A Unified Scheme for Generalized Sectors based on Selection Criteria.
I. Thermal situations, unbroken symmetries and criteria as classifying categorical adjunctions

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Abstract

A unified scheme for treating generalized superselection sectors is proposed on the basis of the notion of selection criteria to characterize states of relevance to each specific domain in quantum physics, ranging from the relativistic quantum fields in the vacuum situations with unbroken and spontaneously broken internal symmetries, through equilibrium and non-equilibrium states to some basic aspects in measurement processes. This is achieved by the help of $c \to q$ and $q \to c$ channels, the former of which determines the states to be selected and to be parametrized by the order parameters defined as the spectrum of the centre constituting the superselection sectors, and the latter of which provides, as classifying maps, the physical interpretations of selected states in terms of order parameters. This formulation extends the traditional range of applicability of the Doplicher-Roberts construction method for recovering the field algebra and the gauge group (of the first kind) from the data of group invariant observables to the situations with spontaneous symmetry breakdown, as will be reported in a succeeding paper.

1 Introduction

The stanrdard way of treating the microscopic world on the basis of quantum field theory (QFT, for short) is to introduce first the *quantum fields* whose characterization is given by means of their behaviours under the various kinds of *symmetries*; e.g., the internal symmetry groups such as the colour $SU(3)$, the chiral $SU(2)$, the electromagnetic $U(1)$, or any other bigger (super)groups of grand unifications (and their corresponding versions of local gauge symmetries), in combination with the spacetime symmetry groups such as the Poincaré group in the Minkowski spacetime, conformal groups in massless theories, or isometry groups of curved spacetimes, and so on. In a word, the basic objects of such a system can essentially be found in an algebra $\mathfrak{F}$ of quantum fields (called a *field algebra*, for short) acted upon by two kinds of symmetries, *internal* and *spacetime* (whose unification has been pursued as one of the ultimate goals of microscopic physics). With respect to the group of an internal symmetry denoted generically by G, the generating elements of $\mathfrak{F}$ (usually called *basic* or *fundamental fields*) are assumed (*by hand*) to belong to certain multiplet(s) transforming covariantly under the action of G, which defines mathematically an action τ of G on $\mathfrak{F}$: $G \underset{\tau}{\curvearrowright} \mathfrak{F}$.

Contrary to this kind of *theoretical* setting, what is to be observed (experimentally) in the real world is believed (or, can be proved in a certain setup; see, for instance, [1]) to be only those elements of $\mathfrak{F}$ *invariant under* G and are usually called *observables* which constitute the algebra $\mathfrak{A}$ of observables:

$$\mathfrak{A} := \mathfrak{F}^G, \tag{1}$$

the fixed-point subalgebra of $\mathfrak{F}$ under the action τ of G. Thus, what we can directly check experimentally is supposed to be only those data described in terms of $\mathfrak{A}$ (and its derived objects) and the rest of the notions appearing in our framework are just mathematical devices whose pertinence can be justified only through the information related to $\mathfrak{A}$. Except for the systematic approaches [2, 3, 4] undertaken by the group of algebraic QFT, however, there have so far been no serious attempts to understand the basic mechanism pertaining to this point as to how a particular choice of $\mathfrak{F}$ and G can be verified, leaving aside the problems of this sort just to the heuristic arguments based on trials and errors. While a particularly chosen combination $G \underset{\tau}{\curvearrowright} \mathfrak{F}$ becomes no doubt meaningless without good agreements of its consequences with the observed data described in terms of $\mathfrak{A}$, the attained agreements support the postulated theoretical assumption only as *one of many possible candidates* of explanations, without justifying it as a *unique* inevitable solution. (Does it not look quite strange that such a kind of problems as this have hardly been examined in the very sophisticated discussions about the unicity of the unification models at the Planck scale?)

Just when restricted to the cases with G an *unbroken* global gauge symmetry (or, gauge symmetry of the first kind), a satisfactory framework in this context has been established in [5, 3], whose physical essence has, unfortunately, not been recognized widely (which may be partly due to its mathematical sophistications, but mainly due to the lack of common understanding of the importance of the above-mentioned problem). This theory enables one to *recover both* $\mathfrak{F}$ *and* G starting only from the data encoded in $\mathfrak{A}$ when supplemented by the so-called DHR *selection criterion* [6, 2] to pick up physically relevant states with localizable charges (which should, in the case of topological charges, be replaced by the one proposed in Buchholz-Fredenhagen [7]). Then, the vacuum representation of the constructed field algebra $\mathfrak{F}$ is decomposed into mutually disjoint irreducible representations of $\mathfrak{A} = \mathfrak{F}^G$, called *superselection sectors*, in one-to-one correspondence with mutually disjoint irreducible unitary representations of the internal symmetry group G which is found to be *compact Lie*. However, the traditional notion of sector structure, hinging strongly to the essential features of unbroken symmetry, has so far allowed only the *discrete sectors*, parametrized by the discrete $\hat{G}$, the dual of a compact group defined as the set of all equivalence classes of finite-dimensional continuous unitary irreducible representations of G. When we start to extend this formalism to the situations with *spontaneous symmetry breakdown* (SSB, for short), we encounter the presence of *continuous sectors* (or, "degenerate vacua" in the traditional terminology) parametrized by continuous *macroscopic order parameters*, as is seen in the second paper of this series ([9], called II hereafter).

