On the exact WKB analysis of operators admitting infinitely many phases

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1 Introduction

The purpose of this article is to propose a reasonably wide class of operators whose WKB solutions admit infinitely many phases and to show that exact WKB analysis, i.e., WKB analysis based on the Borel resummation, for such operators can be performed at least near their turning points. Intuitively speaking, the operators to be studied in this article are appropriate infinite sums of linear ordinary differential operators whose coefficients depend on a large parameter η. Their precise definition is given in Section 2; the definition is designed so that the "wildness" of the differentiation is ameliorated by the effect of the inverse of the large parameter. As the reader will find in Section 2, microlocal analysis ([SKK]; see also [K^3] for some elementary exposition) of the Borel transform of the operators in question enables us to describe the delicate balance between the differentiation and the effect of the large parameter. We call such well-balanced operators "operators of WKB type", in order to distinguish them from linear differential operators of infinite order, which are effectively used in microlocal analysis (cf., e.g., [SKK, Chap. II]).

Operators whose WKB solutions admit infinitely many phases cannot be a linear ordinary differential operator of finite order, because the phase function is then a solution of an algebraic equation; however, such operators become necessary in fluid mechanics (cf., e.g., [BP], [BRS], [S] and [WSW]). Still, there has been no mathematically serious attempt for WKB analysis of such operators as far as we know. Hence we start our discussion with the definition and construction of WKB solutions for operators of WKB type (Sections 3 and 4). To make the reasoning systematic, we use the so-called "exponential calculus of microdifferential operators" (cf., [A1], [A2] and [A3]) to find out the Riccati-type equation associated with an operator of WKB type. After defining the notion of a turning point and that of its rank for an operator of WKB type, we prove a decomposition theorem for such operators near a turning point (Section 5). In the proof, we employ the idea of H. Cartan ([C]) in proving a precise version of the Späth division theorem. Using the decomposition theorem we find where and how a disrupt change occurs in the Borel sum of WKB solutions, i.e., the location of a Stokes curve near a simple turning point and the connection formula around it. The method of the proof is essentially the same as that used in [AKT1]; that is, we use the decomposition theorem to reduce the problem to the case of the second order operators (cf. [V], [DDP], [AKT2] and references cited there).

Some computer-assisted study (that uses Mathematica) is given in Appendix to see what happens at points far away from turning points. As far as we have examined, no substantially new phenomena seem to occur; the notion of new turning points (cf., [AKT1], or that of virtual turning points in our latest terminology in [AKT5]) seems to be still effective for operators of WKB type. But no serious attempt has been made to find an algorithm to describe the complete connection formula. Finding out such an algorithm is one of the most important open problems in exact WKB analysis, even for finite ($\gneqq 2$) order operators. We hope examples in Appendix will be helpful for the reader to obtain a concrete picture of operators of WKB type.

In ending this Introduction, we call the attention of the reader to the fact that, in general, an operator $P(x, d/dx, \eta)$ of WKB type does not determine an infinite order operator (in the sense of microlocal analysis) when the parameter η is fixed to be some finite value however large it is. As a simple example let us consider the following operator L_0:

$$\begin{aligned} L_0(x, \eta^{-1}d/dx) &= \exp(\eta^{-1}d/dx) - x \\ &= \sum_{n\geq 0} \frac{1}{n!}\eta^{-n}(\frac{d}{dx})^n - x. \end{aligned} \tag{1.1}$$

If we fix η to be η_0 the resulting operator is a translation operator: $f(x) \mapsto f(x+\eta_0^{-1})$. Since an infinite order differential operator acts on the sheaf of holomorphic functions as a sheaf homomorphism, $L_0(x, \eta_0^{-1}d/dx)$ cannot be an infinite order differential operator; one important lesson we learn from this example is that the arbitrariness of the large parameter η is critically important in ameliorating the wildness of differentiation. Needless to say, there are operators of infinite order of WKB type that determine infinite order differential operators when we fix the large parameter η at a finite value η_0. A typical simple example is given by the following operator L_1:

$$\begin{aligned} L_1(x, \eta^{-1}d/dx) &= \cosh(\sqrt{(i\eta)^{-1}d/dx}) - x \\ &= \sum_{n\geq 0} \frac{1}{(2n)!}(i\eta)^{-n}(\frac{d}{dx})^n - x. \end{aligned} \tag{1.2}$$

As is well-known, $\cosh(\sqrt{(i\eta_0)^{-1}d/dx})$ is an infinite order differential operator for any η_0 ($\neq 0$).

The operator L_1 (Example A.1), a variant of L_0 (Example A.2), and L_2 given by

$$L_2 = \exp(\eta^{-2}d^2/dx^2) - \exp(-x^2) \tag{1.3}$$

(Example A.3) together with an operator related to a non-adiabatic level crossing problem in quantum mechanics (Example A.4) are concretely analyzed in Appendix.

2 Differential operators of WKB type

Let U be an open set in $\mathbb{C}$. We set $X = U \times \mathbb{C}_y$. Let (x, y) be a coordinate system in X. Here x (resp. y) is a coordinate in U (resp. $\mathbb{C}_y$). Let T^*X denote the cotangent bundle of X and let $(x, y; \xi, \eta)$ denote a coordinate system in T^*X. We denote by Ω the open subset in T^*X defined by

$$\{(x, y; \xi, \eta) \in T^*X; \eta \neq 0\}.$$

We consider a subring of the ring $\mathcal{E}(\Omega)$ of microdifferential operators defined on Ω (cf. [K^3]):

Definition 2.1 The set of all microdifferential operators P of order 0 which are defined on Ω and which satisfy

$$[P, \partial_y] := P\,\partial_y \,-\, \partial_y\, P = 0$$

is denoted by $\mathcal{E}_{\mathrm{WKB}}(U)$. Here we set $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$. An element P in $\mathcal{E}_{\mathrm{WKB}}(U)$ is called a differential operator of WKB type defined on U.

For a differential operator P of WKB type, the total symbol $\sigma(P)$ of P may be written in the following form of a formal power series:

$$\sigma(P) = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, \xi/\eta). \tag{2.1}$$

Here each $P_j(x, \xi/\eta)$ is a holomorphic function which is defined on Ω and homogeneous of degree 0 in (ξ, η). That is, the homogeneous part of degree $-j$ of P is $\eta^{-j} P_j(x, \xi/\eta)$. Hence we can write

$$\sigma(P) = P(x, \xi/\eta, \eta) = P(x, \zeta, \eta)$$

and

$$P = P(x, \partial_x/\partial_y, \partial_y).$$

Here we set $\zeta = \xi/\eta$ and we regard ζ as an inhomogeneous coordinate in $\Omega/(\mathbb{C} - \{0\})$. Following the traditional terminologies in microlocal analysis, we call $P_0(x,\zeta)$ the principal symbol of P. Each coefficient $P_j(x,\zeta)$ of η^{-j} in $\sigma(P)$ is an entire function of ζ with holomorphic coefficients $a_{j,k}$ defined on U:

$$P_j(x,\zeta) = \sum_{k=0}^{\infty} a_{j,k}(x)\zeta^k. \tag{2.2}$$

By the definition of microdifferential operators, $P_j(x,\zeta)$ $(j = 0, 1, 2, \dots)$ satisfy the following estimates: For every compact set K in $U \times \mathbb{C}_\zeta$, there exists a positive constant C so that

$$|P_j(x,\zeta)| \le C^{j+1} j! \tag{2.3}$$

holds for each $(x,\zeta) \in K$ and for $j = 0, 1, 2, \dots$. Conversely, if $P_j(x,\zeta)$ are holomorphic on $U \times \mathbb{C}$ and they satisfy (2.3), then the formal series

$$\sum_{j=0}^{\infty} \eta^{-j} P_j(x, \xi/\eta) \tag{2.4}$$

defines a differential operator of WKB type.

It follows from the composition rule for microdifferential operators (cf. [SKK, Chap. II]) that the total symbol $\sigma(R)$ of the composite operator R of two differential operators P, Q of WKB type is written in the form

$$\sigma(R) = \sum_{l=0}^{\infty} \eta^{-l} R_l(x,\zeta) \tag{2.5}$$

with

$$R_l(x,\zeta) = \sum_{j+k+m=l} \frac{1}{m!} (\partial_\zeta^m P_j(x,\zeta))(\partial_x^m Q_k(x,\zeta)), \tag{2.6}$$

where

$$\sum_{j=0}^{\infty} \eta^{-j} P_j(x,\zeta)$$

and

$$\sum_{k=0}^{\infty} \eta^{-k} Q_k(x,\zeta)$$

are the total symbols of P and Q, respectively. The composite symbol $\sigma(R) = R(x,\zeta,\eta)$ can be also written in the following form (cf. [A1], [A2]):

$$R(x,\zeta,\eta) = \exp(\eta^{-1}\partial_w\partial_z)P(x,\zeta+w,\eta)Q(x+z,\zeta,\eta)|_{w=z=0}. \tag{2.7}$$

Since a differential operator P of WKB type is independent of y, we identify ∂_y with its symbol η in what follows. Thus we often write P with the total symbol (2.1) in the form

$$P = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, \partial_x/\eta). \tag{2.8}$$

We regard η (or ∂_y) as a large parameter. Then P can be considered to be a formal ordinary differential operator of (finite or) infinite order with the large parameter η. This is the reason why we call P a *differential* operator of WKB type.

