots\to} & \mathbf{P}_{D'} & \end{array}$$

where Ψ_I is the standard Cremona transformation whose center is $\{p_i \,|\, i \in I\}$ and cocenter is $\{p'_i \,|\, i \in I\}$. Any point of C other than $\{p_i \,|\, i \in I\}$ does not lie in the indeterminacy locus of Ψ_I. Let $\pi : X \longrightarrow \mathbf{P}_D$ be the blowing up at the n points $p_1, \ldots, p_n$ and $\pi' : X \longrightarrow \mathbf{P}_{D'}$ at $p'_1, \ldots, p'_n$. Then Ψ_I induces $\tilde{\Psi}_I$ between X and X' and we have the commutative diagram:

$$\begin{array}{ccccc} & C & = & C & \\ & \Big\downarrow & & \Big\downarrow & \\ & X & \overset{\tilde{\Psi}_I}{\cdots\longrightarrow} & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & \mathbf{P}_D & \underset{\Psi_I}{\cdots\cdots\to} & \mathbf{P}_{D'} & \end{array}$$

By our choice of $(D; c_1, \ldots, c_n)$, the images $p'_1, \ldots, p'_n$ of $c_1, \ldots, c_n$ are distinct and any subset of cardinality r spans $\mathbf{P}_{D'}$. Hence we can perform the standard Cremona transformation with any r of $p'_1, \ldots, p'_n$ as center. We can continue this as many times as we like. Hence we have the following by (13) and (16):

Lemma 5 *If an $(n+1)$-tuple $(D; c_1, \ldots, c_n)$ belongs to the open subset* (15) *of Λ_C and if α is in the orbit $W \cdot e_n$, then there exists a (-1)-divisor D whose linear equivalence class is α.*

It is obvious that the same holds for the blow-up $\tilde{X}$ at $\tilde{p}_1, \ldots, \tilde{p}_n$ if the n-tuple $(\tilde{p}_1, \ldots, \tilde{p}_n) \in \mathbf{P}^{r-1} \times \cdots \times \mathbf{P}^{r-1}$ belongs to a neighborhood of $(p_1, \ldots, p_n)$ in the classical topology. Hence, by virtue of Lemma 4, $\tilde{X}$ contains infinitely many classes of (-1)-divisors if (2) holds. Therefore, S^G for a general $G \subset \mathbf{C}^n$ is not finitely generated by Corollary and two lemmas in §2. $\square$

Remark 1 Following [5], Steinberg [10] and independently the author [4] consider the diagonal subring

$$S^{T\cdot G} := R[x] + \sum_{b\geq 0}(I_1^b\cap\cdots\cap I_n^b)x^{-b}\subset R[x^{\pm 1}],\quad x=\prod_{i=1}^n x_i,$$

of (9), which is isomorphic to

$$\bigoplus_{a,b\in\mathbf{Z}} H^0(X_G,\mathcal{O}_X(ah-b(e_1+\cdots+e_n))),\tag{17}$$

in the case where $n=9$ and $G\subset\mathbf{C}^9$ is of codimension 3. They show that this is not finitely generated if $3D-\sum_{i=1}^9 c_i\in C$ is of infinite order. The infinite generation of S^G follows from this easily. Note that $S^{T\cdot G}$ becomes finitely generated if $3D-\sum_{i=1}^9 c_i$ is torsion but still S^G is not finitely generated if the differences c_i-c_j are general. Note also that $\kappa=3h-\sum_{i=1}^9 e_i\in\Lambda$ corresponding to $3D-\sum_{i=1}^9 c_i$ is an imaginary root of the affine root system $\kappa^\perp$ of type $T_{2,3,6}$.

Remark 2 If (2) holds and if $c_1,\ldots,c_n\in C$ are general, then the image of the restriction map

$$S^G=\mathcal{TC}(X_G)\longrightarrow\mathcal{TC}(C|D;c_1,\ldots,c_n):=\bigoplus_{a,b_1,\ldots,b_n\in\mathbf{Z}} H^0(C,\mathcal{O}_C(aD-\sum_{i=1}^n b_ic_i))$$

is not finitely generated. This gives another proof of Theorem. The image is similar to the bi-graded ring

$$\bigoplus_{m,n\in\mathbf{Z}} H^0(C,\mathcal{O}_C(mc+nd))$$

obtained from two points $c,d\in C$. If the difference $c-d\in C$ is of infinite order, then the support is $\{m+n>0\}\cup\{(0,0)\}$, which is not finitely generated as semi-group (cf. [7]).

References

[1] Dolgachev, I.: Weyl groups and Cremona transformations, Proc. Symp. Pure Math. **40**(1983), 283-294.

[2] Hilbert, D.: Über die Theorie der algebraischen Formen, Math. Ann., **36** (1890), 473-534.

[3] Kac, V.G.: *Infinite dimensional Lie algebras*, 2nd. ed., Cambridge Univ. Press., 1983.

[4] Mukai, S.: *Moduli Theory I, II*, Iwanami Shoten, 1998, 2000, Tokyo. (English translation : *An introduction to invariants and moduli*, to appear.)

[5] Nagata, M.: On the fourteenth problem of Hilbert, Int'l Cong. Math., Edingburgh, 1958.

[6] ——: On rational surfaces, II, Mem. Coll. Sci. Univ. Kyoto. Ser. A, **33**(1960), 271-293.

[7] Rees, D.: On a problem of Zariski, Illinois J. Math. **2**(1958), 145-149.

[8] Roberts, P.: An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's 14th problem, J. Algebra, **132**(1990), 461-473.

[9] Seshadri, C.S.: On a theorem of Weitzenböck in invariant theory, J. Math. Kyoto Univ., **1**(1962), 403-409.

[10] Steinberg, R.: Nagata's example, in '*Algebraic Groups and Lie Groups*', Austral. Math. Soc. Lect. Ser. **9**, Cambridge Univ. Press, 1997, pp. 375–384.

[11] Weitzenböck, R.: Über die Invarianten von Linearen Gruppen, Acta. Math., **58**(1932), 230-250.

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