Aside from such very fundamental issues as the "ultimate" unifications, we have so far faced with so many different levels and areas in the directions from microscopic worlds to macroscopic ones, ranging from the vacuum situations (the standard QFT relevant to particle physics), thermal equilibria (QFT at finite temperatures or quantum statistical mechanics), non-equilibrium ones and so on. In [8], a general framework is proposed for defining non-equilibrium local states in relativistic QFT and for describing their thermodynamic properties in terms of the associated macroscopic observables found in the *centre* of relevant representations of observables. From the general standpoint, one easily notices that the thermal equilibria at different temperatures can also be seen to constitute families of continuous sectors parametrized by such thermodynamic variables as temperatures, chemical potentials and pressure and so on. In view of such roles of central observables associated with continuous sectors appearing in SSB cases as well as the above various kinds of thermal states, it seems appropriate to extend the notion of sectors so as to incorporate and to try the possibility of unified ways of treating these different cases, just regarding the traditional discrete ones as special cases; this is simply parallel to the extension of the traditional *eigenvalue* problems for linear operators with *discrete spectra* to the general *spectral decompositions* admitting the appearance of *continuous spectra*.

The aim of the present series of articles (in combination with [9]) is to propose a scheme to unify such a generalized notion of sectors from the viewpoint of the key roles played by the *selection criteria* at the starting point of theory in defining and choosing *physically relevant family of states* as well as in providing a systematic way for *describing* and *interpreting* the relevant physical properties. In this paper (called I), we introduce the necessary ingredients for formulating the scheme through the discussions on the basic structures found in thermal situations of equilibrium (Sec.2) and of the extension to non-equilibrium (Sec.3) and in the reformulated version of DHR superselection theory (Sec.4). At the end, we explain the general mathematical meaning of the proposed scheme, in relation with the categorical adjunctions, especially with the geometric notions of classifying spaces and classifying maps. The analysis is to be continued in the second paper [9], where we develop a systematic treatment of spontaneously broken symmetries from the viewpoint of proposed scheme and close the series by examining the operational meaning of our basic ingredients in relation with certain basic notions relevant to the quantum measurements.

2 Equilibrium states and thermal interpretations: roles of $c\to q$ and $q\to c$ channels

To draw a clear picture of the idea, we briefly sketch the essense of the scheme proposed in [8] for defining and describing non-equilibrium local states in a relativistic QFT. From the present standpoint, it can be reformulated as follows according to [10]. To characterize an unknown state ω as a non-equilibrium local state, we prepare the following basic ingredients.

i) The set $\mathcal{K}$ of *thermal reference states* consisting of all global thermal equilibrium states defined as the relativistic KMS states ω_β [11] (with inverse temperature 4-vectors $\beta = (\beta^\mu) \in V_+ := \{x \in \mathbb{R}^4; x^0 > 0, x^2 = (x^0)^2 - \vec{x}^2 > 0\}$) and their suitable convex combinations: $\mathcal{K}$ plays the role of a *model space* whose analogue in the definition of a manifold M can be found in a Euclidean space $\mathbb{R}^n$ as the value space of local charts.

ii) The set $\mathcal{T}_x$ of *local thermal observables* consisting of suitable combinations of quantum observables at a spacetime point x defined mathematically as linear forms on states with suitable regularity in their energy contents (which are quantum observables in an extended sense) [12, 8]. Along the above analogy to a manifold M in differential geometry, their role is to relate our unknown state ω to the known reference states in $\mathcal{K}$, just in parallel to the local coordinates which relate locally a generic curved space M to the known space $\mathbb{R}^n$. As explained just below, the physical interpretations of local thermal observables $\hat{A}$ are given by macroscopic *thermal functions* corresponding to $\hat{A}$, through which our unknown ω can be *compared* with thermal reference states in $\mathcal{K}$.