The ring $\mathcal{E}_{\mathrm{WKB}}(U)$ contains various types of operators which can be analyzed by using exact WKB analysis. We give some examples of operators in $\mathcal{E}_{\mathrm{WKB}}(\mathbb{C})$.

Example 2.1 $\eta^{-2}\partial_x^2 - x$. (Airy operator with a large parameter.)

Example 2.2 $\eta^{-3}\partial_x^3 - 3\eta^{-1}\partial_x + x$. (Cf. [AKT1], [AKT3], [BNR].)

Example 2.3 $\exp(\eta^{-1}\partial_x) - x$. (Cf. Introduction.)

Example 2.4 $\cosh(\sqrt{(i\eta)^{-1}\partial_x}) - x$. (Cf. Introduction and Appendix.)

For the convenience of our later discussions, we further introduce an extension of the ring $\mathcal{E}_{\mathrm{WKB}}(U)$: We denote by $\hat{\mathcal{E}}_{\mathrm{WKB}}(U)$ the set of all formal sums of the form (2.8), where $P_j(x,\zeta)$ are holomorphic in $U \times \mathbb{C}$, without any growth condition on P_j. The product (composition of operators) in this set is defined by (2.7). With this product and the sum as the formal power series, the set $\hat{\mathcal{E}}_{\mathrm{WKB}}(U)$ becomes a ring that contains $\mathcal{E}_{\mathrm{WKB}}(U)$. An element of $\hat{\mathcal{E}}_{\mathrm{WKB}}(U)$ is called a formal differential operator of WKB type. If U is not simply connected, we denote the universal covering of U by $\tilde{U}$. We will also use the rings $\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{U})$ and $\mathcal{E}_{\mathrm{WKB}}(\tilde{U})$, where we admit multivalued analytic functions as coefficients of powers of ζ in P_j.

3 WKB solutions, turning points and Stokes curves for operators of WKB type

Let

$$P = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, \partial_x/\eta)$$

be a differential operator of WKB type defined on U with the principal symbol $P_0(x, \zeta)$. We consider a differential equation

$$P\psi = 0. \tag{3.1}$$

To define a WKB solution of the equation, we introduce several notations.

Definition 3.1 (i) A formal WKB symbol is a formal series with an exponential factor that has the form

$$f = \exp(\eta a(x)) \sum_{j=0}^{\infty} \eta^{-j-\alpha} f_j(x), \tag{3.2}$$

where $a(x)$, $f_j(x)$ ($j = 0, 1, 2, \dots$) are holomorphic functions in an open set V in $\mathbb{C}$ and α is a real number.
(ii) A formal WKB symbol (3.2) is said to be Borel transformable if for every compact set K in V, there exists a positive constant C for which

$$\sup_{x \in K} |f_j(x)| \le j!\, C^{j+1}, \quad j = 0, 1, 2, \dots$$

hold. If α is not contained in the set $\{0, -1, -2, \dots\}$, the Borel transform of f is a (possibly multi-valued) analytic function $f_B(x, y)$ defined by the series

$$\sum_{j=0}^{\infty} \frac{f_j(x)}{\Gamma(\alpha + j)} (y + a(x))^{\alpha + j - 1}$$

which is locally uniformly convergent in

$$\{(x, y) |\, x \in V \text{ and } 0 < |y + a(x)| < r\}$$

for some $r > 0$.

Remark 3.1 (i) The formal WKB symbol (3.2) is Borel transformable if

$$\overset{\circ}{f} := \sum_{j=0}^{\infty} \eta^{-j} f_j(x)$$

belongs to $\mathcal{E}_{\mathrm{WKB}}(V)$.

(ii) If a formal series $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$ (a formal WKB symbol without exponential factor) is Borel transformable, then we find $\exp(\int^x S dx)$ is also Borel transformable by expanding

$$\exp\left(\sum_{j=0}^{\infty} \eta^{-j} \int^x S_j(x) dx\right)$$

into a formal series in η^{-1} by brute force.
(iii) If α is a negative integer or 0, we consider $(\eta^{-1/2} f)_B$ instead of f_B (cf. [V]).

Let V be an open set in U and $\tilde{V}$ the universal covering of V. Let S denote a formal power series of the form

$$S = S(x, \eta) = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x),$$

where $S_j(x)$ $(j = -1, 0, 1, 2, \dots)$ are analytic functions defined on $\tilde{V}$. Let us consider a formal WKB symbol ψ defined by

$$\psi = \exp\left(\int^x S(x, \eta) dx\right). \tag{3.3}$$

Then ψ, or more precisely, the multiplication operator by ψ, induces an automorphism

$$\phi_S : \hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V}) \longrightarrow \hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})$$

defined by

$$\phi_S(P) = \psi^{-1} P \psi.$$

The right-hand side is well-defined as a quantized contact transform of P (cf. [A4], [SKK]). Actually the geometric aspect of this quantized contact transformation is quite simple; it is a contact (=homogeneous symplectic) transformation induced by a change of variables in y. In terms of symbols, we may write

$$\phi_S(P) = P(x, (\partial_x + S)/\eta, \eta). \tag{3.4}$$

The corresponding contact transformation is given by

$$\begin{cases} x \longrightarrow x, \\ y \longrightarrow y - \int^x S_{-1}(x)dx, \\ \xi \longrightarrow \xi + \eta S_{-1}(x), \\ \eta \longrightarrow \eta. \end{cases}$$

We consider a left ideal of $\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})$ generated by ∂_x:

$$\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})\partial_x := \{Q\,\partial_x\,;\, Q \in \hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})\}.$$

An operator P in $\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})$ belongs to $\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})\partial_x$ if and only if $\sigma(P)|_{\zeta=0} = 0$.

Definition 3.2 A formal WKB symbol ψ defined by (3.3) is said to be a WKB solution of (3.1) if $\phi_S(P)$ belongs to $\hat{\mathcal{E}}_{\mathrm{WKB}}(\tilde{V})\partial_x$.

The background idea of this definition is as follows: if we re-arrange the order of the multiplication operator by $\exp(\int^x S(x,\eta)dx)$ and the differential operator ∂_x by the application of the rule

$$\partial_x \exp(\int^x S(x,\eta)dx) = S(x,\eta)\exp(\int^x S(x,\eta)dx) + (\exp(\int^x S(x,\eta)dx))\partial_x$$

in the composite $P\psi$ of operators P and ψ so that the differential operator ∂_x always stand to the right of the multiplication operators (the so-called normal ordering), then the part which is free from ∂_x should coincide with the formal WKB symbol obtained by letting P act on ψ regarded as a "function of x", not an operator, by the rule

$$\partial_x\psi = S(x,\eta)\psi,$$

due to the fact that ψ is free from ∂_x. Thus intuitively speaking, the requirement in Definition 3.2 amounts to saying that ψ regarded as a "function of x" satisfies the differential equation $P\psi = 0$. Note that the left-hand side of this equation is different from the composite $P\psi$ of operators P and ψ. For example, if P is a differential operator of the second order of the form

$$P = \eta^{-2}\partial_x^2 - Q(x),$$

the requirement of Definition 3.2 reads as

$$\frac{dS}{dx} + S^2 - \eta^2 Q(x) = 0,$$

which is the traditional Riccati equation. In fact, we have

$$\phi_S(P) \ = \ \eta^{-2}\left(\frac{dS}{dx} + S^2 - \eta^2 Q(x)\right) \ + \ \eta^{-2}(\partial_x + 2S)\partial_x$$

in this case. Since ψ does not contain the differential operator ∂_x, the normal ordering of $P\psi$ can be immediately obtained by applying ψ from the left to the normally ordered $\phi_S(P)$.

We believe the employment of the quantized contact transformation $\phi_S(P)$ is a neat way to formulate this intuitive picture in a mathematically rigorous manner. In particular, the explicit form (3.4) of $\phi_S(P)$ clearly explains how natural is our starting assumption that the symbol of a differential operator of WKB type should be entire in $\zeta = \xi/\eta$.

Next we define the notion of turning points for operators of WKB type. Recall that $P_0(x,\zeta)$ denotes the principal symbol of P.

