RIMS Preprint No. 1358, Kyoto University, June 2002

Proximity Theorems of Discrete Convex Functions †

Kazuo Murota and Akihisa Tamura

Abstract

A proximity theorem is a statement that, given an optimization problem and its relaxation, an optimal solution to the original problem exists in a certain neighborhood of a solution to the relaxation. Proximity theorems have been used successfully, for example, in designing efficient algorithms for discrete resource allocation problems. After reviewing the recent results for L-convex and M-convex functions, this paper establishes proximity theorems for larger classes of discrete convex functions, L_2-convex functions and M_2-convex functions, that are relevant to the polymatroid intersection problem and the submodular flow problem.

Kazuo Murota
Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
University of Tokyo, Tokyo 113-8656, Japan.
phone: +81-3-5841-6920, facsimile: +81-3-5841-6879
e-mail: murota@mist.i.u-tokyo.ac.jp.

Akihisa Tamura
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.
phone: +81-75-753-7236, facsimile: +81-75-753-7272
e-mail: tamura@kurims.kyoto-u.ac.jp.

† This work is supported by a Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

1 Introduction

In the area of discrete optimization, nonlinear optimization problems have been investigated as well as linear optimization problems. Submodular (set) functions and separable convex functions are well-known examples of tractable nonlinear functions, in that the submodular function minimization problem can be solved in polynomial time (see [14, 15, 27]), and separable convex functions have been treated successfully in many different discrete optimization problems (see [12]).

Recently, certain classes of "discrete convex functions" were proposed: integrally-convex functions of Favati and Tardella [4] and {L,M,L_2,M_2}-convex functions of Murota [19, 20]. L-convex functions contain the class of submodular set functions. M-convex functions possess structures of matroids and polymatroids. Separable discrete convex functions can be characterized as functions with both L-convexity and M-convexity (in their variants). L_2-convex functions and M_2-convex functions constitute larger classes of discrete convex functions that are relevant to the polymatroid intersection problem, where an L_2-convex function is, by definition, the infimal convolution of two L-convex functions and an M_2-convex function is the sum of two M-convex functions. The M_2-convex function minimization problem is equivalent to the M-convex submodular flow problem [21] which is an extension of the submodular flow problem [3]. The class of integrally-convex functions contains all of the above classes.

Those classes C of discrete convex functions f possess the following features in common:

Discreteness: f is defined on an integral lattice $\mathbf{Z}^n$, i.e., $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, where $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integers and reals, respectively.

Convex Extendibility: There exists a continuous convex function $\overline{f}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$.

Optimality Criterion: There exists a neighborhood $N_C(x^*) \subset \mathbf{Z}^n$ with center x^* such that

$$f(x^*) \leq f(x) \ (\forall x \in \mathbf{Z}^n) \ \Leftrightarrow \ f(x^*) \leq f(x) \ (\forall x \in N_C(x^*)).$$

Optimality criterion says that global minimality is implied by local minimality defined in terms of the neighborhood $N_C(x^*)$. This is a significant feature inherited from continuous convex functions.

Moreover, L-/M-convex functions have a "proximity property" described as

Proximity Property: Given a positive integer α and a point $x^\alpha \in \mathbf{Z}^n$, there exists a function $d_C(n, \alpha)$ such that

$$f(x^\alpha) \leq f(x) \ (\forall x \in N_C^\alpha(x^\alpha)) \ \Rightarrow \ \exists x^* \in \arg\min f : \ ||x^* - x^\alpha||_\infty \leq d_C(n, \alpha),$$

where $N_C^\alpha(x^\alpha) = \{x^\alpha + \alpha(x - x^\alpha) \mid x \in N_C(x^\alpha)\}$ and $\arg\min f$ denotes the set of all minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{Z}^n \mid f(x) \leq f(y)\ (\forall y \in \mathbf{Z}^n)\}.$$

The proximity property says that a locally minimal solution x^α of a "scaled" function

$$f^\alpha(x) = f(x^\alpha + \alpha x) \qquad (x \in \mathbf{Z}^V)$$

is close to a minimizer x^* of f in terms of $d_C(n, \alpha)$. For L-/M-convex functions, $d_C(n, \alpha) = (n - 1)(\alpha - 1)$ is a valid choice ([16] and [17], respectively). The proximity property can be exploited in developing an efficient scaling algorithm for minimizing f. In fact, the L-convex function minimization problem can be solved in polynomial-time by combining submodular set function minimization algorithms and the proximity property [13] (see also [23]). For the M-convex function minimization, polynomial-time scaling algorithms based on the proximity property and its generalization are known [28, 29]. Proximity theorems for separable discrete convex functions are found in [9, 10, 18] in developing efficient algorithms for resource allocation problems. Different types of theorems on proximity have also been investigated: proximity between integral and real optimal solutions in [1, 2, 8, 10, 11] and proximity for a number of resource allocation problems with min-max type objective functions in [6].

This paper addresses proximity properties of L_2-/M_2-convex functions. Our main results say:

- for an essentially bounded L_2-convex function f and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies
$$f(x^\alpha) \leq f(x^\alpha + \alpha\chi_S)$$
for all $S \subseteq V$, then there exists $x^* \in \arg\min f$ such that
$$\|x^* - x^\alpha\|_\infty \leq 2(n-1)(\alpha-1),$$

- for an M_2-convex function f represented as the sum of two M-convex functions f_1 and f_2, and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies
$$\sum_{i=1}^{k}(f_1(x^\alpha - \alpha\chi_{u_i} + \alpha\chi_{w_i}) - f_1(x^\alpha)) + \sum_{i=1}^{k}(f_2(x^\alpha - \alpha\chi_{u_{i+1}} + \alpha\chi_{w_i}) - f_2(x^\alpha)) \geq 0$$
for any ordered sets $U = \{u_1, \ldots, u_k\}, W = \{w_1, \ldots, w_k\} \subset V$ with $U \cap W = \emptyset$ where $u_{k+1} = u_1$, then there exists $x^* \in \arg\min f$ such that
$$\|x^* - x^\alpha\|_\infty \leq \frac{n^2}{2}(\alpha-1).$$

We also discuss a proximity property of integrally-convex functions and briefly survey proximity properties of discrete convex functions. Section 2 states definitions, optimality criteria and proximity properties for several classes of discrete convex functions, including our new results which are proven in Section 3.

2 Definitions, Optimality Criteria and Proximity Theorems

In this section, we introduce five classes of discrete convex functions, namely, {L, M, L_2, M_2, integrally}-convex functions with respect to definitions, optimality criteria and proximity theorems. While other variants of these classes, e.g., $L^\natural$-/$L_2^\natural$-convex functions due to [7] and $M^\natural$-/$M_2^\natural$-convex functions due to [24], are known, we concentrate on the above five classes because the results can be easily extended to the variants.

Subsections 2.3 and 2.4 present new results, an optimality criterion (Theorem 2.11) and a proximity property (Theorem 2.12) for L_2-convex functions, and proximity properties (Theorems 2.17 and 2.18) for M_2-convex functions. Subsection 2.2 also gives a new proximity property (Theorem 2.7) for M-convex functions in terms of ℓ_1-norm. Subsections 2.1 and 2.2 explain known results, optimality criteria and proximity theorems for L-convexity and M-convexity, respectively. Subsection 2.4 introduce optimality criteria for M_2-convexity, which are direct consequences of results for the M-convex submodular flow problem.

We first introduce notations. Let V be a nonempty finite set and put $n = |V|$. We denote by $\mathbf{Z}^V$ the set of all integral vectors $x = (x(v) : v \in V)$ indexed by V, and by $\mathbf{Z}_{++}$ the set of all positive integers. Given a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, the *effective domain* of f is defined by

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) \neq \pm\infty\}.$$

For each $S \subseteq V$, we denote by χ_S the *characteristic vector* of S defined by

$$\chi_S(v) = \begin{cases} 1 & (v \in S) \\ 0 & (v \notin S) \end{cases} \qquad (v \in V)$$

and write simply χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. We also denote by $\mathbf{0}$ and $\mathbf{1}$ the vectors of all zeros and ones, respectively. For two vectors $x, y \in \mathbf{Z}^V$ with $x \leq y$, $[x, y]_{\mathbf{Z}}$ denotes the set $\{z \in \mathbf{Z}^V \mid x \leq z \leq y\}$.

2.1 L-convex Functions

For any $x, y \in \mathbf{Z}^V$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbf{Z}^V$ are such that

$$(x \wedge y)(v) = \min\{x(v), y(v)\}, \quad (x \vee y)(v) = \max\{x(v), y(v)\} \quad (v \in V).$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies the following two conditions:

(SBF) f is *submodular*, i.e.,

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y) \quad (\forall x, y \in \mathbf{Z}^V),$$

(TRF) $\exists r \in \mathbf{R}$ such that $f(x + \mathbf{1}) = f(x) + r \quad (\forall x \in \mathbf{Z}^V)$.

Global optimality of an L-convex function is characterized by local optimality.

Theorem 2.1 (L-optimality criterion, [23])
For an L-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have

$$f(x^*) \leq f(x) \quad (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad \begin{cases} f(x^*) \leq f(x^* + \chi_S) & (\forall S \subseteq V), \\ f(x^* + \mathbf{1}) = f(x^*). \end{cases}$$

The above local optimality criterion can be checked in polynomial time because the first condition can be verified by using submodular function minimization algorithms and the second condition is easy.

We next introduce a proximity theorem of L-convex functions.

Theorem 2.2 (L-proximity theorem, [16])
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L-convex function with $f(x + \mathbf{1}) = f(x)$ $(\forall x \in \mathbf{Z}^V)$ and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha + \alpha \chi_S) \quad (\forall S \subseteq V),$$

then $\arg\min f \neq \emptyset$ *and there exists* $x^* \in \arg\min f$ *with*

$$x^\alpha \leq x^* \leq x^\alpha + (n-1)(\alpha - 1)\mathbf{1}.$$

Remark 2.3 The bound $(n-1)(\alpha-1)$ in Theorem 2.2 is tight. Let $V = \{1, \ldots, n\}$ and let $V_i = \{1, \ldots, i\}$ for $i = 1, \ldots, n$. Assume $\alpha \in \mathbf{Z}_{++}$. We define a set X by

$$X = \left\{ \sum_{i=1}^{n} \mu_i \chi_{V_i} \;\middle|\; \begin{array}{ll} \mu_i \in [0, \alpha - 1]_{\mathbf{Z}} & (i = 1, \ldots, n-1) \\ \mu_n \in \mathbf{Z} & \end{array} \right\}.$$

Any $x \in X$ can be uniquely represented as

$$x = \sum_{i=1}^{n} \mu_i \chi_{V_i} \quad (\mu_i \in [0, \alpha - 1]_{\mathbf{Z}}\ (i = 1, \ldots, n-1), \quad \mu_n \in \mathbf{Z}).$$

By using the representation, we define a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} -\sum_{i=1}^{n-1} \mu_i & (x \in X) \\ +\infty & (x \notin X) \end{cases} \qquad (x \in \mathbf{Z}^V).$$

It is easy to show that

- f is L-convex (linear on X),
- $f(\mathbf{0}) = f(\alpha\mathbf{1}) = 0$,
- $f(\alpha\chi_S) = +\infty \quad (\emptyset \subset \forall S \subset V)$,
- $\arg\min f = \{((n-1)(\alpha-1), (n-2)(\alpha-1), \ldots, (\alpha-1), 0) + \beta\mathbf{1} \mid \beta \in \mathbf{Z}\}$.

The assumption of Theorem 2.2 holds for $x^\alpha = \mathbf{0}$ and

$$x^* = ((n-1)(\alpha-1), (n-2)(\alpha-1), \ldots, (\alpha-1), 0)$$

is the unique minimizer satisfying the bound in Theorem 2.2.

Remark 2.4 Theorems 2.1 and 2.2 are extended to a more general class of "quasi" L-convex functions [26].