Before going into the discussion of non-equilibrium, we need first to establish the physical roles of the above ingredients for describing the relevant thermal properties of states and quantum observables in the realm $\mathcal{K}$ of generalized thermal equilibria. To this end, we introduce

Definition 1 ***Thermal functions*** *are defined for each quantum observables* $\hat{A}(\in \mathcal{T}_x)$ *by the map*

$$\Xi : \hat{A} \longmapsto \Xi(\hat{A}) \in C(B_{\mathcal{K}})$$
$$\text{with } B_{\mathcal{K}} \ni (\beta, \mu) \longmapsto \Xi(\hat{A})(\beta, \mu) := \omega_{\beta,\mu}(\hat{A}), \qquad (2)$$

where $B_{\mathcal{K}}$ *is the classifying space to parameterize thermodynamic pure phases, consisting of inverse temperature 4-vectors* $\beta \in V_+$ *in addition to any other thermodynamic parameters (if any) generically denoted by* μ *(e.g., chemical potentials) necessary to exhaust and discriminate all the thermodynamic pure phases.*

Since this map Ξ is easily seen to be a unital (completely) positive linear map, $\Xi(\mathbf{1}) = 1, \Xi(\hat{A}^*\hat{A}) \geq 0,$ its dual map Ξ^* on states becomes a ***c****lassical-****q****uantum (c→q) channel* $\Xi^* : M_1(B_{\mathcal{K}}) \ni \rho \longmapsto \Xi^*(\rho) \in \mathcal{K}$ given by

$$\Xi^*(\rho)(\hat{A}) = \rho(\Xi(\hat{A})) = \int_{B_{\mathcal{K}}} d\rho(\beta,\mu)\Xi(\hat{A})(\beta,\mu) = \int_{B_{\mathcal{K}}} d\rho(\beta,\mu)\omega_{\beta,\mu}(\hat{A}),$$
$$\Longrightarrow \Xi^*(\rho) := \int_{B_{\mathcal{K}}} d\rho(\beta,\mu)\omega_{\beta,\mu} = \omega_\rho \in \mathcal{K}. \qquad (3)$$

Here $M_1(B_{\mathcal{K}})$ is the space of probability measures ρ on $B_{\mathcal{K}}$ describing the mean values of thermodynamic parameters (β, μ) together with their fluctuations. One can see that thermal interpretation of local quantum thermal observables $\hat{A} \in \mathcal{T}_x$ is given in all thermal reference states of the form $\Xi^*(\rho) = \omega_\rho \in \mathcal{K}$ by the corresponding thermal observable $\Xi(\hat{A})$ evaluated in the classical probability ρ describing the thermodynamic configurations of ω_ρ through the relation

$$\omega_\rho(\hat{A}) = \int_{B_{\mathcal{K}}} d\rho(\beta,\mu)\,\omega_{\beta,\mu}(\hat{A}) = \int_{B_{\mathcal{K}}} d\rho(\beta,\mu)[\,\Xi(\hat{A})](\beta,\mu) = \rho(\Xi(\hat{A})). \quad (4)$$

This applies to the case where ρ is already known. What we need in the actual situations is how to determine the unknown ρ from the given data set $\Phi \longmapsto \rho(\Phi)$ of expectation values of thermal functions Φ (which is the problem of state estimation): this problem can be solved if $\mathcal{T}_x$ has sufficiently many local thermal observables so that the totality $\Xi(\mathcal{T}_x)$ of the corresponding thermal functions can approximate arbitrary continuous functions of $(\beta, \mu) \in B_{\mathcal{K}}$. In this case ρ is given as the unique solution to a (generalized) "moment problem". Thus we see:

- ★ If the set $\mathcal{T}_x$ of local thermal observables are sufficient for discriminating all the thermal reference states in $\mathcal{K}$, then any reference state $\in \mathcal{K}$ can be written as $\Xi^*(\rho)$ in terms of a uniquely determined probability measure $\rho \in B_{\mathcal{K}}$. Then local thermal observables $\hat{\Phi} \in \mathcal{T}_x$ provide the same information on the thermal properties of states in $\mathcal{K}$ as that provided by the corresponding classical macroscopic thermal functions $\Phi = \Xi(\hat{\Phi})$ [e.g., internal energy, entropy density, etc.]: $\omega_\rho(\hat{\Phi}) = \rho(\Phi)$.