Definition 3.3 Suppose that the system of equations

$$P_0(x,\zeta) = \partial_\zeta P_0(x,\zeta) = 0 \tag{3.5}$$

has a solution $(x,\zeta) = (x_*,\zeta_*) \in U \times \mathbb{C}_\zeta$ and $P_0(x_*,\zeta)$ does not vanish identically as a function of ζ. Then x_* is called a turning point of P with a characteristic value ζ_*. The smallest positive integer m so that $\partial_\zeta^m P_0(x_*,\zeta_*)$ does not vanish is called the rank of the turning point x_* with the characteristic value ζ_*.

Note that there may exist ζ_*' which does not equal ζ_* for which $P_0(x_*,\zeta_*') = \partial_\zeta P(x_*,\zeta_*') = 0$. (Cf. Examples A.1 and A.2.)

Let x_* be a turning point of P of rank m with a characteristic value ζ_*. By using the Weierstrass preparation theorem, we see that the principal symbol $P_0(x,\zeta)$ of P is uniquely decomposed into the following form:

$$P_0(x,\zeta) = q(x,\zeta)r(x,\zeta), \tag{3.6}$$

where $r(x,\zeta)$ is a Weierstrass polynomial of degree m in ζ with the center at (x_*, ζ_*) and $q(x,\zeta)$ is a holomorphic function defined on a neighborhood $U_0 \times \omega_0$ of (x_*, ζ_*) so that $q(x_*, \zeta_*) \neq 0$. By the definition, r has the following form:

$$r(x,\zeta) = (\zeta - \zeta_*)^m + f_1(x)(\zeta - \zeta_*)^{m-1} + \cdots + f_m(x),$$

where $f_j(x)$ vanishes at $x = x_*$ for $j = 1, \ldots, m$.

Definition 3.4 The Weierstrass polynomial $r(x,\zeta)$ is called the vanishing factor of P at (x_*, ζ_*). There are m analytic solutions $\zeta = \zeta_1(x), \ldots, \zeta_m(x)$ of the equation $r(x,\zeta) = 0$ satisfying $\zeta_j(x_*) = \zeta_*$ for $j = 1,\ldots,m$. These solutions are called characteristic roots passing through (x_*, ζ_*).

We introduce the notion of Stokes curves for turning points of rank 2.

Definition 3.5 Let x_* be a turning point of P of rank 2 with a characteristic value ζ_* and let $\zeta_+(x)$ and $\zeta_-(x)$ be the characteristic roots of P passing through (x_*, ζ_*). A Stokes curve emanating from x_* is a curve defined locally by the following equation:

$$\mathrm{Im} \int_{x_*}^{x} (\zeta_+(s) - \zeta_-(s))ds = 0. \qquad (3.7)$$

Definition 3.6 (i) A turning point x_* of P of rank 2 with a characteristic value ζ_* is said to be simple if

$$\partial_x P_0(x_*, \zeta_*) \neq 0. \qquad (3.8)$$

(ii) A turning point x_* of P of rank 2 with a characteristic value ζ_* is said to be double if

$$\partial_x P_0(x_*, \zeta_*) = 0 \qquad (3.9)$$

and

$$(\partial_x \partial_\zeta P_0(x_*, \zeta_*))^2 - \partial_x^2 P_0(x_*, \zeta_*) \partial_\zeta^2 P_0(x_*, \zeta_*) \neq 0 \qquad (3.10)$$

hold.

Infinitesimal configuration of Stokes curves at a simple (resp. double) turning point x_* is the same as in the case of second order differential operators. That is, they consist of three (resp. four) half-lines with starting point at x_* and the angle between any adjacent two rays is equal to $2\pi/3$ (resp. $\pi/2$).

Remark 3.2 If P is of third order with simple discriminant (cf. [AKT1]), then all turning points are simple.

In the case of rank 2, we can define the notion of turning points with higher multiplicity (cf. [P] for the second-order case). Since the main interest of this paper is the case of simple or double turning points, we do not give them here.

4 Construction of WKB solutions

Let us construct a WKB solution of (3.1). We assume that there is a solution $\zeta = \zeta(x)$ of the equation $P_0(x, \zeta) = 0$. We suppose $\zeta(x)$ is analytic on an open set U_1 in U. Using $\zeta(x)$, we will construct a formal WKB symbol

$$\psi = \exp\left(\int^x S(x,\eta)dx\right),$$

for which $\phi_S(P)$ belongs to $\hat{\mathcal{E}}_{\mathrm{WKB}}(U_1)\partial_x$. Here

$$S(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$$

is a formal series with analytic coefficients defined on U_1.

We set $\tilde{S} := \eta^{-1}S(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j-1} S_j(x)$ and $T(x,\eta) = \int^x \tilde{S}(x,\eta)dx$. It then follows from the definition of ϕ_S that the total symbol $\sigma(\phi_S(P))$ has the following form (cf. [A4]):

$$\exp(\eta^{-1}\partial_\zeta\partial_z)P(x,\zeta,\eta)\exp(\eta(T(x+z,\eta) - T(x,\eta)))|_{z=0}. \tag{4.1}$$

If this vanishes at $\zeta = 0$, then ψ is a WKB solution of (3.1). To calculate (4.1), we use the following lemma which is a special case of Sublemma of Lemma 1.3 in [A3] (see also [M]).

Lemma 4.1 *Let $A(x,z,\eta) = \sum_{j=0}^{\infty} A_j(x,z)\eta^{-j}$ be a formal power series of η^{-1}, where $A_j(x,z)$ $(j = 0, 1, 2, \ldots)$ are holomorphic on $U_1 \times \{|z| < c\}$ for some constant $c > 0$. Then the following relation of formal power series holds:*

$$\begin{aligned}\exp(\eta^{-1}\partial_\zeta\partial_z)&P(x,\zeta,\eta)\exp(\eta z A(x,z,\eta))|_{z=0} \\ &= \exp(\eta^{-1}\partial_\zeta\partial_z)P(x,\zeta + A(x,z,\eta),\eta)|_{z=0}.\end{aligned} \tag{4.2}$$

We set

$$A(x,z,\eta) = \frac{T(x+z,\eta) - T(x,\eta)}{z}$$

and apply Lemma 4.1 to (4.1). Since we have

$$\begin{aligned} A(x,z,\eta) &= \int_0^1 \tilde{S}(x+zt,\eta)dt \\ &= \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \frac{\partial^k \tilde{S}}{\partial x^k}(x,\eta), \end{aligned}$$

we arrive at the following

Proposition 4.1 *The formal WKB symbol* $\psi = \exp\left(\eta \int^x \tilde{S}(x,\eta)dx\right)$ *is a WKB solution of* (3.1) *if*

$$\exp(\eta^{-1}\partial_\zeta \partial_z) P\left(x, \zeta + \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \frac{\partial^k \tilde{S}}{\partial x^k}(x,\eta), \eta\right)\Bigg|_{z=\zeta=0} = 0 \tag{4.3}$$

holds.

Equation (4.3) is a counterpart of the Riccati equation in the second-order case. To construct in a recursive manner $\tilde{S}(x,\eta)$ satisfying (4.3), we need some more reduction of the condition (4.3). For the sake of simplicity of indices, we set $\tilde{S}_m = S_{m-1}$ $(m = 0, 1, 2, \ldots)$. Recall that $P(x,\zeta,\eta)$ has the form $\sum_{n=0}^{\infty} \eta^{-n} P_n(x,\zeta)$. Substitute this and $\tilde{S} = \sum \eta^{-m} \tilde{S}_m$ into (4.3) and expand it in the powers of η^{-1}. Then we find that (4.3) is equivalent to

$$\sum_{l=0}^{\infty} \eta^{-l} \sum \frac{\partial_\zeta^{j+k} P_n(x,\tilde{S}_0)}{j!} \frac{\partial_x^{k_1} \tilde{S}_{m_1} \cdots \partial_x^{k_j} \tilde{S}_{m_j}}{(k_1+1)! \cdots (k_j+1)!} = 0, \tag{4.4}$$

where the second summation is taken over all indices k, n, m; $k_1, \ldots, k_j$; $m_1, \ldots, m_j$ satisfying $k \geq 0$, $n \geq 0$, $m \geq 0$, $k+n+m = l$; $k_1 + \cdots + k_j = k$, $m_1 + \cdots + m_j = m$, $k_i + m_i > 0$ $(i = 1, \ldots, j)$, $j \leq k+m$. Note that this is a finite sum. The leading term of the relation is

$$P_0(x, \tilde{S}_0) = 0. \tag{4.5}$$

Thus we take $\tilde{S}_0 = \zeta(x)$. In the coefficient of η^{-l} $(l = 1, 2, \dots)$ in (4.4), $\tilde{S}_l$ appears in exactly one term which is corresponding to the set of indices $m = l$, $k = n = 0$, $j = 1$, $k_1 = 0$, $m_1 = l$ and which has the form

$$\partial_\zeta P_0(x, \tilde{S}_0)\tilde{S}_l.$$

All other terms contain $\tilde{S}_j$ and their derivatives for $j < l$. Hence, if $\tilde{S}_j$ for $j < l$ are known, we can determine $\tilde{S}_l$ uniquely so far as $\partial_\zeta P_0(x, \tilde{S}_0)$ does not vanish identically. Namely,

$$\tilde{S}_l = -\frac{1}{\partial_\zeta P_0(x, \tilde{S}_0)} \sum \frac{\partial_\zeta^{j+k} P_n(x, \tilde{S}_0)}{j!} \frac{\partial_x^{k_1} \tilde{S}_{m_1} \cdots \partial_x^{k_j} \tilde{S}_{m_j}}{(k_1+1)! \cdots (k_j+1)!}. \tag{4.6}$$