2.2 M-convex Functions

We define the *positive support* and *negative support* of a vector $x = (x(v) : v \in V) \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\} \quad \text{and} \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies

(M-EXC) for $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

We note that (M-EXC) is also represented as: for $x, y \in \mathrm{dom}\, f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\,],$$

where the maximum and the minimum over an empty set are $-\infty$ and $+\infty$, respectively. From (M-EXC), the effective domain $\mathrm{dom}\, f$ lies on a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = \text{constant}\}$, where $x(V) = \sum_{v \in V} x(v)$. It is also known that $\mathrm{dom}\, f$ is the set of integer points of the base polyhedron of an integral submodular system (see [5] for submodular systems).

The minimizers of an M-convex function have a nice characterization which can be checked efficiently.

Theorem 2.5 (M-optimality criterion, [19, 20])
For an M-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have

$$f(x^*) \leq f(x) \ \ (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad f(x^*) \leq f(x^* - \chi_u + \chi_v) \quad (\forall u, v \in V).$$

We next introduce a proximity theorem of M-convex functions.

Theorem 2.6 (M-proximity theorem, [17])
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function and let* $\alpha \in \mathbf{Z}_{++}$. *If* $x^\alpha \in \mathrm{dom}\, f$ *satisfies*

$$f(x^\alpha) \leq f(x^\alpha - \alpha\chi_u + \alpha\chi_v) \quad (\forall u, v \subseteq V),$$

then $\arg\min f \neq \emptyset$ *and there exists* $x^* \in \arg\min f$ *with*

$$|x^\alpha(v) - x^*(v)| \leq (n-1)(\alpha - 1) \quad (\forall v \in V).$$

By slightly modifying the proof of [17], we also obtain the following proximity theorem in terms of ℓ_1-norm.

Theorem 2.7 *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function and let* $\alpha \in \mathbf{Z}_{++}$. *If* $x^\alpha \in \mathrm{dom}\, f$ *satisfies*

$$f(x^\alpha) \leq f(x^\alpha - \alpha\chi_u + \alpha\chi_v) \quad (\forall u, v \subseteq V), \tag{2.1}$$

then $\arg\min f \neq \emptyset$ *and there exists* $x^* \in \arg\min f$ *with*

$$||x^* - x^\alpha||_1 \leq \frac{n^2}{2}(\alpha - 1). \tag{2.2}$$

Remark 2.8 The bound $(n-1)(\alpha-1)$ in Theorem 2.6 is tight. Let $V = \{1, \ldots, n\}$ and let $\alpha \in \mathbf{Z}_{++}$. We define a set X by

$$X = \left\{ \sum_{i=2}^{n} \mu_i(\chi_i - \chi_1) \mid \mu_i \in [0, \alpha - 1]_{\mathbf{Z}} \quad (i = 2, \ldots, n) \right\}$$

and a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} x(1) & (x \in X) \\ +\infty & (x \notin X) \end{cases} \qquad (x \in \mathbf{Z}^V).$$

It is easy to show that

- f is M-convex (linear on X),
- $f(\mathbf{0}) = 0$,
- $f(\alpha\chi_v - \alpha\chi_u) = +\infty \quad (\forall u, v \in V, u \neq v)$,
- $\arg\min f = \{(-(n-1)(\alpha-1), (\alpha-1), \ldots, (\alpha-1))\}$.

The assumption of Theorem 2.6 holds for $x^\alpha = \mathbf{0}$ and the unique minimizer of f shows the tightness of $(n-1)(\alpha-1)$.

Remark 2.9 The bound $\frac{n^2}{2}(\alpha - 1)$ in Theorem 2.7 is also tight. In the same way as Remark 2.8, for a positive integer m and sets

$$V^+ = \{1^+, \ldots, m^+\}, \quad V^- = \{1^-, \ldots, m^-\}, \quad V = V^+ \cup V^-,$$

the tightness of the bound can be shown by a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} x(V^-) & (x \in X) \\ +\infty & (x \notin X) \end{cases} \qquad (x \in \mathbf{Z}^V),$$

where

$$X = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij}(\chi_{i^+} - \chi_{j^-}) \;\middle|\; \lambda_{ij} \in [0, \alpha - 1]_{\mathbf{Z}} \ (i, j = 1, \ldots, m) \right\}.$$

Remark 2.10 Theorems 2.5 and 2.6 are extended to a more general class of "quasi" M-convex functions [26].

2.3 L$_2$-convex Functions

For any functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *infimal convolution* of f_1 and f_2, denoted by $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, is defined by

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x, \ x_1, x_2 \in \mathbf{Z}^V\} \quad (x \in \mathbf{Z}^V).$$

It is easy to show that if $f_1 \Box f_2 > -\infty$ then the effective domain of $f_1 \Box f_2$ coincides with the Minkowski sum of the effective domains of f_1 and f_2, that is,

$$\mathrm{dom}\,(f_1 \Box f_2) = (\mathrm{dom}\, f_1) + (\mathrm{dom}\, f_2) \equiv \{x_1 + x_2 \mid x_1 \in \mathrm{dom}\, f_1, \ x_2 \in \mathrm{dom}\, f_2\}.$$

It is known that the infimal convolution of two M-convex functions is also M-convex, but the infimal convolution of two L-convex functions may not be L-convex [19]. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L*$_2$*-convex* if $\mathrm{dom}\, f \neq \emptyset$ and $f = f_1 \Box f_2$ for some L-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We say that an L-/L$_2$-convex function f is *essentially bounded* if $\mathrm{dom}\, f \cap \{x \in \mathbf{Z}^V \mid x(v) = 0\}$ is bounded for some $v \in V$. If an L$_2$-convex function $f = f_1 \Box f_2$ is essentially bounded, then f_1 and f_2 are also essentially bounded, because $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) + (\mathrm{dom}\, f_2)$ holds for L$_2$-convex function f.

The following optimality criterion and the proximity theorem for L$_2$-convex functions are new results. We emphasize that the optimality criterion is the same as that for L-convex functions stated in Theorem 2.1 and that the proximity theorem is almost the same as that stated in Theorem 2.2.

Theorem 2.11 (L_2-optimality criterion)
For an L_2-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have

$$f(x^*) \leq f(x) \quad (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad \begin{cases} f(x^*) \leq f(x^* + \chi_S) & (\forall S \subseteq V), \\ f(x^* + \mathbf{1}) = f(x^*). \end{cases}$$

Theorem 2.12 (L_2-proximity theorem)
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an essentially bounded L_2-convex function with $f(x + \mathbf{1}) = f(x)$ $(\forall x \in \mathbf{Z}^V)$ and let $\alpha \in \mathbf{Z}_{++}$. If $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha + \alpha\chi_S) \quad (\forall S \subseteq V),$$

then $\arg\min f \neq \emptyset$ and there exists $x^* \in \arg\min f$ with

$$x^\alpha \leq x^* \leq x^\alpha + 2(n-1)(\alpha - 1)\mathbf{1}.$$

Remark 2.13 The bound $2(n-1)(\alpha-1)$ in Theorem 2.12 is almost tight. We can construct an example such that $x^* \geq x^\alpha$ and $||x^* - x^\alpha||_\infty = (2n-3)(\alpha-1)$ as follows. We continue to use the notations in Remark 2.3 and assume that $n \geq 2$. Let $V_i' = V_i \cup \{n\}$ for $i = 1, \ldots, n-1$. We consider two sets defined by

$$X_1 = \left\{ \sum_{i=1}^{n} \mu_i \chi_{V_i} \;\middle|\; \begin{array}{ll} \mu_i \in [0, \alpha-1]_{\mathbf{Z}} & (i = 1, \ldots, n-1) \\ \mu_n \in \mathbf{Z} & \end{array} \right\}$$

and

$$X_2 = \left\{ \sum_{i=1}^{n-1} \lambda_i \chi_{V_i'} \;\middle|\; \begin{array}{ll} \lambda_i \in [0, \alpha-1]_{\mathbf{Z}} & (i = 1, \ldots, n-2) \\ \lambda_{n-1} \in \mathbf{Z} & \end{array} \right\}.$$

Any $x \in X_1$ can be uniquely represented as

$$x = \sum_{i=1}^{n} \mu_i \chi_{V_i} \quad (\mu_i \in [0, \alpha-1]_{\mathbf{Z}}\ (i = 1, \ldots, n-1), \quad \mu_n \in \mathbf{Z}),$$

and any $x \in X_2$ can be uniquely represented as

$$x = \sum_{i=1}^{n-1} \lambda_i \chi_{V_i'} \quad (\lambda_i \in [0, \alpha-1]_{\mathbf{Z}}\ (i = 1, \ldots, n-2), \quad \lambda_{n-1} \in \mathbf{Z}).$$

By using these representations, we define L-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_1(x) = \begin{cases} -\sum_{i=1}^{n-1} \mu_i & (x \in X_1) \\ +\infty & (x \notin X_1) \end{cases} \qquad (x \in \mathbf{Z}^V),$$

$$f_2(x) = \begin{cases} -\sum_{i=1}^{n-2} \lambda_i & (x \in X_2) \\ +\infty & (x \notin X_2) \end{cases} \qquad (x \in \mathbf{Z}^V).$$

In the same way as in Remark 2.3, we can show that the assumption of Theorem 2.12 holds for $x^\alpha = \mathbf{0}$ and that there is no minimizer x^* of $f_1 \Box f_2$ such that $\mathbf{0} \leq x^* < (2n-3)(\alpha-1)\mathbf{1}$.

Remark 2.14 Let us call a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ an *L_k-convex function* if $\mathrm{dom}\, f \neq \emptyset$ and

$$f(x) = \inf\left\{\sum_{i=1}^{k} f_i(x_i) \mid \sum_{i=1}^{k} x_i = x,\ x_i \in \mathbf{Z}^V (i = 1, \ldots, k)\right\} \quad (x \in \mathbf{Z}^V)$$

for some L-convex functions $f_1, \ldots, f_k : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. It is known that an L_k-convex function f with $k \geq 3$ is not necessarily integrally-convex and the convex full of $\mathrm{dom}\, f$ may include an integral point not belonging to $\mathrm{dom}\, f$. The optimality criterion for L-convex functions, however, is also valid for L_k-convex functions with $k \geq 3$. Moreover, the proximity property with $2(n-1)(\alpha-1)$ replaced by $k(n-1)(\alpha-1)$ holds for L_k-convex functions. We can show those by slightly modifying our proofs of Theorems 2.11 and 2.12.

2.4 M_2-convex Functions

It is known that the sum of two M-convex functions is not necessarily M-convex. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *M_2-convex* if $\mathrm{dom}\, f \neq \emptyset$ and $f = f_1 + f_2$ for some M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. It is easy to show that $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$. Obviously, if $\mathrm{dom}\, f_1 = \mathrm{dom}\, f_2$ and f_2 is identically zero, then $f = f_1$ is M-convex, and hence, the class of M_2-convex functions includes that of M-convex functions. The M_2-convex function minimization problem contains the polymatroid intersection problem as a special case. Thus, optimality criteria for M_2-convexity below are extensions of known results for the matroid intersection problem and the polymatroid intersection problem.

For a vector $p \in \mathbf{R}^V$, let us define functions $\langle p, x\rangle$ and $f[p](x)$ by

$$\langle p, x\rangle = \sum_{v \in V} p(v)x(v) \quad \text{and} \quad f[p](x) = f(x) + \langle p, x\rangle \qquad (x \in \mathbf{Z}^V).$$

If f is M-convex, then $f[p]$ is also M-convex.

Several results on optimality of M_2-convexity are known.

Theorem 2.15 (M-convex intersection theorem, [19])

For M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a point $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, we have

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \quad (\forall x \in \mathbf{Z}^V)$$

if and only if there exists $p^* \in \mathbf{R}^V$ such that

$$\begin{aligned} f_1[-p^*](x^*) &\leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \\ f_2[+p^*](x^*) &\leq f_1[+p^*](x) \quad (\forall x \in \mathbf{Z}^V), \end{aligned}$$

and furthermore, we have

$$\arg\min(f_1 + f_2) = \arg\min(f_1[-p^*]) \cap \arg\min(f_2[+p^*])$$

for such p^*.

Optimality criteria of M_2-convex functions can be transformed from those of the M-convex submodular flow problem in [20], because the M_2-convex function minimization and the M-convex submodular flow problem are equivalent to each other. The following theorem is a direct consequence of the results in [20].