Here the sum is taken as in the second summation in (4.4) except for the term corresponding to $m = l$, $k = n = 0$, $j = 1$, $k_1 = 0$, $m_1 = l$. The series $\tilde{S} = \sum_{j=0}^{\infty} \eta^{-j} \tilde{S}_j$ thus constructed clearly satisfies (4.3). Therefore we have

Theorem 4.1 *Suppose that $\partial_\zeta P_0(x, \zeta(x))$ never vanishes in an open set V in U_1. If we set $\tilde{S}_0(x) = \zeta(x)$ and define $\{\tilde{S}_l(x)\}$ $(l = 1, 2, \dots)$ by* (4.6)*, then a formal WKB symbol*

$$\psi = \exp\left(\int^x S(x, \eta) dx \right)$$

defined by

$$S(x, \eta) = \eta \tilde{S} = \eta \sum_{l=0}^{\infty} \eta^{-j} \tilde{S}_l(x)$$

is a WKB solution of (3.1)*. Here $\tilde{S}_l$ $(l \geq 1)$ are uniquely determined as* (*possibly multi-valued*) *analytic functions defined on V once $\tilde{S}_0(x) = \zeta(x)$ is fixed.*

Remark 4.1 The subleading term of $\tilde{S}$ is given as follows:

$$\tilde{S}_1(x) = -\frac{1}{\partial_\zeta P_0(x, \zeta(x))} \left(\frac{1}{2} \partial_\zeta^2 P_0(x, \zeta(x)) \zeta'(x) + P_1(x, \zeta(x)) \right).$$

Our construction of a WKB solution only requires that P to be a *formal* differential operator of WKB type. If we assume that P is a differential operator of WKB type, we have

Theorem 4.2 *The WKB solution ψ constructed in Theorem 4.1 is Borel transformable.*

Proof It suffices to show that $\tilde{S}$ is Borel transformable. Let V' be an open set satisfying $\bar{V}' \subset\subset V$. There is a constant $C_0 > 0$ so that

$$\left| \frac{1}{\partial_\zeta P_0(x, \tilde{S}_0(x))} \right| \leq C_0 \tag{4.7}$$

holds for each $x \in V'$. Since $P_n(x, \zeta)$ is entire in ζ for every fixed $x \in V'$, its derivatives can be written in the form

$$\partial_\zeta^j P_n(x, \zeta) = \frac{j!}{2\pi\sqrt{-1}} \int_{|\zeta' - \zeta| = 1/\delta} \frac{P_n(x, \zeta')}{(\zeta' - \zeta)^{j+1}} d\zeta', \tag{4.8}$$

where δ is an arbitrary constant and $j = 0, 1, 2, \ldots$. Hence there is a constant $C_1 = C_1(\delta)$ for which

$$|\partial_\zeta^j P_n(x, \tilde{S}_0(x))| \;\leq\; j!\, n!\, C_1^{n+1}\, \delta^j \tag{4.9}$$

holds for $x \in V'$, $j, n = 0, 1, 2, \ldots$. Let x_0 be a point in V'. Let $B(x_0, \rho)$ denote a closed disk of radius $\rho > 0$ with center at x_0. Let ρ_0 be a positive number satisfying $B(x_0, \rho_0) \subset V'$. We shall show the following estimates for $\tilde{S}_m(x)$ by induction: There exist constants $A > 0$, $C_2 > 0$ and $M > 1$ so that

$$|\tilde{S}_m(x)| \;\leq m!\, A\, C_2^m\, \varepsilon^{-Mm} \tag{4.10}$$

holds for any sufficiently small ε, $x \in B(x_0, \rho_0 - \varepsilon)$ and $m = 0, 1, 2, \ldots$. This holds for $m = 0$ if we take $A \geq \sup\limits_{x \in V'} |\tilde{S}_0(x)|$. Suppose that (4.10) holds for $m = 0, 1, \ldots, l-1$. Then we have

$$|\partial_x^k \tilde{S}_m(x)| \;\leq\; (m+k)!\, A\, C_2^m\, \varepsilon^{-M(m+k)} \tag{4.11}$$

for $x \in B(x_0, \rho_0 - \varepsilon)$, $k = 0, 1, 2, \ldots$. In fact, if the above estimate holds for k, writing $\partial_x^{k+1} \tilde{S}_m(x)$ in the form

$$\partial_x^{k+1} \tilde{S}_m(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|x' - x| = \varepsilon/(k+m+1)} \frac{\partial_x^k \tilde{S}_m(x')}{(x' - x)^2} dx' \tag{4.12}$$

and using (4.11) for ε to be $(1-1/(m+k+1))\varepsilon$, we have

$$|\partial_x^{k+1}\tilde{S}_m(x)| \le (k+m+1)!\,A\,C_2^m\left(1-\frac{1}{m+k+1}\right)^{-M(m+k)}\varepsilon^{-M(m+k)-1} \tag{4.13}$$

for $x \in B(x_0, \rho_0-\varepsilon)$. Since we have

$$\left(1-\frac{1}{m+k+1}\right)^{-M(m+k)} = \left(1+\frac{1}{m+k}\right)^{M(m+k)} \le e^M$$

and we may assume ε is so small that $e^M\varepsilon^{M-1} \le 1$, the right hand side of (4.13) is dominated by

$$(m+k+1)!\,A\,C_2^m\varepsilon^{-M(m+k+1)}.$$

This implies (4.11) holds for $k+1$. Combining (4.6) with (4.7), (4.8) and (4.11), we see that $|\tilde{S}_l(x)|$ is dominated by

$$C_0\sum\frac{(j+k)!n!(k_1+m_1)!\cdots(k_j+m_j)!}{j!(k_1+1)!\dots(k_j+1)!}C_1^{n+1}\delta^{j+k}A^jC_2^m\varepsilon^{-M(k+m)} \tag{4.14}$$

in $B(x_0,\rho_0-\varepsilon)$. Here and hereafter the summation is taken as in (4.6) unless otherwise stated. Clearly (4.14) is less than

$$C_0\sum\frac{(j+k)!n!(k_1+m_1)!\cdots(k_j+m_j)!}{j!k_1!\dots k_j!}C_1^{n+1}\delta^{j+k}A^jC_2^m\varepsilon^{-M(k+m)}. \tag{4.15}$$

Since

$$\frac{(k_1+m_1)!\cdots(k_j+m_j)!}{k_1!\dots k_j!} \tag{4.16}$$

can be rewritten in the form

$$\frac{(\sum_1^j(k_i+m_i))!}{(\sum_1^j k_i)!}\prod_{\nu=2}^{j}\frac{\binom{\sum_1^\nu k_i}{\sum_1^{\nu-1}k_i}}{\binom{\sum_1^\nu(k_i+m_i)}{\sum_1^{\nu-1}(k_i+m_i)}} \tag{4.17}$$

and $\binom{a}{b}\binom{c}{d} \le \binom{a+c}{b+d}$ holds, we see that

$$\sum_{\substack{k_1+\cdots+k_j=k\\ m_1+\cdots+m_j=m}}\frac{(k_1+m_1)!\cdots(k_j+m_j)!}{k_1!\dots k_j!} \tag{4.18}$$

is not greater than

$$\sum_{m_1+\cdots+m_j=m} \frac{m_1!\cdots m_j!}{m!}\frac{(k+m)!}{k!}2^{k+j-1}. \tag{4.19}$$

It is easy to see that the following inequality holds:

$$\sum_{m_1+\cdots+m_j=m} \frac{m_1!\cdots m_j!}{m!} \le 2^j.$$

Hence (4.15) is dominated by

$$\frac{C_0C_1}{2} \sum_{\substack{0\le j\le k+m, 0\le m<l\\ k+m+n=l}} (8\delta A)^j \left(\frac{4\delta}{C_2}\right)^k \left(\frac{C_1\varepsilon^M}{C_2}\right)^n l!\, C_2^l\, \varepsilon^{-Ml}. \tag{4.20}$$

First we take $\delta > 0$ so that $8\delta A < 1$. Next we choose $C_2 > 0$ for which $4\delta/C_2 < 1$, $C_1\varepsilon^M/C_2 < 1$ and

$$\frac{C_0C_1}{2C_2}\frac{4\delta + C_1\varepsilon^M}{(1-8\delta A)\left(1-\dfrac{4\delta}{C_2}\right)\left(1-\dfrac{C_1\varepsilon^M}{C_2}\right)} \le A$$

hold. Then we have

$$|\tilde{S}_l(x)| \le l!\, A\, C_2^l \varepsilon^{-Ml}$$

for $x \in B(x_0, \rho_0 - \varepsilon)$. This completes the proof of Theorem 4.2.

5 Local theory near a turning point

In this section we analyze WKB solutions of (3.1) near a turning point of rank 2.

Theorem 5.1 *Let P be a differential operator of WKB type defined on an open set U in $\mathbb{C}$. Suppose that $x_* \in U$ is a turning point of rank 2 of P with a characteristic value ζ_*. Let $r(x,\zeta)$ be the vanishing factor of P at (x_*, ζ_*). Let U_0 be a sufficiently small open disk with center at x_*. Then there uniquely exist differential operators Q and R of WKB type defined on U_0 which satisfy the relation*

$$P = QR \tag{5.1}$$

and the following conditions:
(i) The principal symbol $R_0(x,\zeta)$ of R coincides with $r(x,\zeta)$.
(ii) For each $j>0$, the coefficient $R_j(x,\zeta)$ of η^{-j} of the symbol of R is of degree at most one in ζ.
(iii) The principal symbol $Q_0(x,\zeta)$ of Q does not vanish at (x_*,ζ_*).

Remark 5.1 The above conditions (i) and (ii) imply that R is of second order. Hence this theorem is a generalization of Theorem 1.4 of [AKT1].

Proof We set

$$Q = Q(x,\partial_x/\eta,\eta) = \sum_{j=0}^{\infty} \eta^{-j} Q_j(x,\partial_x/\eta), \tag{5.2}$$

$$R = R(x,\partial_x/\eta,\eta) = \sum_{j=0}^{\infty} \eta^{-j} R_j(x,\partial_x/\eta) \tag{5.3}$$

and we will construct holomorphic functions $Q_j(x,\zeta)$ and $R_j(x,\zeta)$ ($j = 0,1,2,\ldots$) defined in $U_0\times\mathbb{C}$ so that (5.1) holds. Taking total symbols, we see that (5.1) is equivalent to the following relation of symbols:

$$\sum_{n=0}^{\infty} \eta^{-n} P_n(x,\zeta) = \sum_{n=0}^{\infty} \eta^{-n} \sum_{i+j+l=n} \frac{1}{l!} (\partial_\zeta^l Q_i(x,\zeta))(\partial_x^l R_j(x,\zeta)). \tag{5.4}$$

Comparing the coefficients of like powers of η^{-1}, we have

$$P_0(x,\zeta) = Q_0(x,\zeta) R_0(x,\zeta) \tag{5.5}$$

for the leading terms, and

$$P_n(x,\zeta) = \sum_{i+j+l=n} \frac{1}{l!} \partial_\zeta^l Q_i(x,\zeta)\, \partial_x^l R_j(x,\zeta) \tag{5.6}$$

for $n = 1,2,3,\ldots$. We take $R_0(x,\zeta) = r(x,\zeta)$ and then,

$$Q_0(x,\zeta) = P_0(x,\zeta)/r(x,\zeta)$$

is holomorphic in $U_0\times\mathbb{C}$ and $Q_0(x_*,\zeta_*)\neq 0$. Suppose that R_0, Q_0, ..., R_{k-1}, Q_{k-1} have been obtained so that they satisfy (5.6) for $n = 0,1,2,\ldots,k-1$

and are holomorphic on $U_0 \times \mathbb{C}$. To find holomorphic functions R_k and Q_k on $U_0 \times \mathbb{C}$ for which (5.6) holds for $n = k$, we set

$$H_k(x,\zeta) = -\sum_{\substack{i+j+l=k \\ i,j<k}} \frac{1}{l!}\partial_\zeta^l Q_i(x,\zeta)\partial_x^l R_j(x,\zeta) + P_k(x,\zeta) \tag{5.7}$$

and

$$F_k(x,\zeta) = \frac{H_k(x,\zeta)}{Q_0(x,\zeta)}. \tag{5.8}$$

Employing the idea of H. Cartan ([C, Appendix]) in proving a precise (i.e., global in ζ) version of the Späth division theorem, we have two holomorphic functions G_k and R_k satisfying

$$F_k(x,\zeta) = G_k(x,\zeta)R_0(x,\zeta) + R_k(x,\zeta), \tag{5.9}$$

where R_k is a polynomial in ζ of degree at most one. We know that $G_k(x,\zeta)$ and $R_k(x,\zeta)$ are written as follows:

$$G_k(x,\zeta) = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_0} \frac{F_k(x,\zeta')}{R_0(x,\zeta')}\frac{1}{\zeta'-\zeta}d\zeta', \tag{5.10}$$

$$R_k(x,\zeta) = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_1} \frac{F_k(x,\zeta')}{R_0(x,\zeta')}(\zeta+\zeta'-(\zeta_+(x)+\zeta_-(x)))d\zeta'. \tag{5.11}$$

Here $\zeta_\pm(x)$ denote the characteristic roots passing through (x_*,ζ_*), γ_0 (resp. γ_1) is a contour which encircles ζ and $\zeta_\pm(x)$ (resp. $\zeta_\pm(x)$) counterclockwise and which does not contain any other zeros of $P_0(x,\zeta)$ in ζ. It is clear that R_k is holomorphic on $U_0 \times \mathbb{C}$. Since F_k is holomorphic outside the set of zeros of Q_0, so is G_k. We now set

$$Q_k(x,\zeta) = G_k(x,\zeta)Q_0(x,\zeta). \tag{5.12}$$

Multiplying the both sides of (5.9) by $Q_0(x,\zeta)$, we have

$$F_k(x,\zeta)Q_0(x,\zeta) = G_k(x,\zeta)Q_0(x,\zeta)R_0(x,\zeta) + Q_0(x,\zeta)R_k(x,\zeta),$$

namely,

$$H_k(x,\zeta) = Q_k(x,\zeta)R_0(x,\zeta) + Q_0(x,\zeta)R_k(x,\zeta).$$

Thus Q_k and R_k satisfy (5.6) for $n = k$. By the definition of G_k, we have

$$Q_k(x,\zeta) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_0} \frac{H_k(x,\zeta')}{R_0(x,\zeta')} \frac{Q_0(x,\zeta)}{Q_0(x,\zeta')} \frac{1}{\zeta'-\zeta} d\zeta'. \tag{5.13}$$

The right-hand side of (5.13) can be decomposed into the sum of

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_0} \frac{H_k(x,\zeta')}{R_0(x,\zeta')} \frac{1}{\zeta'-\zeta} d\zeta' \tag{5.14}$$

and

$$-\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_0} \frac{H_k(x,\zeta')}{R_0(x,\zeta')} \frac{1}{Q_0(x,\zeta')} \frac{Q_0(x,\zeta') - Q_0(x,\zeta)}{\zeta'-\zeta} d\zeta'. \tag{5.15}$$

Since the integrand of the integral (5.14) is holomorphic at the set of zeros of Q_0, (5.14) can be holomorphically continued to $\mathbb{C}$ in ζ. The integrand of (5.15) is holomorphic at $\zeta = \zeta'$. Hence $Q_k(x,\zeta)$ can be also continued to the whole $\mathbb{C}$ in ζ. Hence we have Q and R satisfying (5.1) and (i)–(iii) in $\hat{\mathcal{E}}_{\mathrm{WKB}}(U_0)$.

Next we prove that $Q_k(x,\zeta)$ and $R_k(x,\zeta)$ satisfy the growth conditions required for WKB type operators. Let x_0 be a point in U_0 and ρ a positive number satisfying

$$\{x \in U_0; |x - x_0| \le \rho\} \subset U_0.$$

Let L be a positive number and let K_0 denote the compact set

$$\{(x,\zeta); |x - x_0| \le \rho,\ |\zeta| \le L\}.$$

Then there is a positive constant C so that

$$|P_j(x,\zeta)| \le j! C^{j+1} \tag{5.16}$$

holds for all $(x,\zeta) \in K_0$. We can take a positive number A_0 for which

$$|Q_0(x,\zeta)| \le A_0 \tag{5.17}$$

and

$$|R_0(x,\zeta)| \le A_0 \tag{5.18}$$

hold for all $(x,\zeta) \in K_0$. Note that, by the construction of $Q_k(x,\zeta)$ and $R_k(x,\zeta)$, there exists a positive constant C_1 that is independent of the choice of γ_0 and γ_1 for which

$$|Q_k(x,\zeta)| \le C_1 \sup_{(x',\zeta') \in K_0} |H_k(x',\zeta')| \tag{5.19}$$

and

$$|R_k(x,\zeta)| \le C_1 \sup_{(x',\zeta')\in K_0} |H_k(x',\zeta')| \tag{5.20}$$

hold for all $(x,\zeta) \in K_0$.