Theorem 2.16 (M_2-optimality criteria, see [20])

For M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and a point* $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, *three conditions below are equivalent:*

(a) $x^* \in \arg\min(f_1 + f_2)$.

(b) *For any ordered sets* $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ *with* $U \cap W = \emptyset$,

$$\sum_{i=1}^{k}(f_1(x^* - \chi_{u_i} + \chi_{w_i}) - f_1(x^*)) + \sum_{i=1}^{k}(f_2(x^* - \chi_{u_{i+1}} + \chi_{w_i}) - f_2(x^*)) \geq 0,$$

where $u_{k+1} = u_1$.

(c) $(f_1 + f_2)(x^*) \leq (f_1 + f_2)(x^* - \chi_U + \chi_W) \quad (\forall U, W \subset V,\ |U| = |W|)$.

The optimality for M_2-convexity can be checked in polynomial time by transforming (b) of Theorem 2.16 to a network problem (see Remark 2.21), although checking condition (c) of Theorem 2.16 seems to be a hard problem. In view of polynomial time verifiability, we relax (b) of Theorem 2.16 to formulate a proximity theorem of M_2-convex functions. This is the main result of this paper.

Theorem 2.17 (M_2-proximity theorem)

Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be M-convex functions and let* $\alpha \in \mathbf{Z}_{++}$. *If* $x^\alpha \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ *satisfies*

$$\sum_{i=1}^{k}(f_1(x^\alpha - \alpha\chi_{u_i} + \alpha\chi_{w_i}) - f_1(x^\alpha)) + \sum_{i=1}^{k}(f_2(x^\alpha - \alpha\chi_{u_{i+1}} + \alpha\chi_{w_i}) - f_2(x^\alpha)) \geq 0$$

for any ordered sets $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ *with* $U \cap W = \emptyset$ *where* $u_{k+1} = u_1$, *then* $\arg\min(f_1 + f_2) \neq \emptyset$ *and there exists* $x^* \in \arg\min(f_1 + f_2)$ *with*

$$||x^* - x^\alpha||_\infty \leq \frac{n^2}{2}(\alpha - 1). \tag{2.3}$$

The proof of Theorem 2.17 relies heavily on the following result.

Theorem 2.18 *Let* $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be M-convex functions with* $\arg\min(f_1 + f_2) \neq \emptyset$. *For a given point* $x \in \mathbf{Z}^V$ *with* $x(V) = y(V)$ *for any* $y \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, *and for* $d \in \mathbf{Z}$, *if there exist* $x^1 \in \arg\min f_1$ *and* $x^2 \in \arg\min f_2$ *such that*

$$||x^1 - x||_1 \leq d, \qquad ||x^2 - x||_1 \leq d, \tag{2.4}$$

then there exists $x^* \in \arg\min(f_1 + f_2)$ *with*

$$||x^* - x||_\infty \leq d.$$

Remark 2.19 The bound $\frac{n^2}{2}(\alpha-1)$ of Theorem 2.17 is tight in the sense that the statement with $\frac{n^2}{2}(\alpha-1)$ replaced by $\frac{(n-2)^2}{4}(\alpha-1)-1$ is false. For any positive integer m, we consider three sets:

$$V^+ = \{1^+, \ldots, m^+\}, \quad V^- = \{1^-, \ldots, m^-\}, \quad V = \{0^+, 0^-\} \cup V^+ \cup V^-.$$

We define two functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_1(x) = \begin{cases} x(V^-) & (x \in X_1) \\ +\infty & (x \notin X_1), \end{cases} \qquad f_2(x) = \begin{cases} x(V^-) & (x \in X_2) \\ +\infty & (x \notin X_2) \end{cases} \qquad (x \in \mathbf{Z}^V),$$

where

$$X_1 = \left\{ \sum_{i=1}^m \sum_{j=1}^m \lambda_{ij}(\chi_{i^+} - \chi_{j^-}) + \lambda_0(\chi_{0^+} - \chi_{0^-}) \;\middle|\; \begin{array}{l} \lambda_{ij} \in [0, \alpha-1]_{\mathbf{Z}} \ (i,j = 1,\ldots,m) \\ \lambda_0 \in [0, m^2(\alpha-1)]_{\mathbf{Z}} \end{array} \right\},$$

$$X_2 = \left\{ \sum_{i=1}^m \mu_i(\chi_{i^+} - \chi_{0^-}) + \sum_{j=1}^m \nu_j(\chi_{0^+} - \chi_{j^-}) \;\middle|\; \begin{array}{l} \mu_i \in [0, m(\alpha-1)]_{\mathbf{Z}} \ (i = 1,\ldots,m) \\ \nu_j \in [0, m(\alpha-1)]_{\mathbf{Z}} \ (j = 1,\ldots,m) \end{array} \right\}.$$

By using (M-EXC), we can easily show that f_1 and f_2 are M-convex; f_1 and f_2 are linear on X_1 and X_2, respectively. Let $x^\alpha = \mathbf{0}$. Obviously, $\mathbf{0} \in X_1 \cap X_2$ holds. By the definition of X_1, if $\alpha\chi_v - \alpha\chi_u \in X_1$ for $u, v \in V$ with $u \neq v$, then we have $u = 0^-$ and $v = 0^+$. On the other hand, $\alpha\chi_{0^+} - \alpha\chi_{0^-}$ is not contained in X_2 by its definition. Thus, the hypothesis of Theorem 2.17 holds for $x^\alpha = \mathbf{0}$. By the definitions of f_1 and f_2, M_2-convex function $f_1 + f_2$ has the unique minimizer x^* defined by

$$x^*(u) = \begin{cases} m(\alpha-1) & (u \in V^+) \\ -m(\alpha-1) & (u \in V^-) \\ m^2(\alpha-1) & (u = 0^+) \\ -m^2(\alpha-1) & (u = 0^-). \end{cases}$$

We have $||x^* - x^\alpha||_\infty = m^2(\alpha-1) = \frac{(n-2)^2}{4}(\alpha-1)$, where $n = |V| = 2(m+1)$.

Remark 2.20 The bound of Theorem 2.18 is tight. Let $V = \{1,2,3\}$ and let $\beta \in \mathbf{Z}_{++}$. We define two functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_1(y) = \begin{cases} y(3) & (y \in X_1) \\ +\infty & (y \notin X_1), \end{cases} \qquad f_2(y) = \begin{cases} y(3) & (y \in X_2) \\ +\infty & (y \notin X_2) \end{cases} \qquad (y \in \mathbf{Z}^V),$$

where

$$X_1 = \{\lambda(\chi_1 - \chi_3) \mid \lambda \in [0, 2\beta]_{\mathbf{Z}}\}, \quad X_2 = \{\mu(\chi_2 - \chi_3) \mid \mu \in [0, 2\beta]_{\mathbf{Z}}\}.$$

Obviously, f_1 and f_2 are M-convex, and furthermore,

$$\arg\min f_1 = \{x^1 \equiv 2\beta(\chi_1 - \chi_3)\}, \quad \arg\min f_2 = \{x^2 \equiv 2\beta(\chi_2 - \chi_3)\}.$$

Since $X_1 \cap X_2 = \{\mathbf{0}\}$, we have

$$\arg\min(f_1 + f_2) = \{x^* \equiv \mathbf{0}\}.$$

By putting $x = \beta\chi_1 + \beta\chi_2 - 2\beta\chi_3$ and $d = 2\beta$, we obtain

$$||x^1 - x||_1 = ||x^2 - x||_1 = ||x^* - x||_\infty = d.$$

Therefore, the bound of Theorem 2.18 is tight.

Remark 2.21 Condition (b) of Theorem 2.16 can be checked in polynomial time. Given two M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, a point $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and a positive integer $\alpha \in \mathbf{Z}_{++}$, we construct a directed graph $G_x^\alpha = (V_1 \cup V_2, A)$ and an arc length $\ell_x^\alpha \in \mathbf{R}^A$ as follows. Let V_1 and V_2 be copies of V, i.e.,

$$V_1 = \{v_1 \mid v \in V\}, \quad V_2 = \{v_2 \mid v \in V\},$$

where v_1 and v_2 are the copies of $v \in V$. Arc set A consists of three disjoint parts:

$$\begin{aligned} A_b &= \{(v_1, v_2) \mid v \in V\} \cup \{(v_2, v_1) \mid v \in V\}, \\ A_1 &= \{(u_1, v_1) \mid u, v \in V,\ u \neq v,\ x - \alpha\chi_u + \alpha\chi_v \in \mathrm{dom}\, f_1\}, \\ A_2 &= \{(v_2, u_2) \mid u, v \in V,\ u \neq v,\ x - \alpha\chi_u + \alpha\chi_v \in \mathrm{dom}\, f_2\}. \end{aligned} \tag{2.5}$$

We define $\ell_x^\alpha \in \mathbf{R}^A$ by

$$\ell_x^\alpha(a) = \begin{cases} 0 & (a \in A_b) \\ f_1(x - \alpha\chi_u + \alpha\chi_v) - f_1(x) & (a = (u_1, v_1) \in A_1) \\ f_2(x - \alpha\chi_u + \alpha\chi_v) - f_2(x) & (a = (v_2, u_2) \in A_2). \end{cases} \tag{2.6}$$

Lemma 2.22 below guarantees that (b) of Theorem 2.16 can be checked in polynomial time by applying shortest path algorithms.

Lemma 2.22 *For two M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, a point $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and $\alpha \in \mathbf{Z}_{++}$, two conditions below are equivalent:*

(a) *There exists no negative cycle in G_x^α with length ℓ_x^α.*

(b) *For any ordered sets $U=\{u_1, \ldots, u_k\}, W=\{w_1, \ldots, w_k\} \subset V$ with $U \cap W = \emptyset$,*

$$\sum_{i=1}^{k}(f_1(x-\alpha\chi_{u_i}+\alpha\chi_{w_i})-f_1(x))+\sum_{i=1}^{k}(f_2(x-\alpha\chi_{u_{i+1}}+\alpha\chi_{w_i})-f_2(x)) \geq 0, \quad (2.7)$$

where $u_{k+1} = u_1$.

2.5 Integrally-Convex Functions

For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, its *convex closure* $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$\overline{f}(x) = \sup\left\{ \langle p, x\rangle + \gamma \;\middle|\; \begin{array}{l} p \in \mathbf{R}^V,\ \gamma \in \mathbf{R} \\ \langle p, y\rangle + \gamma \leq f(y)\ (\forall y \in \mathbf{Z}^V) \end{array} \right\} \quad (x \in \mathbf{R}^V).$$

For $x \in \mathbf{R}^V$, we define a neighborhood $N(x)$ of x by

$$N(x) = \{y \in \mathbf{Z}^V \mid \lfloor x \rfloor \leq y \leq \lceil x \rceil\},$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the vectors obtained from x by rounding down and up the components of x, respectively. The *local convex extension* $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = \sup\left\{ \langle p, x\rangle + \gamma \;\middle|\; \begin{array}{l} p \in \mathbf{R}^V,\ \gamma \in \mathbf{R} \\ \langle p, y\rangle + \gamma \leq f(y)\ (\forall y \in N(x)) \end{array} \right\} \quad (x \in \mathbf{R}^V).$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *integrally-convex* if $\overline{f} = \tilde{f}$. It is known that the class of integrally-convex functions contains all classes of $\{$L,M,L_2,$\mathrm{M}_2\}$-convex functions [25].

Theorem 2.23 (Optimality criterion for integral-convexity, [4]) *For an integrally-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x^* \in \mathrm{dom}\, f$, we have*

$$f(x^*) \leq f(x) \ \ (\forall x \in \mathbf{Z}^V) \quad \Longleftrightarrow \quad f(x^*) \leq f(x^* - \chi_A + \chi_B) \quad (\forall A, B \subseteq V).$$

Checking the above local optimality criterion is very hard because any function on $\{0, 1\}^V$ is integrally-convex.