Let ε be a positive number. Let M and N be real numbers satisfying $1 < N \le M/2$.

Lemma 5.1 *Let $\{A_k\}$ $(k = 0, 1, 2, \dots)$ be a sequence of positive numbers defined by the following recurrence formula:*

$$A_k = C_1 \Big(\sum_{\substack{i+j\le k \\ i,j<k}} A_i A_j + (\varepsilon^M C)^k C \Big), \quad k > 0. \tag{5.21}$$

Then the formal power series $f(t)$ defined by

$$f(t) = \sum_{k=0}^{\infty} A_k t^k \tag{5.22}$$

is a convergent power series of t.

Proof of Lemma 5.1 The formal power series $f(t)$ satisfies the following quadratic equation:

$$\frac{1}{1-t} f(t)^2 = \left(\frac{1}{C_1} + 2A_0 \right) f(t) - \frac{\varepsilon^M C^2 t}{1-\varepsilon^M C t} - A_0^2 - \frac{A_0}{C_1}. \tag{5.23}$$

Since the discriminant of (5.23) does not vanish at $t = 0$, (5.23) has a unique analytic solution $f(t)$ which satisfies $f(0) = A_0$. This proves the lemma.

Let $D(\varepsilon)$ denote the compact set

$$\{(x,\zeta); |x - x_0| \le \rho - \varepsilon, |\zeta| \le L - \varepsilon\}.$$

We shall prove the following estimates by induction:

$$|Q_j(x,\zeta)| \le j! \varepsilon^{-Mj} A_j \tag{5.24}$$

and

$$|R_j(x,\zeta)| \le j! \varepsilon^{-Mj} A_j \tag{5.25}$$

hold for all $(x,\zeta) \in D(\varepsilon)$, $j = 0, 1, 2, \dots$.

Suppose that (5.24) and (5.25) hold for $j = 0, 1, 2, \ldots, k-1$. A similar argument as in the proof of (4.11) shows the following: we may assume ε is so small that $e^N \varepsilon^{N-1} \le 1$, and then

$$|\partial_\zeta^l Q_j(x, \zeta)| \le (j+l)! \varepsilon^{-Mj-Nl} A_j \tag{5.26}$$

and

$$|\partial_x^l R_j(x, \zeta)| \le (j+l)! \varepsilon^{-Mj-Nl} A_j \tag{5.27}$$

hold for all $(x, \zeta) \in D(\varepsilon)$, $j = 0, 1, 2, \ldots, k-1$ and for all $l = 0, 1, 2, \ldots$. Then, by the definition of H_k, we see that $|H_k(x, \zeta)|$ is dominated in $D(\varepsilon)$ by

$$\sum_{\substack{i+j+l=k \\ i,j<k}} \frac{(i+l)!(j+l)!}{l!} \varepsilon^{-M(i+j+l)} A_i A_j + k! C^{k+1}. \tag{5.28}$$

Since we have

$$\frac{(i+l)!(j+l)!}{k!l!} \le 1$$

for $k = i + j + l$, we see that (5.28) is not greater than

$$k! \varepsilon^{-Mk} \Big(\sum_{\substack{i+j \le k \\ i,j<k}} A_i A_j + (\varepsilon^M C)^k C \Big).$$

Thus, by the definition of A_k, $|H_k(x, \zeta)|$ is dominated by $k! \varepsilon^{-Mk} A_k / C_1$ in $D(\varepsilon)$. Combining this with the preceding remark, we see that (5.24) and (5.25) hold for $j = k$. By Lemma 5.1, there exist a constant $C_2 > 0$ so that

$$A_k \le C_2^{k+1}$$

holds for $k = 0, 1, 2, \ldots$. Hence Q and R are differential operators of WKB type. The uniqueness follows from the uniqueness of Cartan-Späth division theorem. This completes the proof.

This theorem can be easily generalized to the case where the rank of a turning point is larger than 2:

Theorem 5.2 *Let P be a differential operator of WKB type defined on an open set U in $\mathbb{C}$. Suppose that $x_* \in U$ is a turning point of rank m of P with a characteristic value ζ_*. Let $r(x, \zeta)$ be the vanishing factor of P at (x_*, ζ_*). Let U_0 be a sufficiently small open disk with center at x_*. Then there uniquely* *exist differential operators Q and R of WKB type defined in U_0 which satisfy the relation*

$$P = Q\,R$$

and the following conditions:
(i) The principal symbol $R_0(x,\zeta)$ of R coincides with $r(x,\zeta)$.
(ii) For each $j > 0$, the coefficient $R_j(x,\zeta)$ of η^{-j} of the symbol of R is of degree at most $m-1$ in ζ. Hence R is of order m.
(iii) The principal symbol $Q_0(x,\zeta)$ of Q does not vanish at (x_*,ζ_*).

It is evident that Theorems 5.1 and 5.2 enable us to reduce the WKB analysis of the operator P to that of a finite order differential operator R, at least near the turning point in question. In ending this paper, we show, as a typical example of such a reduction, how to analyze the connection phenomena of Borel transformed WKB solutions near a simple turning point; the reasoning is essentially the same as that in [AKT1], where the exact WKB analysis for the third order differential operator is discussed through the reduction to that of the second order differential operator.

Let x_* be a simple turning point of P. Then the differential operator R constructed in Theorem 5.1 near x_* has the form

$$R = \eta^{-2}\partial_x^2 + A(x,\eta)\eta^{-1}\partial_x + B(x,\eta), \tag{5.29}$$

where A and B are formal series of η^{-1} with holomorphic coefficients defined on a neighborhood of x_* and the leading terms of A and B are $-(\zeta_+ + \zeta_-)$ and $\zeta_+\zeta_-$, respectively. Hence x_* is a simple turning point of the second-order operator R. Let $\psi_\pm$ and $\phi_\pm$ respectively denote WKB solutions of $P\psi = 0$ and $R\phi = 0$ of the form

$$\psi_\pm = \exp\left(\int^x S_\pm(x,\eta)dx\right) \quad \text{and} \quad \phi_\pm = \exp\left(\int^x T_\pm(x,\eta)dx\right), \tag{5.30}$$

where $S_\pm(x,\eta) = \sum_{j=-1}^{\infty}\eta^{-j}S_{\pm,j}(x)$, $T_\pm(x,\eta) = \sum_{j=-1}^{\infty}\eta^{-j}T_{\pm,j}(x)$, and $S_{\pm,-1}(x) = T_{\pm,-1}(x) = \zeta_\pm(x)$. It follows from Theorem 5.1 that

$$P\phi_\pm = Q\,R\phi_\pm = 0.$$

Hence $S_\pm$ and $T_\pm$ must coincide since both $\psi_\pm$ and $\phi_\pm$ are solutions of $P\psi = 0$ and their leading terms $S_{\pm,-1}(x)$ and $T_{\pm,-1}(x)$ are the same. In other words,

$\psi_\pm$ also satisfies the second-order equation $R\phi = 0$. For a second-order equation we have a "good" normalization of WKB solutions near a simple turning point (cf. [AKT1]). That is, letting $S_{\rm odd}$ and $S_{\rm even}$ respectively denote

$$S_{\rm odd} = \frac{1}{2}(S_+ - S_-) \quad \text{and} \quad S_{\rm even} = \frac{1}{2}(S_+ + S_-), \tag{5.31}$$

we have

$$S_{\rm even} = -\frac{1}{2S_{\rm odd}}\frac{\partial S_{\rm odd}}{\partial x} - \frac{1}{2}\eta A(x,\eta). \tag{5.32}$$

Then the following gives well-normalized WKB solutions near x_*:

$$\psi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\int_{x_*}^{x}(\pm S_{\rm odd} - \frac{1}{2}\eta A)dx\right). \tag{5.33}$$

By using the same reasoning as in [AKT1, Theorem 1.8], we can then find a formal coordinate transformation which brings $R\phi = 0$ to the Airy equation

$$(\eta^{-2}\partial_{\tilde{x}}^2 - \tilde{x})\varphi = 0$$

near x_* (cf. [AY]), and consequently obtain the following connection formula for (5.33).