From the optimality criterion, we may suppose that a proximity property for integral-convexity employs the following form:

> For an integrally-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, $\alpha \in \mathbf{Z}_{++}$ and x^α with $f(x^\alpha) \leq f(x^\alpha - \alpha\chi_A + \alpha\chi_B)$ for all $A, B \subseteq V$, there exists $x^* \in \arg\min f$ such that $||x^* - x^\alpha||_\infty \leq d_C(n, \alpha)$ for some $d_C : \mathbf{Z}_{++}^2 \to \mathbf{Z}$.

We can verify that $d_C(n, \alpha)$ is bounded by $n!(\alpha-1)$, although its proof is not so simple and the bound is not likely to be tight. A tighter bound for the above proximity property for integral-convexity remains an open question.

3 Proofs

In this section, we will give proofs of our new results.

3.1 Proof of Theorem 2.7

It is sufficient to show that for any $\gamma \in \mathbf{R}$ with $\gamma > \inf f$, there exists $x^* \in \mathrm{dom}\, f$ satisfying $f(x^*) \leq \gamma$ and (2.2). Assume that $x^* \in \mathrm{dom}\, f$ minimizes $||x^* - x^\alpha||_1$ among all vectors satisfying $f(x^*) \leq \gamma$. We fix $v \in \mathrm{supp}^+(x^\alpha - x^*)$ and put $k = x^\alpha(v) - x^*(v)$. The following claims are shown in [17].

Claim 1: There exist $w_1, \ldots, w_k \in \mathrm{supp}^-(x^\alpha - x^*)$ and $y_0(= x^\alpha), y_1, \ldots, y_k \in \mathrm{dom}\, f$ such that $y_i = y_{i-1} - \chi_v + \chi_{w_i}$ and $f(y_i) < f(y_{i-1})$ $(i = 1, \ldots, k)$.

Claim 2: For any $w \in \mathrm{supp}^-(x^\alpha - x^*)$ with $y_k(w) > x^\alpha(w)$ and $\mu \in [0, y_k(w) - x^\alpha(w) - 1]_{\mathbf{Z}}$, we have $f(x^\alpha - (\mu+1)(\chi_v - \chi_w)) < f(x^\alpha - \mu(\chi_v - \chi_w))$.

Claim 2 and (2.1) imply

$$f(x^\alpha - \mu_w(\chi_v - \chi_w)) < \cdots < f(x^\alpha - (\chi_v - \chi_w)) < f(x^\alpha) \leq f(x^\alpha - \alpha(\chi_v - \chi_w))$$

for any w with $\mu_w = y_k(w) - x^\alpha(w) > 0$. Therefore, $y_k(w) - x^\alpha(w) \leq \alpha - 1$ holds for all $w \in \mathrm{supp}^-(x^\alpha - x^*)$. Then we obtain

$$\begin{aligned} x^\alpha(v) - x^*(v) &= x^\alpha(v) - y_k(v) = \sum_{w \in \mathrm{supp}^-(x^\alpha - x^*)} (y_k(w) - x^\alpha(w)) \\ &\leq |\mathrm{supp}^-(x^\alpha - x^*)| \cdot (\alpha - 1), \end{aligned}$$

where the second equality is by $x^\alpha(V) = y_k(V)$. Similarly, we can show

$$|x^\alpha(v) - x^*(v)| \leq |\mathrm{supp}^+(x^\alpha - x^*)| \cdot (\alpha - 1)$$

for $v \in \mathrm{supp}^-(x^\alpha - x^*)$. Hence, we have

$$||x^* - x^\alpha||_1 \leq 2|\mathrm{supp}^+(x^\alpha - x^*)| \cdot |\mathrm{supp}^-(x^\alpha - x^*)| \cdot (\alpha - 1) \leq \frac{n^2}{2}(\alpha - 1).$$

3.2 Proofs of Theorems 2.11 and 2.12

This subsection gives proofs of Theorems 2.11 and 2.12.

Proposition 3.1 *Assume that $f = f_1 \Box f_2$ for some functions f_1 and f_2 such that $f_1(y + \mathbf{1}) = f_1(y) + r_1$ and $f_2(y + \mathbf{1}) = f_2(y) + r_2$ for all $y \in \mathbf{Z}^V$. If $f(x + \mathbf{1}) = f(x)$ for some $x \in \mathrm{dom}\, f$, then $r_1 = r_2 = 0$ holds, and furthermore, $f(y + \mathbf{1}) = f(y)$ for any $y \in \mathrm{dom}\, f$.*

Proof. By the definition of the infimal convolution, we have

$$\begin{aligned} f(x+\mathbf{1}) &\le \inf\{f_1(x_1+\alpha_1\mathbf{1})+f_2(x_2+\alpha_2\mathbf{1}) \mid x_1+x_2=x,\ \alpha_1+\alpha_2=1\} \\ &= \inf\{f_1(x_1)+r_1\alpha_1+f_2(x_2)+r_2\alpha_2 \mid x_1+x_2=x,\ \alpha_1+\alpha_2=1\} \\ &= f(x)+\inf\{r_1\alpha_1+r_2\alpha_2 \mid \alpha_1+\alpha_2=1\} \\ &= f(x)+r_2+\inf\{(r_1-r_2)\alpha_1 \mid \alpha_1\in\mathbf{R}\}. \end{aligned}$$

This says that if $f(x+\mathbf{1})=f(x)\in\mathbf{R}$ then $r_1=r_2\ge 0$ must hold. Analogously, we have

$$\begin{aligned} f(x) &\le \inf\{f_1(x_1+\alpha_1\mathbf{1})+f_2(x_2+\alpha_2\mathbf{1}) \mid x_1+x_2=x+\mathbf{1},\ \alpha_1+\alpha_2=-1\} \\ &= f(x+\mathbf{1})-r_2. \end{aligned}$$

Since $f(x+\mathbf{1})=f(x)\in\mathbf{R}$ is satisfied, we obtain $r_2\le 0$. Therefore, $r_1=r_2=0$ holds. In the same as above, we can show that $f(y+\mathbf{1})\le f(y)$ and $f(y)\le f(y+\mathbf{1})$ for any $y\in\operatorname{dom} f$. ∎

If there exist $x_1\in\operatorname{dom} f_1$ and $x_2\in\operatorname{dom} f_2$ such that

$$f(x^*)=f_1(x_1)+f_2(x_2),\quad x_1+x_2=x,$$

then Theorem 2.11 can be easily proven (see the proof of Theorem 2.12 below), whereas the following proof works in the general case.

Proof of Theorem 2.11 Let $f=f_1\square f_2$ for some L-convex functions f_1 and f_2 with $f_1(x+\mathbf{1})=f_1(x)+r_1$ and $f_2(x+\mathbf{1})=f_2(x)+r_2$ for $x\in\mathbf{Z}^V$.

($\Rightarrow$) For any $S\subseteq V$, $f(x^*)\le f(x^*+\chi_S)$ trivially holds. In the same way as the proof of Proposition 3.1, we can show that $r_1=r_2$, $f(x^*+\mathbf{1})\le f(x^*)+r_2$ and $f(x^*-\mathbf{1})\le f(x^*)-r_2$. Since $f(x^*)\le f(x^*+\mathbf{1}), f(x^*-\mathbf{1})$ holds, we have $r_2=0$ and $f(x^*+\mathbf{1})=f(x^*)$.

($\Leftarrow$) Since $f(x^*+\mathbf{1})=f(x^*)$ holds, Proposition 3.1 yields that

$$f(y+\mathbf{1})=f(y)\quad(\forall y\in\operatorname{dom} f). \tag{3.1}$$

Suppose to the contrary that there exists $y^*\in\operatorname{dom} f$ with $f(y^*)<f(x^*)$. By (3.1), we can assume that $y^*\ge x^*$, and assume, in addition, that y^* minimizes $||y^*-x^*||_1$ among all points $y\in\operatorname{dom} f$ with $y\ge x^*$ and $f(y)<f(x^*)$.

Let γ be an arbitrary positive number. By the definition of the infimal convolution, there exist $x_1,y_1\in\operatorname{dom} f_1$ and $x_2,y_2\in\operatorname{dom} f_2$ such that

$$f(x^*)+\gamma\ge f_1(x_1)+f_2(x_2),\quad x_1+x_2=x^*, \tag{3.2}$$

$$f(y^*)+\gamma\ge f_1(y_1)+f_2(y_2),\quad y_1+y_2=y^*, \tag{3.3}$$

$$x_1\le y_1,\quad x_2\ge y_2, \tag{3.4}$$

where (3.4) follows from (3.1).

Let $\beta = \max\{y_1(v) - x_1(v) \mid v \in \mathrm{supp}^+(y^* - x^*)\} - 1$. It follows from the assumptions that $||y^* - x^*||_\infty \geq 2$. By (3.4), there exists $u \in \mathrm{supp}^+(y^* - x^*)$ with $y_1(u) - x_1(u) \geq 2$, and hence, β must be positive. We now consider points defined by

$$x_1' = (x_1 + \beta\mathbf{1}) \vee y_1, \quad x_2' = (x_2 - \beta\mathbf{1}) \wedge y_2, \quad x' = x_1' + x_2',$$
$$y_1' = (x_1 + \beta\mathbf{1}) \wedge y_1, \quad y_2' = (x_2 - \beta\mathbf{1}) \vee y_2, \quad y' = y_1' + y_2'.$$

We will show that

$$x^* \leq x' \leq x^* + \mathbf{1}, \quad x' \neq x^*, \quad y' \geq x^*, \quad x' + y' = x^* + y^*. \tag{3.5}$$

Obviously, $x' + y' = x^* + y^*$ holds. By the definitions of x_1' and x_2', we have

$$\begin{array}{lcl} x_1(v) + \beta \geq y_1(v) & \Rightarrow & x_1'(v) = x_1(v) + \beta, \\ x_1(v) + \beta < y_1(v) & \Rightarrow & x_1'(v) = y_1(v), \\ x_2(v) - \beta \leq y_2(v) & \Rightarrow & x_2'(v) = x_2(v) - \beta, \\ x_2(v) - \beta > y_2(v) & \Rightarrow & x_2'(v) = y_2(v) \end{array} \tag{3.6}$$

for each $v \in V$. Let v be any element of V. If $x^*(v) = y^*(v)$ holds, we have

$$\begin{array}{lcl} x_1(v) + \beta \geq y_1(v) & \Rightarrow & x_2(v) - \beta \leq y_2(v), \\ x_1(v) + \beta < y_1(v) & \Rightarrow & x_2(v) - \beta > y_2(v) \end{array}$$

by (3.3), and therefore, $x'(v) = x^*(v)$ is satisfied by (3.2) and (3.6). Suppose that $x^*(v) < y^*(v)$. If $x_1(v) + \beta \geq y_1(v)$, then $x_2(v) - \beta \leq y_2(v)$ must hold, and hence, $x'(v) = x^*(v)$ is obtained. Assume that $x_1(v) + \beta < y_1(v)$. In this case, the definition of β states that $x_1(v) + \beta = y_1(v) - 1$. Moreover, we have $x_2(v) - \beta \leq y_2(v)$; since otherwise, we would obtain $x_1(v) + x_2(v) > y_1(v) + y_2(v) - 1$ which contradicts the assumption $x^*(v) < y^*(v)$. Thus, $x'(v) = y_1(v) + x_2(v) - \beta = x_1(v) + x_2(v) + 1 = x^*(v) + 1$ holds. From the above discussion, we obtain $x^* \leq x' \leq x^* + \mathbf{1}$ and $y' \geq x^*$. The definition of β guarantees that there exists $u \in \mathrm{supp}^+(y^* - x^*)$ with $x_1(u) + \beta < y_1(u)$. Hence, $x' \neq x^*$.