Theorem 5.3 *Let P be the operator considered in Theorem 5.1. Assume that x_* is a simple turning point with a characteristic value ζ_*. Let $\zeta_\pm(x)$ denote the characteristic roots passing through (x_*, ζ_*). Let $\psi_\pm(x,\eta)$ be the WKB solutions of Equation* (3.1) *normalized as* (5.33) *with S_{-1} being $\zeta_\pm(x)$. Let $\psi_{\pm,B}(x,y)$ denote their Borel transforms. Then on a sufficiently small neighborhood of the origin of $\mathbb{C}_x \times \mathbb{C}_y$ $\psi_{\pm,B}(x,y)$ have their singularities only along $\Gamma_+ \cup \Gamma_-$, where*

$$\Gamma_\pm = \{(x,y); y + \int_{x_*}^{x}\zeta_\pm(x)dx = 0\}.$$

Furthermore the singular part of $\psi_{+,B}(x,y)$ (resp., $\psi_{-,B}(x,y)$) along Γ_- (resp., Γ_+) coincides with $i\psi_{-,B}(x,y)$ (resp., $-i\psi_{+,B}(x,y)$).

A Appendix

In this appendix, to see what happens at points far away from turning points, we present the global Stokes geometry for some concrete operators of WKB type which admit infinitely many phases; the Stokes geometry is described with the help of a computer. Throughout this appendix all operators are considered on $\mathbb{C}$ (i.e., all operators belong to $\mathcal{E}_{\mathrm{WKB}}(\mathbb{C})$).

Example A.1

$$P(x,\eta^{-1}\frac{d}{dx}) = \cosh(\sqrt{\frac{1}{i\eta}\frac{d}{dx}}) - x \tag{A.1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{1}{i\eta}\frac{d}{dx}\right)^n - x. \tag{A.2}$$

A straightforward calculation shows that solutions of equations $P(x,\zeta) = (\partial P/\partial\zeta)(x,\zeta) = 0$ are given by

$$(x,\zeta) = (1, -4n^2 i\pi^2), (-1, -(2n+1)^2 i\pi^2) \quad (n \in \mathbb{Z}). \tag{A.3}$$

Hence turning points are located at $x = 1$ and $x = -1$. These turning points are shown to be simple in the sense of Definition 3.6. We can also verify that characteristic roots of P passing through $(1, -4n^2 i\pi^2)$ are given by f_n and f_{-n} for each $n \in \mathbb{Z}$, where

$$f_n(x) = i\left(2n\pi i + \log(x + \sqrt{x^2-1})\right)^2. \tag{A.4}$$

Here, to specify the branch of f_n, we place cuts along the intervals $[1,\infty)$ and $(-\infty,-1]$ in x-plane and choose the branch so that the following relations hold:

$$\sqrt{x^2-1}\Big|_{x=0} = i \quad \text{and} \quad \log(x+\sqrt{x^2-1})\Big|_{x=0} = i\pi/2. \tag{A.5}$$

Hence we find that Stokes curves (cf. Definition 3.5) emanating from $x = 1$ are given by

$$\mathrm{Im}\int_1^x (f_n(x) - f_{-n}(x))dx = 0. \tag{A.6}$$

Note that (A.6) is equivalent to the following form, which does not depend on n:

$$\mathrm{Im}\int_1^x \log(x+\sqrt{x^2-1})dx = 0. \tag{A.7}$$

Hence all Stokes curves emanating from $x = 1$ sit on the same curve. Similarly, since characteristic roots of P passing through $(-1, -(2n+1)^2 i\pi^2)$

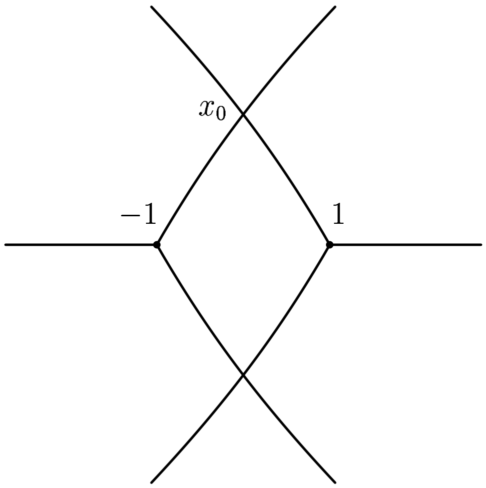

Figure 1

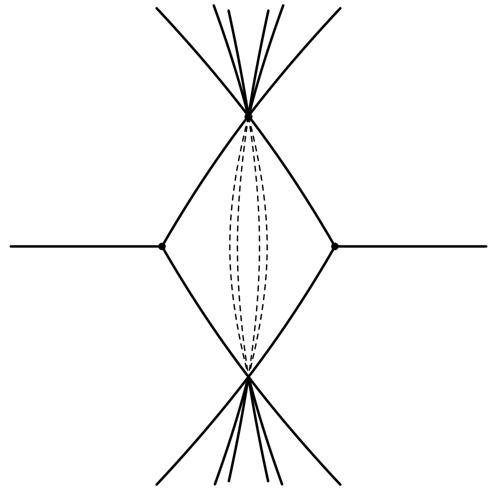

Figure 2

($n \in \mathbb{Z}$) are given by f_n and f_{-n-1}, we find that Stokes curves emanating from $x = -1$ are given by

$$\mathrm{Im} \int_{-1}^{x} (f_n(x) - f_{-n-1}(x))dx = 0, \tag{A.8}$$

which is equivalent to

$$\mathrm{Im} \int_{-1}^{x} \left(\log(x + \sqrt{x^2 - 1}) - i\pi\right) dx = 0. \tag{A.9}$$

Thus (A.8) is also independent of n, like (A.6). Stokes curves defined by (A.6) and (A.8) are shown in Fig. 1.

Since Stokes curves in Fig. 1 have crossing points, we need to introduce virtual turning points and new Stokes curves. (See [AKKT] for the details. See also [BNR], [AKT2].) For the operator (A.1) virtual turning points are given by $x = \pm \cos w$, where w is a nonzero solution of $w = \tan w$. New Stokes curves which should be added to Fig. 1 are given by

$$\mathrm{Im} \int_{x_0}^{x} (f_n(x) - f_m(x))dx = 0, \tag{A.10}$$

where x_0 is a crossing point in the upper half plane, and n, m are integers satisfying $n \neq -m$, $n \neq -m-1$. (Each of new Stokes curves passes through one of virtual turning points.) Eq. (A.10) is equivalent to

$$\operatorname{Im} \int_{x_0}^{x} (ik\pi + \log(x + \sqrt{x^2-1}))dx = 0, \tag{A.11}$$

where $k = m + n$. We show in Fig. 2 these new Stokes curves for $k = -3, -2, 1, 2$.

Example A.2

$$P(x, \eta^{-1}\frac{d}{dx}) = \cosh(\frac{1}{\eta}\frac{d}{dx}) - x. \tag{A.12}$$

This is a variant of the operator L_0 discussed in Introduction. Since solutions of equations $P(x, \zeta) = (\partial P/\partial \zeta)(x, \zeta) = 0$ are

$$(x, \zeta) = (1, 2in\pi), (-1, (2n+1)i\pi) \quad (n \in \mathbb{Z}), \tag{A.13}$$

turning points for (A.12) are $x = 1$ and $x = -1$. These turning points are simple. Characteristic roots of P passing through $(1, 2in\pi)$ are given by $f_n^{(+)}$ and $f_n^{(-)}$ $(n \in \mathbb{Z})$, where

$$\begin{aligned} f_n^{(+)}(x) &= \log(x + \sqrt{x^2-1}) + 2in\pi, & \text{(A.14)} \\ f_n^{(-)}(x) &= -\log(x + \sqrt{x^2-1}) + 2in\pi. & \text{(A.15)} \end{aligned}$$

Here we choose the branch of $f_n^{(\pm)}$ in a similar way as in Example A.1. We also find that the characteristic roots of P passing through $(-1, (2n+1)i\pi)$ are given by $f_n^{(+)}$ and $f_{n+1}^{(-)}$. Hence Stokes curves emanating from $x = 1$ are given by

$$\operatorname{Im} \int_{1}^{x} (f_n^{(+)}(x) - f_n^{(-)}(x))dx = 0, \tag{A.16}$$

and Stokes curves emanating from $x = -1$ are given by

$$\operatorname{Im} \int_{-1}^{x} (f_n^{(+)}(x) - f_{n+1}^{(-)}(x))dx = 0. \tag{A.17}$$

We can show that (A.16) and (A.17) are equivalent to (A.7) and (A.9), respectively. Hence Fig. 1 also shows the configuration of Stokes curves defined by (A.16) and (A.17).

We should introduce virtual turning points and new Stokes curves to find a complete Stokes geometry, just in the same manner as in Example A. 1; virtual turning points are given by $x = 0$ and $x = \pm\cos w$, where w is a nonzero solution of $w = \tan w$, and new Stokes curves are defined by

$$\mathrm{Im}\int_{x_0}^{x}(f_n^{(+)}(x) - f_m^{(+)}(x))dx = 0 \quad (n \neq m), \tag{A.18}$$

$$\mathrm{Im}\int_{x_0}^{x}(f_n^{(-)}(x) - f_m^{(-)}(x))dx = 0 \quad (n \neq m), \tag{A.19}$$

$$\mathrm{Im}\int_{x_0}^{x}(f_n^{(+)}(x) - f_m^{(-)}(x))dx = 0 \quad (n \neq m, m-1). \tag{A.20}$$

We can show that (A.18) and (A.19) become the imaginary axis, and (A.20) is equivalent to (A.11). These new Stokes curves (A.18), (A.19) and (A.20) are shown in Fig. 3.