By (3.1), $x_1 + \beta\mathbf{1}$ and $x_2 - \beta\mathbf{1}$ also satisfy (3.2). Because f_1 and f_2 are L-convex, we have

$$f_1(x_1) + f_1(y_1) + f_2(x_2) + f_2(y_2) \geq f_1(x_1') + f_1(y_1') + f_2(x_2') + f_2(y_2'). \tag{3.7}$$

From (3.2), (3.3), (3.5) and (3.7), for any $\gamma > 0$, there exists a nonempty subset $S_\gamma \subseteq V$ such that

$$f(x^*) + f(y^*) + 2\gamma \geq f(x^* + \chi_{S_\gamma}) + f(y^* - \chi_{S_\gamma}), \quad y^* - \chi_{S_\gamma} \geq x^*. \tag{3.8}$$

Since γ is an arbitrary positive number, (3.8) implies that there exists a nonempty subset $S \subseteq V$ such that

$$f(x^*) + f(y^*) \geq f(x^* + \chi_S) + f(y^* - \chi_S), \quad y^* - \chi_S \geq x^*. \tag{3.9}$$

The hypothesis and (3.9) yield $f(y^*) \geq f(y^* - \chi_S)$ which contradicts the definition of y^*. Therefore, x^* must be a minimizer of f. ∎

Proof of Theorem 2.12 Let f be defined by two L-convex functions f_1 and f_2. By Proposition 3.1, we have $f_1(x + \mathbf{1}) = f_1(x)$ and $f_2(x + \mathbf{1}) = f_2(x)$ for all $x \in \mathbf{Z}^V$. Since f is essentially bounded, there exist $x_1^\alpha, x_2^\alpha \in \mathbf{Z}^V$ such that $f(x^\alpha) = f_1(x_1^\alpha) + f_2(x_2^\alpha)$ and $x^\alpha = x_1^\alpha + x_2^\alpha$. By the definition of the infimal convolution, we have

$$f(x^\alpha + \alpha\chi_S) \leq \min\left\{f_1(x_1^\alpha + \alpha\chi_S) + f_2(x_2^\alpha), \quad f_1(x_1^\alpha) + f_2(x_2^\alpha + \alpha\chi_S)\right\}.$$

This inequality and the assumption that $f(x^\alpha) = f_1(x_1^\alpha) + f_2(x_2^\alpha) \leq f(x^\alpha + \alpha\chi_S)$ yield

$$f_1(x_1^\alpha) \leq f_1(x_1^\alpha + \alpha\chi_S), \quad f_2(x_2^\alpha) \leq f_2(x_2^\alpha + \alpha\chi_S)$$

for any $S \subseteq V$. By Theorem 2.2, there exist $x_1^* \in \arg\min f_1$ and $x_2^* \in \arg\min f_2$ such that

$$x_1^\alpha \leq x_1^* \leq x_1^\alpha + (n-1)(\alpha-1)\mathbf{1}, \quad x_2^\alpha \leq x_2^* \leq x_2^\alpha + (n-1)(\alpha-1)\mathbf{1}.$$

The above inequalities guarantee that $x^* = x_1^* + x_2^*$ satisfies $x^\alpha \leq x^* \leq x^\alpha + 2(n-1)(\alpha-1)\mathbf{1}$. Moreover, x^* must be a minimizer of f because $x_1^* \in \arg\min f_1$ and $x_2^* \in \arg\min f_2$. ∎

3.3 Proofs of Theorem 2.17

In this subsection, we prove Theorem 2.17 by using Theorem 2.18 and Lemma 2.22 which will be proven in the following subsections.

Since $x^\alpha \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ satisfies (b) of Lemma 2.22, graph $G_{x^\alpha}^\alpha$ has no negative cycle with length $\ell_{x^\alpha}^\alpha$. Thus, there exists a potential $\widehat{p} \in \mathbf{R}^{V_1 \cup V_2}$ satisfying

$$\widehat{p}(u) + \ell_{x^\alpha}^\alpha(u, v) \geq \widehat{p}(v) \quad (\forall (u, v) \in A). \tag{3.10}$$

Definitions (2.5) and (2.6) say that (3.10) is equivalent to

$$\begin{aligned}
\widehat{p}(v_1) &= \widehat{p}(v_2) && (\forall v \in V),\\
f_1[-p](x^\alpha) &\leq f_1[-p](x^\alpha - \alpha\chi_u + \alpha\chi_v) && (\forall u, v \in V),\\
f_2[+p](x^\alpha) &\leq f_2[+p](x^\alpha - \alpha\chi_u + \alpha\chi_v) && (\forall u, v \in V),
\end{aligned}$$

where p is a vector in $\mathbf{R}^V$ defined by $p(v) = \frac{1}{\alpha}\hat{p}(v_1)$ for $v \in V$.

Since $f_1[-p]$ and $f_2[+p]$ are also M-convex, Theorem 2.7 guarantees that

$$\exists x^1 \in \arg\min f_1[-p] \quad : \quad ||x^1 - x^\alpha||_1 \leq \frac{n^2}{2}(\alpha - 1),$$
$$\exists x^2 \in \arg\min f_2[+p] \quad : \quad ||x^2 - x^\alpha||_1 \leq \frac{n^2}{2}(\alpha - 1).$$

For a sufficiently large $\gamma > 0$, let $\hat{f}$ be the function obtained from $f_1 + f_2 = f_1[-p] + f_2[+p]$ by restricting the effective domain to $[x^\alpha - \gamma\mathbf{1}, x^\alpha + \gamma\mathbf{1}]_{\mathbf{Z}}$. It is easy to show that $\hat{f}$ is also M_2-convex. Since $\mathrm{dom}\,\hat{f}$ is bounded, it has a minimizer. Therefore, Theorem 2.18 implies that $\hat{f}$ has a minimizer x^* with $||x^* - x^\alpha||_\infty \leq \frac{n^2}{2}(\alpha - 1)$, which is also a minimizer of $f_1 + f_2$.

3.4 Proof of Theorem 2.18

We first introduce useful properties of M-convex functions. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function. For a pair (x, y) of integer points satisfying $x \in \mathrm{dom}\, f$ and $||x - y||_\infty = 1$, we consider a bipartite graph $G(x, y) = (V^+, V^-; A)$ with vertex sets $V^+ = \mathrm{supp}^+(x - y)$, $V^- = \mathrm{supp}^-(x - y)$ and edge set

$$A = \{(u, v) \mid u \in V^+,\ v \in V^-,\ x - \chi_u + \chi_v \in \mathrm{dom}\, f\},$$

and associate $c(u, v) = f(x - \chi_u + \chi_v) - f(x)$ with arc $(u, v) \in A$ as its weight. Let $\check{f}(x, y)$ denote the minimum weight of a perfect matching in $G(x, y)$, where $\check{f}(x, y) = +\infty$ if no perfect matching exists. The following lemma is a reformulation of "unique-max lemma" for valuated matroids (see [22]).

Lemma 3.2 (Unique-min lemma, see [22])
Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, assume $x \in \mathrm{dom}\, f$, $y \in \mathbf{Z}^V$ and $||x - y||_\infty = 1$. If graph $G(x, y)$ has exactly one minimum weight perfect matching with respect to c, then $y \in \mathrm{dom}\, f$ and $f(y) = f(x) + \check{f}(x, y)$.

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and two points $a, b \in \mathbf{Z}^V$ with $a \leq b$, we define a function $f^{[a,b]}$ by

$$f^{[a,b]}(x) = \begin{cases} f(x) & (x \in [a, b]_{\mathbf{Z}}) \\ +\infty & (x \notin [a, b]_{\mathbf{Z}}) \end{cases} \qquad (x \in \mathbf{Z}^V).$$

It is easy to show that if f is M-convex then $f^{[a,b]}$ is also M-convex.

Proposition 3.3 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, assume $x \in \arg\min f$ and $y \in \mathrm{dom}\, f$.*

(a) *For any* $u \in \mathrm{supp}^+(x-y)$*, there exists* $v \in \mathrm{supp}^-(x-y)$ *such that*

$$\hat{x} = x - \chi_u + \chi_v \in \arg\min f^{[\hat{x}\wedge y, \hat{x}\vee y]}.$$

(b) *For any* $u \in \mathrm{supp}^-(x-y)$*, there exists* $v \in \mathrm{supp}^+(x-y)$ *such that*

$$\hat{x} = x + \chi_u - \chi_v \in \arg\min f^{[\hat{x}\wedge y, \hat{x}\vee y]}.$$

Proof. We will prove (a) only; we can prove (b) similarly. Let $v \in V$ be such that

$$\hat{x} = x - \chi_u + \chi_v \in \arg\min\{f(x - \chi_u + \chi_j) \mid j \in \mathrm{supp}^-(x-y)\}.$$

We note that $\hat{x}$ is well-defined since $\mathrm{supp}^-(x-y) \neq \emptyset$ by (M-EXC). Let z be any point in $[\hat{x}\wedge y, \hat{x}\vee y]_{\mathbf{Z}} \cap \mathrm{dom}\, f$. We have $u \in \mathrm{supp}^+(x-z)$ because $y(u) < x(u)$, $\hat{x}(u) = x(u) - 1$ and $z(u) \leq \max\{\hat{x}(u), y(u)\}$. By (M-EXC) for x, z and $u \in \mathrm{supp}^+(x-z)$, there exists $j \in \mathrm{supp}^-(x-z) \subseteq \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(z) \geq f(x - \chi_u + \chi_j) + f(z + \chi_u - \chi_j).$$

We have $f(x) \leq f(z + \chi_u - \chi_j)$ because $x \in \arg\min f$. The above two inequalities yield

$$f(z) \geq f(x - \chi_u + \chi_j) \geq f(\hat{x}).$$

Therefore we obtain $\hat{x} \in \arg\min f^{[\hat{x}\wedge y, \hat{x}\vee y]}$. ∎

We start a discussion about Theorem 2.18. For the specified point $x \in \mathbf{Z}^V$, we fix $x^* \in \arg\min(f_1 + f_2)$ such that $||x^* - x||_1$ is minimized among all minimizers of $(f_1 + f_2)$, in the rest of the subsection. We will show that x^* satisfies the assertion of Theorem 2.18, i.e., $||x^* - x||_\infty \leq d$. Given two minimizers $x^1 \in \arg\min f_1$ and $x^2 \in \arg\min f_2$, we consider the following partition of V (since x^1 and x^2 will be modified in the sequel, we attach arguments x^1, x^2 to each part):

$$\begin{aligned}
V_0(x^1,x^2) &= \left\{ v \in V \mid x^*(v) = x^1(v) = x^2(v) \right\}, \\
V_\epsilon(x^1,x^2) &= \left\{ v \in V \,\middle|\, \begin{array}{l} \max\{x^1(v), x^2(v)\} < x^*(v) \leq x(v) \text{ or} \\ x(v) \leq x^*(v) < \min\{x^1(v), x^2(v)\} \end{array} \right\}, \\
V_b(x^1,x^2) &= \left\{ v \in V \,\middle|\, \begin{array}{l} \min\{x^1(v), x^2(v)\} \leq x^*(v) \leq \max\{x^1(v), x^2(v)\}, \\ x^1(v) \neq x^2(v) \end{array} \right\}, \\
V_+(x^1,x^2) &= \left\{ v \in V \mid x^*(v) < \min\{x^1(v), x^2(v), x(v)\} \right\}, \\
V_-(x^1,x^2) &= \left\{ v \in V \mid \max\{x^1(v), x^2(v), x(v)\} < x^*(v) \right\}.
\end{aligned}$$

Proposition 3.4 *Suppose that* $x^1 \in \arg\min f_1$ *and* $x^2 \in \arg\min f_2$ *for two M-convex functions* f_1 *and* f_2*. If* $V_\epsilon(x^1,x^2) \cup V_b(x^1,x^2) = \emptyset$ *then* $x^1 = x^2 = x^*$ *holds.*

Proof. It is sufficient to show that $V_+(x^1, x^2) = V_-(x^1, x^2) = \emptyset$. Suppose to the contrary that $V_+(x^1, x^2) \cup V_-(x^1, x^2) \neq \emptyset$. It follows from $x^1(V) = x^2(V) = x^*(V)$ that both $V_+(x^1, x^2)$ and $V_-(x^1, x^2)$ are nonempty. By the hypothesis, we also have

$$\begin{aligned} V_+(x^1, x^2) &= \mathrm{supp}^+(x^1 - x^*) = \mathrm{supp}^+(x^2 - x^*), \\ V_-(x^1, x^2) &= \mathrm{supp}^-(x^1 - x^*) = \mathrm{supp}^-(x^2 - x^*). \end{aligned}$$

Since $x^* \in \arg\min(f_1 + f_2)$ holds, there exists $p \in \mathbf{R}^V$ such that

$$x^* \in (\arg\min f_1[-p]) \cap (\arg\min f_2[+p]), \tag{3.11}$$

(see, Theorem 2.15). We consider a bipartite digraph $G = (V_+, V_-; A)$ with vertex set $V_+ = V_+(x^1, x^2)$, $V_- = V_-(x^1, x^2)$ and arc set

$$\begin{aligned} A \;\; = \;\; & \{(u, v) \mid u \in V_+,\ v \in V_-,\ x^* + \chi_u - \chi_v \in \mathrm{dom}\, f_1\} \\ & \cup \{(v, u) \mid u \in V_+,\ v \in V_-,\ x^* + \chi_u - \chi_v \in \mathrm{dom}\, f_2\}, \end{aligned}$$

and arc weight $c \in \mathbf{R}^A$ defined by

$$c(a) = \begin{cases} f_1[-p](x^* + \chi_u - \chi_v) - f_1[-p](x^*) & (a = (u, v),\ u \in V_+,\ v \in V_-) \\ f_2[+p](x^* + \chi_u - \chi_v) - f_2[+p](x^*) & (a = (v, u),\ u \in V_+,\ v \in V_-). \end{cases}$$

By (M-EXC), for any $u \in V_+ = \mathrm{supp}^+(x^1 - x^*)$, there exists $j \in V_- = \mathrm{supp}^-(x^1 - x^*)$ such that

$$f_1(x^1) + f_1(x^*) \geq f_1(x^1 - \chi_u + \chi_j) + f_1(x^* + \chi_u - \chi_j).$$

It follows from $x^1 \in \arg\min f_1$ that

$$f_1(x^1) \leq f_1(x^1 - \chi_u + \chi_j).$$

By the above two inequalities,

$$f_1(x^*) \geq f_1(x^* + \chi_u - \chi_j)$$

holds, and hence, arc $(u, j) \in A$ satisfies

$$c(u, j) = f_1(x^* + \chi_u - \chi_j) - f_1(x^*) - p(u) + p(j) \leq -p(u) + p(j). \tag{3.12}$$

Analogously, for any $v \in V_-$, there exists $i \in V_+$ such that $(v, i) \in A$ and

$$c(v, i) = f_2(x^* + \chi_i - \chi_v) - f_2(x^*) + p(i) - p(v) \leq p(i) - p(v). \tag{3.13}$$

By the above discussion, every vertex of $V_+ \cup V_-$ has an arc satisfying either (3.12) or (3.13). Thus, G has a directed cycle C consisting of these arcs. By (3.12) and (3.13), the amount of weights of all arcs in C must be less than or equal to zero.

On the other hand, (3.11) guarantees that each arc of G has nonnegative weight. Hence, C consists of arcs of weight zero.

Let Q be a shortest cycle, with respect to the number of arcs, consisting of arcs of weight zero, and let $Q_+ = V_+ \cap V(Q)$ and $Q_- = V_- \cap V(Q)$, where $V(Q)$ denotes the set of vertices of Q. Because Q is a shortest cycle, the subgraph $G[Q]$ of G induced by $Q_+ \cup Q_-$ has no arc of weight zero other than those of Q. This says that the subgraph of $G[Q]$ induced by the arcs from Q_+ to Q_- has exactly one minimum weight perfect matching of weight zero. By Lemma 3.2, we have $x^* - \chi_{Q_-} + \chi_{Q_+} \in \mathrm{dom}\, f_1[-p]$ and $f[-p](x^* - \chi_{Q_-} + \chi_{Q_+}) = f[-p](x^*)$. This says that $x^* - \chi_{Q_-} + \chi_{Q_+} \in \arg\min f_1[-p]$. Similarly, it can be shown that $x^* - \chi_{Q_-} + \chi_{Q_+}$ is also a minimizer of $f_2[+p]$. By Theorem 2.15, we obtain $x^* - \chi_{Q_-} + \chi_{Q_+} \in \arg\min(f_1 + f_2)$, and furthermore, $||(x^* - \chi_{Q_-} + \chi_{Q_+}) - x||_1 < ||x^* - x||_1$. This, however, contradicts the definition of x^*. Hence, $V_+(x^1, x^2)$ and $V_-(x^1, x^2)$ must be empty, that is, $x^1 = x^2 = x^*$. ∎

We can easily show $|x^*(v) - x(v)| \leq \frac{d}{2}$ for any $v \in V_\epsilon(x^1, x^2) \cup V_b(x^1, x^2)$.

Proposition 3.5 *For $v \in V_\epsilon(x^1, x^2) \cup V_b(x^1, x^2)$, $|x^*(v) - x(v)| \leq \frac{d}{2}$ holds.*

Proof. For any $v \in V_\epsilon(x^1, x^2) \cup V_b(x^1, x^2)$, we have

$$|x^*(v) - x(v)| \leq \max\{|x^1(v) - x(v)|, |x^2(v) - x(v)|\}.$$

It follows from $x^1(V) = x^2(V) = x(V)$ that $|x^1(v) - x(v)| \leq \frac{d}{2}$ and $|x^2(v) - x(v)| \leq \frac{d}{2}$. Therefore, $|x^*(v) - x(v)| \leq \frac{d}{2}$ is obtained. ∎

In the rest of the subsection, we will show $|x^*(v) - x(v)| \leq d$ for any $v \in V_+(x^1, x^2) \cup V_-(x^1, x^2)$. For each $v \in V$ and $\ell \in \{1, 2\}$, we have

$$|x^*(v) - x(v)| \leq |x^*(v) - x^\ell(v)| + |x^\ell(v) - x(v)| \leq |x^*(v) - x^\ell(v)| + \frac{d}{2},$$

where the second inequality follows from $x^\ell(V) = x(V)$. We estimate the distance between x^* and x^ℓ with aid of the following algorithm that transforms x^1 and x^2 to x^* by generating a pair of sequences starting from the given x^1 and x^2 and reaching x^*. We note that Transformation below modifies M-convex functions f_1, f_2 as well as x^1, x^2 maintaining $x^1 \in \arg\min f_1$, $x^2 \in \arg\min f_2$ and $x^* \in \arg\min(f_1 + f_2)$ at each iteration.

algorithm Transformation
while $V_\epsilon(x^1, x^2) \cup V_b(x^1, x^2) \neq \emptyset$ **do** {
  take u from $V_\epsilon(x^1, x^2) \cup V_b(x^1, x^2)$;
  for $\ell \in \{1, 2\}$ **do** {

while $x^\ell(u) \neq x^*(u)$ **do** {
  if $x^\ell(u) > x^*(u)$ **then**
    take $v \in \mathrm{supp}^-(x^\ell - x^*)$ with $\hat{x} = x^\ell - \chi_u + \chi_v \in \arg\min f_\ell^{[\hat{x}\wedge x^*, \hat{x}\vee x^*]}$;
  if $x^\ell(u) < x^*(u)$ **then**
    take $v \in \mathrm{supp}^+(x^\ell - x^*)$ with $\hat{x} = x^\ell + \chi_u - \chi_v \in \arg\min f_\ell^{[\hat{x}\wedge x^*, \hat{x}\vee x^*]}$;
  $x^\ell \leftarrow \hat{x}$;
  $f_\ell \leftarrow f_\ell^{[\hat{x}\wedge x^*, \hat{x}\vee x^*]}$;
  ($\star$) (to be added later for the analysis of the algorithm)
} (end of while)
} (end of for)
}. (end of while)

We first verify the correctness of algorithm Transformation.

Proposition 3.6 Transformation *transforms both* x^1 *and* x^2 *to* x^*.

Proof. Proposition 3.3 guarantees that $\hat{x}$ exists, and that the current x^1 and x^2 are minimizers of the current f_1 and f_2, respectively. Since either $||x^1 - x^*||_1$ or $||x^2 - x^*||_1$ is strictly decreased at each iteration, the algorithm must terminate in finite steps. Since x^* minimizes $\| x^* - x \|_1$ among all minimizers of $f_1 + f_2$ for the current f_1 and f_2, the assertion follows from Proposition 3.4. ■

We note that for each $v \in V$ the following transitions are possible during Transformation:

$$
\begin{array}{ccccc}
V_+(x^1,x^2) & \longrightarrow & V_b(x^1,x^2) & \longrightarrow & V_0(x^1,x^2) \\
 & \nearrow & & \nearrow & \\
V_-(x^1,x^2) & & V_\epsilon(x^1,x^2) & &
\end{array}
$$

To analyze Transformation, we utilize a diagram as in Figure 1. The horizontal axis labeled "level of x^*" is indexed by a pair $(v,\ell) \in V \times \{1,2\}$, and each pair (v,ℓ) is called a *column*. For each column, we consider "positions" which are vertically assigned at regular intervals, and call the distance from the level of x^* to a position P the *height* of P (the height may be negative). Each position has one of four states: null, with a box (without a stone), with a stone (without a box), or with both a box and a stone; in Figure 1 a stone is depicted by a solid circle. For each $v \in V$ and $\ell \in \{1,2\}$, we initially stack $|x^\ell(v) - x^*(v)|$ boxes at column (v,ℓ) upward if $x^\ell(v) > x^*(v)$; downward if $x^\ell(v) < x^*(v)$, from the level of x^*. That is, the number of boxes at column (v,ℓ) denotes the difference between $x^\ell(v)$ and $x^*(v)$. For each $\ell \in \{1,2\}$, let $\overline{\ell}$ be defined by

$$
\overline{\ell} = \begin{cases} 1 & (\ell = 2) \\ 2 & (\ell = 1). \end{cases}
$$

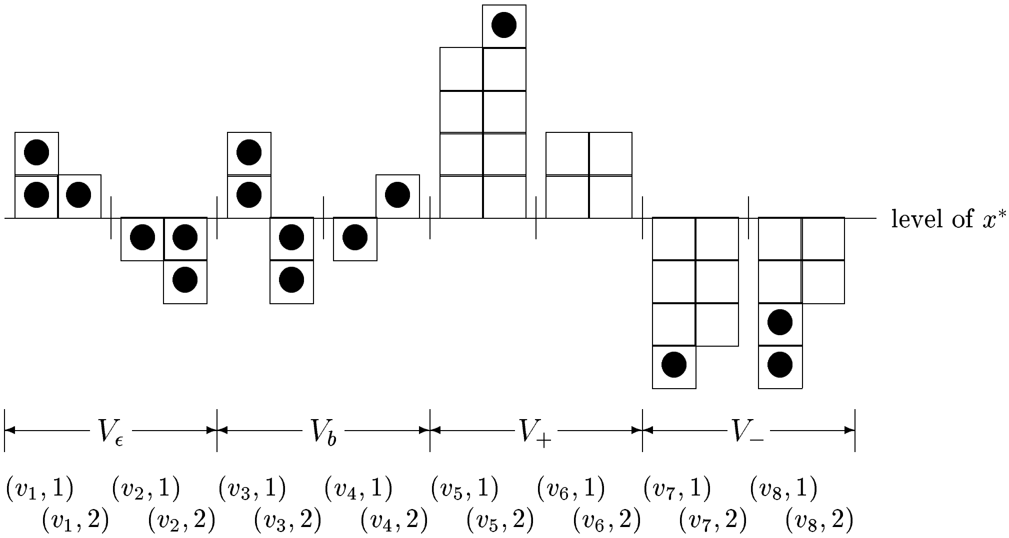

Figure 1: An initial state: $x^1 - x^* = (2,-1,2,-1,4,2,-4,-4)$ and $x^2 - x^* = (1,-2,-2,1,5,2,-3,-2)$.

We say that two positions of the same height at columns (v,ℓ) and $(v,\overline{\ell})$ are *adjacent* to each other. For two boxes (or positions) b_1 and b_2 at the same column, we simply say that b_1 is *farther/nearer* than b_2, if b_1 is farther/nearer than b_2 from/to the level of x^*. The *farthest/nearest* box is defined accordingly.