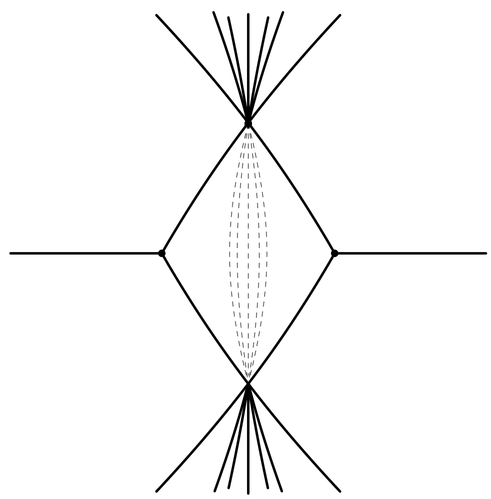

Figure 3

Example A.3

$$P(x, \eta^{-1}\frac{d}{dx}) = \exp(\eta^{-2}d^2/dx^2) - \exp(-x^2). \tag{A.21}$$

For (A.21) we find that a double turning point is located at $x = 0$, and simple turning points are located at $x = a_n$ and $x = -a_n$, where

$$a_n = \begin{cases} e^{i\pi/4}\sqrt{2n\pi} & (n > 0), \\ ie^{i\pi/4}\sqrt{-2n\pi} & (n < 0). \end{cases} \tag{A.22}$$

We define functions $g_n^{(\pm)}(x)$ $(n \in \mathbb{Z})$ by

$$g_0^{(+)}(x) = ix, \qquad g_n^{(+)}(x) = \sqrt{-x^2 + 2in\pi} \ \ (n \neq 0), \tag{A.23}$$

and $g_n^{(-)} = -g_n^{(+)}$. Then Stokes curves emanating from $x = 0$ are given by

$$\mathrm{Im} \int_0^x (g_0^{(+)}(x) - g_0^{(-)}(x))dx = 0. \tag{A.24}$$

We also find that Stokes curves emanating from $x = a_n$ and $x = -a_n$ are given by

$$\mathrm{Im} \int_{a_n}^x (g_n^{(+)}(x) - g_n^{(-)}(x))dx = 0 \quad \text{and} \quad \mathrm{Im} \int_{-a_n}^x (g_n^{(+)}(x) - g_n^{(-)}(x))dx = 0, \tag{A.25}$$

respectively. Stokes curves defined by (A.24) and (A.25) are shown in Fig. 4.

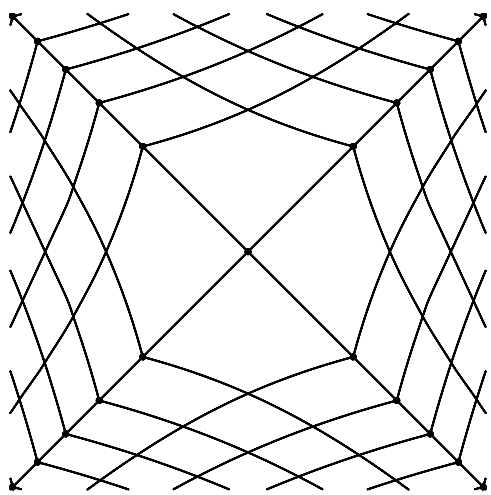

Figure 4

An important observation is that none of the crossing points of Stokes curves in Fig. 4 are ordered (in the sense of [BNR]); thus there is no need to add new Stokes curves to obtain a complete Stokes geometry.

Example A.4

$$P = P_0 + \eta^{-1} P_1, \tag{A.26}$$

where

$$P_0(x,\zeta) = (i\zeta - x)\prod_{n=1}^{\infty}\left(1-\frac{i\zeta}{n^2}\right), \tag{A.27}$$

$$P_1(x,\zeta)(=P_1(\zeta)) = \sum_{n=1}^{\infty}\frac{|c_n|^2}{n^2}\prod_{p\geq 1, p\neq n}\left(1-\frac{i\zeta}{p^2}\right) \quad (c_j \in \mathbb{C}). \tag{A.28}$$

Here we assume that $\sum_{j=1}^{\infty}|c_j|^2$ should converge. One can readily confirm that, for each fixed $\eta(\neq 0)$, both $P_0(x,\eta^{-1}d/dx)$ and $P_1(\eta^{-1}d/dx)$ determine differential operators of infinite order in the sense of [SKK].

This operator is related to a level crossing problem in quantum mechanics in an non-adiabatic approximation; consider the following system of equations, which is a generalization of the n-level model discussed in [BE]:

$$i\eta^{-1}\frac{d\psi}{dx} = \left(H_0 + \eta^{-1/2}H_{1/2}\right)\psi, \tag{A.29}$$

where

$$\psi = {}^t(\psi_0, \psi_1, \ldots), \tag{A.30}$$

$$H_0 = \mathrm{diag}(x, 1, 4, 9, \ldots), \tag{A.31}$$

$$H_{1/2} = \begin{bmatrix} 0 & c_1 & c_2 & \cdots \\ \overline{c_1} & & & \\ \overline{c_2} & & 0 & \\ \vdots & & & \end{bmatrix}. \tag{A.32}$$

Then a straightforward computation shows that ψ_0 satisfies $P\psi_0 = 0$.

We find that turning points are located at $x = \alpha_n$ for $n = 1, 2, \ldots$, where $\alpha_n = n^2$. These turning points $\{\alpha_n\}$ are shown to be double. Stokes curves emanating from $x = \alpha_n$ are given by

$$\mathrm{Im}\int_{\alpha_n}^{x}\frac{(x-\alpha_n)}{i}dx = 0, \tag{A.33}$$

or, equivalently,

$$\mathrm{Re}\int_{\alpha_n}^{x}(x-\alpha_n)dx = 0. \tag{A.34}$$

These Stokes curves are given in Fig. 5.

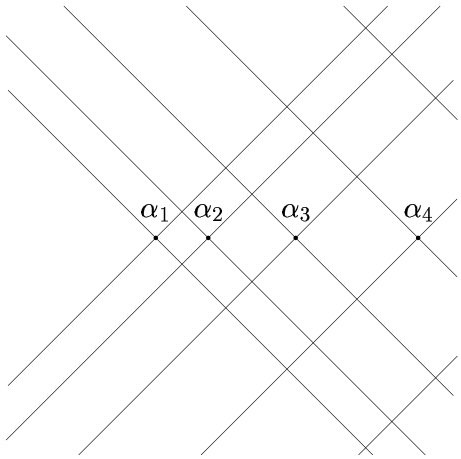

Figure 5

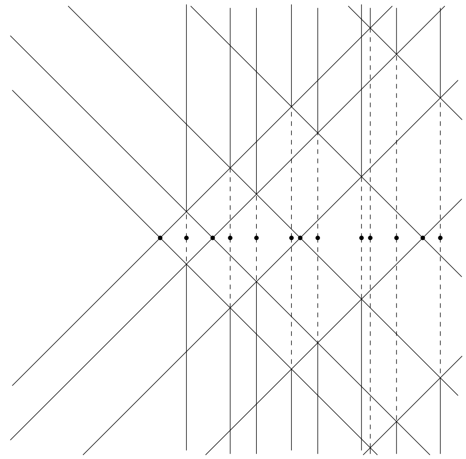

Figure 6

Stokes curves drawn in Fig. 5 have crossing points, and all of them are ordered crossing points. To detect virtual turning points needed to find a complete Stokes geometry, we use the method given in [AKT5], which is based on the notion of bicharacteristic diagrams (cf. [AKT5, Section 3]). We then find that needed virtual turning points $\alpha_{n,m}$ $(n, m = 0, 1, 2, \ldots, n \neq m)$ are given by

$$\int_{\alpha_n}^{\alpha_{n,m}} \alpha_n \, dx = \int_{\alpha_n}^{\alpha_m} x \, dx + \int_{\alpha_m}^{\alpha_{n,m}} \alpha_m \, dx. \tag{A.35}$$

By solving (A.35) we obtain $\alpha_{n,m} = (\alpha_n + \alpha_m)/2$. New Stokes curves which should be added to Fig. 6 are given by

$$\mathrm{Re} \int_{\alpha_{n,m}}^{x} (\alpha_n - \alpha_m) dx = 0 \quad (n, m = 0, 1, 2, \ldots, n \neq m). \tag{A.36}$$

Fig. 6 shows the resulting Stokes geometry of (A.26).

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