Before starting TRANSFORMATION, we put stones into boxes according to the following rules (see Figure 1):

- For $v \in V_\epsilon(x^1,x^2) \cup V_b(x^1,x^2)$ and $\ell \in \{1,2\}$, we put one stone into every box at column (v,ℓ).
- For $v \in V_+(x^1,x^2) \cup V_-(x^1,x^2)$ and $\ell \in \{1,2\}$ with $x^\ell(v) > x^{\overline{\ell}}(v)$, we put one stone into each of $(x^\ell(v) - x^{\overline{\ell}}(v))$ boxes from the farthest one at column (v,ℓ).

At the place $(\star)$ in TRANSFORMATION, we modify the arrangement of boxes and stones as follows. We emphasize that u, v and ℓ are fixed at $(\star)$, and that b_u always contains a stone and the position adjacent to b_v has an empty box if b_v is empty, which will be shown later in Proposition 3.8, where b_u and b_v are the farthest boxes at (u,ℓ) and (v,ℓ), respectively.

at $(\star)$ in TRANSFORMATION

  let b_u be the farthest box at column (u,ℓ) ;
  let b_v be the farthest box at column (v,ℓ) ;
  if b_v contains a stone **then**

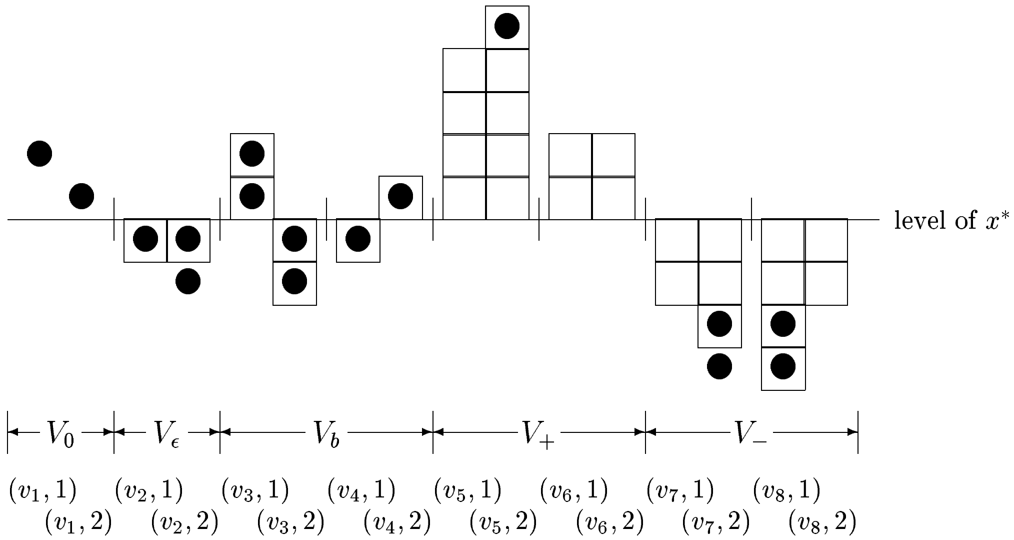

Figure 2: After modifications for $(v_1,1,v_7),(v_1,1,v_7),(v_1,2,v_2)$.

eliminate the boxes b_u and b_v leaving the stones in the same positions ;
else (b_v is empty) {
move the stone in b_u into the box adjacent to b_v ;
eliminate the boxes b_u and b_v ;
shift each stone at column (v,ℓ) to its adjacent position ;
} ;

For example, after three iterations for $(u,\ell,v)=(v_1,1,v_7)$, $(v_1,1,v_7)$, $(v_1,2,v_2)$,

$$x^1-x^*=(2,-1,2,-1,4,2,-4,-4),\quad x^2-x^*=(1,-2,-2,1,5,2,-3,-2)$$

are transformed to

$$x^1-x^*=(0,-1,2,-1,4,2,-2,-4),\quad x^2-x^*=(0,-1,-2,1,5,2,-3,-2)$$

which are represented by boxes in diagrams, v_1 is moved into $V_0(x^1,x^2)$ and the arrangement of boxes and stones in Figure 1 is modified as Figure 2.

Let x_0^1 and x_0^2 denote the initial x^1 and x^2, respectively. We can estimate the distance between x^* and x_0^ℓ in terms of the number of stones.

Proposition 3.7 *For $u\in V_+(x_0^1,x_0^2)\cup V_-(x_0^1,x_0^2)$ and $\ell\in\{1,2\}$, let s_ℓ denote the number of stones at column (u,ℓ) at the time when u is taken at an outer while iteration. Then we have:*

$$\max\{s_1,s_2\}=\max\{|x_0^1(u)-x^*(u)|,|x_0^2(u)-x^*(u)|\}. \tag{3.14}$$

Proof. Suppose that $|x_0^1(u)-x^*(u)| \geq |x_0^2(u)-x^*(u)|$ without loss of generality.

If $|x_0^1(u)-x^*(u)| > |x_0^2(u)-x^*(u)|$ then at the initial state, the farthest box at column $(u,1)$ contains a stone. Till the algorithm reaches this point, the stone is located at either $(u,1)$ or $(u,2)$ and does not change its height. Therefore, (b), (e) and (f) of Proposition 3.8 below shows that $\max\{s_1, s_2\} = |x_0^1(u)-x^*(u)|$.

Suppose that $|x_0^1(u)-x^*(u)| = |x_0^2(u)-x^*(u)|$. At the initial state, none of the boxes at $(u,1)$ or at $(u,2)$ have a stone. At the first time when a box at (u,ℓ) (with $\ell = 1$ or 2) is eliminated, a stone is put into the farthest box at $(u,\overline{\ell})$. The argument thereafter is similar to the above case. ∎

The following proposition analyzes the movement of stones and show properties of the modification at $(\star)$.

Proposition 3.8 *At the beginning of each inner while iteration of* TRANSFORMATION*, the following statements hold, where* $\ell' \in \{1,2\}$*:*

(a) *The total number of stones remains the same as that at the initial state. For any column* (v',ℓ')*, boxes are located in consecutive positions from the level of* x^**.*

(b) *Each position has at most one stone.*

(c) *Any empty box is adjacent to an empty box.*

(d) *For column* (v',ℓ') *with* $v' \in V_+(x^1,x^2) \cup V_-(x^1,x^2)$*, any stone is adjacent to a null position.*

(e) *For column* (v',ℓ') *with* $v' \in V \setminus (V_0(x^1,x^2) \cup \{u\})$*,* (e1) *stones are put in consecutive positions,* (e2) *no farther position than a position* P *with a stone has an empty box, and* (e3) *all nearer positions than* P *are nonnull.*

Moreover, at the beginning of each outer while iteration, the following statement holds:

(f) *For column* (v',ℓ') *with* $v' \in V_\epsilon(x^1,x^2) \cup V_b(x^1,x^2)$*, any box contains a stone.*

Proof. (a) is obviously satisfied by the above modification at $(\star)$. Statements from (b) to (f) are initially satisfied.

We first show (b) to (e) according to the two cases in the modification at $(\star)$. Suppose that conditions (b) to (f) hold at the beginning of an iteration.

Assume that b_v contains a stone. By the hypotheses, both b_u and b_v contain exactly one stone. In this case, statements (b) to (e) trivially hold because no stone is moved and no empty box is eliminated.

Assume that b_v is empty. We first consider the time just after the elimination of b_u and b_v. By (f) and (c), b_u contained exactly one stone s and the position adjacent to b_v had an empty box b. Since the stone s was moved into b, (b) is preserved. Although empty box b_v was eliminated, (c) is satisfied because the stone s is located in b, and furthermore, the position adjacent to b is null, i.e., (d) is not violated. We now verify (e). This is obviously true for (v, ℓ') with $v' \neq v$. We will focus on (e) for $(v, \overline{\ell})$, where we note that (e) for (v, ℓ) may not true at this moment. By (a) and (d), any nearer position than b has an empty box, and hence (e3). Since b_v was the farthest box at (v, ℓ), no farther position than b has an empty box by (c), which shows (e2). By (e), if there existed stones at $(v, \overline{\ell})$ before the modification then these were consecutively located from the position farther than b by one. Hence, (e1) is satisfied.

We next consider the time just after shifting stones in the case of empty b_v. By (e), the stones at column (v, ℓ), if any, were consecutively located from the position which is farther than b_v by one. By (d), these stones were adjacent to a null position. Hence, shifting these stones preserves (b), (c) and (d). Since these stones and the stone s are located consecutively, (e) remains to be true for $(v', \ell') \neq (v, \ell)$. Moreover, (e) for (v, ℓ) is also satisfied because column (v, ℓ) has no stone.

We finally consider (f). Since (f) holds initially, we deal with the case where $v' \in V_+(x^1, x^2)$ was moved into $V_b(x^1, x^2)$, without loss of generality. This means that either $x^1(v') = x^*(v')$ or $x^2(v') = x^*(v')$. Here we assume $x^2(v') = x^*(v')$. Suppose to the contrary that there is an empty box at column $(v', 1)$. By (e), the nearest box $b_{(v',1)}$ at $(v', 1)$ is empty. Let us consider the time when the box $b_{(v',2)}$ adjacent to $b_{(v',1)}$ was eliminated. By (c), $b_{(v',2)}$ had to be empty. However, a stone was put in $b_{(v',1)}$, a contradiction. Hence, any box at column $(v', 1)$ has a stone. ∎

By Proposition 3.7, $\| x^* - x_0^\ell \|_\infty$ is bounded by the number of stones. We next estimate the left-hand side of (3.14) more precisely. Let us call a stone in a box to be *black* and a stone not in a box *white*, and classify stones into six categories (where $\ell \in \{1, 2\}$):

$Ba\ell$: the set of <u>b</u>lack stones located <u>a</u>bove the level of x^* at column $(v, \underline{\ell})$ for some v,

$Bb\ell$: the set of <u>b</u>lack stones located <u>b</u>elow the level of x^* at column $(v, \underline{\ell})$ for some v,

Wa : the set of <u>w</u>hite stones located <u>a</u>bove the level of x^*,

Wb : the set of <u>w</u>hite stones located <u>b</u>elow the level of x^*.

The partition dynamically changes during TRANSFORMATION under the following restrictions.

Proposition 3.9 *If a stone changes its category, then this is one of the following transitions:*

$$Ba1 \to Bb2 \cup Wa, \qquad Bb2 \to Ba1 \cup Wb,$$
$$Ba2 \to Bb1 \cup Wa, \qquad Bb1 \to Ba2 \cup Wb.$$

Therefore, no stone in $Wa \cup Wb$ changes its category.

Proof. The modification at ($\star$) contains three alterations of states of stones: color change of black stones in boxes b_u and b_v, movement of the black stone in b_u and shift of white stones at (v, ℓ). Let ℓ denote 1 or 2. The first alteration is either $Ba\ell \to Wa$ or $Bb\ell \to Wb$. The second one is either $Ba\ell \to Bb\overline{\ell}$ or $Bb\ell \to Ba\overline{\ell}$. The third one means that white stones do not change the category. ∎

Proposition 3.10 *At the beginning of each inner while iteration, we have*

$$|Ba1| + |Bb2| = |Ba2| + |Bb1|, \tag{3.15}$$
$$|Wa| = |Wb|, \tag{3.16}$$

and therefore, for any $u \in V_+(x_0^1, x_0^2) \cup V_-(x_0^1, x_0^2)$,

$$\max\{|x_0^1(u) - x^*(u)|, |x_0^2(u) - x^*(u)|\} \leq \frac{s}{2}, \tag{3.17}$$

where s denotes the total number of stones.

Proof. We first consider the initial state. Since $x_0^1(V) = x_0^2(V) = x^*(V)$, we have

$$\begin{aligned} & \sum_{v \in \mathrm{supp}^+(x_0^1 - x^*)} (x_0^1(v) - x^*(v)) + \sum_{v \in \mathrm{supp}^-(x_0^2 - x^*)} (x^*(v) - x_0^2(v)) \\ = & \sum_{v \in \mathrm{supp}^+(x_0^2 - x^*)} (x_0^2(v) - x^*(v)) + \sum_{v \in \mathrm{supp}^-(x_0^1 - x^*)} (x^*(v) - x_0^1(v)). \end{aligned} \tag{3.18}$$

The left-hand side of (3.18) minus the number of relevant empty boxes is equal to $|Ba1| + |Bb2|$, and a similar relation between the right-hand side of (3.18) and $|Ba2| + |Bb1|$ also holds. On the other hand, for $v \in V_+(x_0^1, x_0^2) \cup V_-(x_0^1, x_0^2)$, the number of empty boxes at $(v, 1)$ is equal to that at $(v, 2)$. Therefore, $|Ba1| + |Bb2| = |Ba2| + |Bb1|$ and $|Wa| = |Wb| = 0$ are satisfied at the initial state.

Color change of black stones in the modification at ($\star$) decreases the cardinalities of $Ba1 \cup Bb2$ and $Ba2 \cup Bb1$ exactly by one, and increases those of Wa and Wb

exactly by one. Thus, Proposition 3.9 guarantees that $|Ba1|+|Bb2| = |Ba2|+|Bb1|$ and $|Wa| = |Wb|$ are preserved during TRANSFORMATION.

Let s_1 and s_2 be the numbers defined in Proposition 3.7. By (3.15) and (3.16), s_1 and s_2 are bounded by $\frac{s}{2}$. Hence, (3.17) is obtained from (3.14). ∎

Proof of Theorem 2.18 We already proved that

$$|x^*(v) - x(v)| \leq \frac{d}{2}$$

holds for any $v \in V_\epsilon(x_0^1, x_0^2) \cup V_b(x_0^1, x_0^2)$ in Proposition 3.5. Here we prove that

$$|x^*(v) - x(v)| \leq d$$

holds for any $v \in V_+(x_0^1, x_0^2) \cup V_-(x_0^1, x_0^2)$. Without loss of generality, we assume that $v \in V_+(x_0^1, x_0^2)$ and $x_0^1(v) \geq x_0^2(v)$. Obviously, we have

$$\begin{array}{lcl} x(v) \leq x_0^1(v) & \Rightarrow & x(v) - x^*(v) \leq x_0^1(v) - x^*(v), \\ x(v) > x_0^1(v) & \Rightarrow & x(v) - x^*(v) = (x_0^1(v) - x^*(v)) + (x(v) - x_0^1(v)). \end{array} \tag{3.19}$$

Let s denote the total number of stones. (3.17) says

$$x_0^1(v) - x^*(v) \leq \frac{s}{2}. \tag{3.20}$$

We finally estimate s. Here we abbreviate $V_\epsilon(x_0^1, x_0^2)$ to V_ϵ, $V_b(x_0^1, x_0^2)$ to V_b, and so on. The number s is bounded as

$$\begin{array}{rcl} s & = & \sum_{\ell\in\{1,2\}} \sum_{i\in V_\epsilon} |x_0^\ell(i) - x^*(i)| + \sum_{i\in V_b\cup V_+\cup V_-} |x_0^1(i) - x_0^2(i)| \\ & \leq & \sum_{\ell\in\{1,2\}} \sum_{i\in V\setminus\{v\}} |x_0^\ell(i) - x(i)| + (x_0^1(v) - x_0^2(v)) \\ & = & ||x_0^1 - x||_1 + ||x_0^2 - x||_1 - |x_0^1(v) - x(v)| - |x_0^2(v) - x(v)| + (x_0^1(v) - x_0^2(v)) \\ & \leq & 2d - |x_0^1(v) - x(v)| - |x_0^2(v) - x(v)| + (x_0^1(v) - x_0^2(v)) \\ & = & \begin{cases} 2(d - (x(v) - x_0^1(v))) & (x(v) > x_0^1(v)) \\ 2d & (x_0^2(v) \leq x(v) \leq x_0^1(v)) \\ 2(d - (x_0^2(v) - x(v))) & (x(v) < x_0^2(v)). \end{cases} \end{array} \tag{3.21}$$

By (3.19), (3.20) and (3.21), we obtain $x(v) - x^*(v) \leq d$. ∎

3.5 Proof of Lemma 2.22

We use notations defined in Remark 2.21.

[(a) $\Rightarrow$ (b)] Given ordered sets $U, W \subset V$, if the left-hand side of (2.7) is finite, then it is equal to the length of some cycle in G_x^α with respect to ℓ_x^α. Assumption (a) says that it must be nonnegative, that is, (b) holds.

[(b) ⇒ (a)] We prove that if there exists a negative cycle C in G_x^α then (2.7) does not hold for some $U, W \subset V$. Without loss of generality, we assume that C is simple, and that C is denoted by a sequence of arcs. Assume that consecutive two arcs $(u,v),(v,w)$ $(u \neq w)$ of C belong to A_1 (the case where these belong to A_2 can be dealt with, similarly). We first show that

$$\ell_x^\alpha(u,v) + \ell_x^\alpha(v,w) \geq \ell_x^\alpha(u,w). \tag{3.22}$$

It follows from $(v,w) \in A_1$ that $x - \beta\chi_v + \beta\chi_w$ is contained in $\mathrm{dom}\, f_1$ for any $\beta \in [0,\alpha]_{\mathbf{Z}}$. By applying (M-EXC), we have the following inequalities:

$$\begin{aligned}
&f_1(x - \chi_v + \chi_w) + f_1(x - \alpha\chi_u + \alpha\chi_v)\\
&\qquad\qquad \geq f_1(x) + f_1(x - \alpha\chi_u + (\alpha{-}1)\chi_v + \chi_w)\\
&f_1(x - 2\chi_v + 2\chi_w) + f_1(x - \alpha\chi_u + (\alpha{-}1)\chi_v + \chi_w)\\
&\qquad\qquad \geq f_1(x - \chi_v + \chi_w) + f_1(x - \alpha\chi_u + (\alpha{-}2)\chi_v + 2\chi_w)\\
&f_1(x - 3\chi_v + 3\chi_w) + f_1(x - \alpha\chi_u + (\alpha{-}2)\chi_v + 2\chi_w)\\
&\qquad\qquad \geq f_1(x - 2\chi_v + 2\chi_w) + f_1(x - \alpha\chi_u + (\alpha{-}3)\chi_v + 3\chi_w)\\
&\qquad\qquad\qquad\qquad \vdots\\
&f_1(x - \alpha\chi_v + \alpha\chi_w) + f_1(x - \alpha\chi_u + \chi_v + (\alpha{-}1)\chi_w)\\
&\qquad\qquad \geq f_1(x - (\alpha{-}1)\chi_v + (\alpha{-}1)\chi_w) + f_1(x - \alpha\chi_u + \alpha\chi_w).
\end{aligned}$$

By summing up both sides of the above inequalities, we obtain

$$f_1(x - \alpha\chi_u + \alpha\chi_v) + f_1(x - \alpha\chi_v + \alpha\chi_w) \geq f_1(x) + f_1(x - \alpha\chi_u + \alpha\chi_w),$$

which is equivalent to (3.22).

Inequality (3.22) guarantees that $C \setminus \{(u,v),(v,w)\} \cup \{(u,w)\}$ is also a negative cycle. While there are consecutive two arcs of C as above, we replace these by the shortcut arc. After the process, we obtain either a negative cycle C in which arcs of $A_1 \cup A_2$ and A_b appear alternately, or a negative cycle of length two contained in one of A_1, A_2 and A_b, because of the structure of G_x^α. The second case, however, cannot occur because the length of the cycle is not negative. Since $\ell_x^\alpha(a) = 0$ holds for any $a \in A_b$, the length of C is expressed as the left-hand side of (2.7) for some ordered sets U and W with $U \cap W = \emptyset$, and furthermore, it must be negative.

References

[1] R. Baldick, Refined proximity and sensitivity results in linearly constrained convex separable integer programming, *Linear Algebra Appl.* **226/228** (1995), 389–407.

[2] W. Cook, A. M. H. Gerards, A. Schrijver and E. Tardos, Sensitivity theorems in integer linear programming, *Math. Program.* **34** (1986), 251–264.

[3] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, *Ann. Discrete Math.* **1** (1977), 185–204.

[4] P. Favati and F. Tardella, Convexity in nonlinear integer programming, *Ricerca Operativa* **53** (1990), 3–44.

[5] S. Fujishige, "Submodular Functions and Optimization," Annals of Discrete Mathematics 47, North-Holland, Amsterdam, 1991.

[6] S. Fujishige, N. Katoh and T. Ichimori, The fair resource allocation problem with submodular constraints, *Math. Oper. Res.* **13** (1988), 164–173.

[7] S. Fujishige and K. Murota, Notes on L-/M-convex functions and the separation theorems, *Math. Program.* **88** (2000), 129–146.

[8] F. Granot and J. Skorin-Kapov, Some proximity and sensitivity results in quadratic integer programming, *Math. Program.* **47** (1990), 259–268.

[9] D. S. Hochbaum, Lower and upper bounds for the allocation problem and other nonlinear optimization problems, *Math. Oper. Res.* **19** (1994), 390–409.

[10] D. S. Hochbaum and J. G. Shanthikumar, Convex separable optimization is not much harder than linear optimization, *J. Assoc. Comput. Mach.* **37** (1990), 843–862.

[11] T. Ibaraki and N. Katoh, "Resource Allocation Problems: Algorithmic Approaches," The MIT Press, Cambridge, 1988.

[12] T. Ibaraki and N. Katoh, Resource Allocation Problems, in "Handbook of Combinatorial Optimization (Vol. 2)," (D.-Z. Du and P. M. Pardalos, Eds.), pp. 159–260, Kluwer Academic Publishers, Boston, 1998.

[13] S. Iwata, Oral presentation at Workshop on Matroids, Matching, and Extensions, University of Waterloo, December, 1999.

[14] S. Iwata, A fully combinatorial algorithm for submodular function minimization, *J. Combin. Theory Ser. B* **84** (2002), 203–212.

[15] S. Iwata, L. Fleischer and S. Fujishige, A combinatorial strongly polynomial algorithm for minimizing submodular functions, *J. Assoc. Comput. Mach.* **48** (2001), 761–777.

[16] S. Iwata and M. Shigeno, Conjugate scaling algorithm for Fenchel-type duality in discrete convex optimization, *SIAM J. Optim.*, to appear.

[17] S. Moriguchi, K. Murota and A. Shioura, Scaling algorithms for M-convex function minimization, *IEICE Trans. Fundamentals* **E85-A** (2002), 922–929.

[18] S. Moriguchi and A. Shioura, On Hochbaum's scaling algorithm for the general resource allocation problem, Research Report B-377, Tokyo Institute of Technology, 2002.

[19] K. Murota, Convexity and Steinitz's exchange property, *Adv. Math.* **124** (1996), 272–311.

[20] K. Murota, Discrete convex analysis, *Math. Program.* **83** (1998), 313–371.

[21] K. Murota, Submodular flow problem with a nonseparable cost function, *Combinatorica* **19** (1999), 87–109.

[22] K. Murota, "Matrices and Matroids for Systems Analysis," Springer-Verlag, Berlin, 2000.

[23] K. Murota, "Discrete Convex Analysis — An Introduction (in Japanese)," Kyoritsu Publ. Co., Tokyo, 2001.

[24] K. Murota and A. Shioura, M-convex function on generalized polymatroid, *Math. Oper. Res.* **24** (1999), 95–105.

[25] K. Murota and A. Shioura, Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella, *Discrete Appl. Math.* **115** (2001), 151–176.

[26] K. Murota and A. Shioura, Quasi M-convex and L-convex functions: Quasiconvexity in discrete optimization, *Discrete Appl. Math.* to appear.

[27] A. Schrijver, A combinatorial algorithm minimizing submodular functions in strongly polynomial time, *J. Combin. Theory Ser. B* **80** (2000), 346–355.

[28] A. Shioura, Fast scaling algorithms for M-convex function minimization with application to the resource allocation problem, Tohoku University, (2002).

[29] A. Tamura, Coordinatewise domain scaling algorithm for M-convex function minimization, in: "Proceedings of the Ninth Conference on Integer Programming and Combinatorial Optimization," Lecture Notes in Computer Science Vol. 2337 (W. J. Cook and A. S. Schulz, Eds.) pp. 21–35, Springer-Verlag, Berlin, 2002.