In this situation, the *inverse* of Ξ^* becomes meaningful on $\mathcal{K}$ and the thermal interpretation of thermal reference states $\in \mathcal{K}$ is just given by this ***q* →*c* channel** $(\Xi^*)^{-1} : \mathcal{K} \ni \omega \longmapsto \rho \in M_1(B_\mathcal{K})$ s.t. $\omega = \Xi^*(\rho)$ [10], which can be regarded as a simple adaptation and extension of the notions of classifying spaces and classifying maps to the context involving (quantum) probability theory. We express the essence of the above situation (★) by

$$\mathcal{K}(\omega, \Xi^*(\rho))/\mathcal{T}_x \overset{q\rightleftarrows c}{\simeq} Th((\Xi^*)^{-1}(\omega), \rho)/\Xi(\mathcal{T}_x). \tag{5}$$

What is to be noted here is that two levels, quatum statistical mechanics with family $\mathcal{K}$ of KMS states and macroscopic thermodynamics with parameter space $B_\mathcal{K}$, are interrelated by means of the channels, $c \to q$ (Ξ^*) and $q \to c$ ($(\Xi^*)^{-1}$). This is an expression of a categorical adjunction at the level of states.

3 Selection criterion for non-equilibrium states

The next step is to examine how to characterize a non-equilibrium state $\omega \notin \mathcal{K}$ so as for the possibility of thermal interpretations to be extended to the outside of $\mathcal{K}$.

Selection criterion and thermal interpretation of non-equilibrium local states in terms of hierarchized zeroth law of local thermodynamics [10]: To meet simultaneously the two requirements of characterizing an unkown state ω as a non-equilibrium local state and of establishing its thermal interpretations, we now compare ω with thermal reference states $\in \mathcal{K} = \Xi^*(M_1(V_+))$ by means of local thermal observables $\mathcal{T}_x$ at x whose thermal meanings are provided by the corresponding thermal functions $\Xi(\mathcal{T}_x)$ as seen above. In view of the above conclusion [$q \to c$ channel $(\Xi^*)^{-1}$ = thermal interpretation of quantum states] and also of the hierarchical structure in T_x, we relax the requirement for ω to agree with $\exists \omega_B := \Xi^*(\rho_x) \in \mathcal{K}$ up to some suitable *sub*space $\mathcal{S}_x$ of all local thermal observables $\mathcal{T}_x$. Then, we characterize ω as a non-equilibrium local state by

iii) a *selection criterion* for ω to be $\mathcal{S}_x$-*thermal* at x, requiring the existence of $\rho_x \in M_1(V_+)$ s.t.

$$\omega(\hat{\Phi}(x)) = \Xi^*(\rho_x)(\hat{\Phi}(x)) \quad \text{for } \forall \hat{\Phi}(x) \in \mathcal{S}_x, \tag{6}$$

or, $\omega \equiv \Xi^*(\rho_x)$ (mod $\mathcal{S}_x$) for short. This means that, as far as the gross thermal properties described by a subspace $\mathcal{S}_x$ of $\mathcal{T}_x$ are concerned,

ω can be identified with a thermal reference state $\Xi^*(\rho_x) \in \mathcal{K}$. The deviations of ω from $\Xi^*(\rho_x)$ in the finer resolutions will characterize the extent to which ω is *non-equilibrium* even locally, and, for this purpose, the hierarchical structure inherent to $\mathcal{T}_x$ plays important roles. In terms of thermal functions $\Phi := \Xi(\hat{\Phi}(x)) \in \Xi(\mathcal{S}_x)$, the above (6) is rewritten as

$$\omega(\Phi)(x) := \omega(\hat{\Phi}(x)) = \rho_x(\Phi), \quad \Phi \in \Xi(\mathcal{S}_x), \tag{7}$$

which allows $\omega \equiv \Xi^*(\rho_x) \pmod{\mathcal{S}_x}$ to be solved in ρ_x as "$(\Xi^*)^{-1}$"$(\omega) = \rho_x \pmod{\Xi(\mathcal{S}_x)}$, providing

iv) local thermal interpretations of a non-equilibrium local state ω by means of the conditional *$q \rightarrow c$ channels* $(\Xi^*)^{-1}$, conditionally meaningful on the selected states [10]. We denote this similarly to (5) by

$$\mathcal{K}(\omega, \Xi^*(\rho))/\mathcal{S}_x \overset{q \rightleftarrows c}{\simeq} Th((\Xi^*)^{-1}(\omega), \rho)/\Xi(\mathcal{S}_x). \tag{8}$$

The reason for mentioning here the expression, *hierarchized zeroth law of local thermodynamics*, is that the above equalities presuppose some measuring processes of local thermal observables which involve and require the *contacts of two bodies*, measured object(s) and measuring device(s), in a local thermal equilibrium, conditional with respect to $\mathcal{S}_x$. The transitivity of this contact relation just corresponds to the localized and hierarchized version of the standard zeroth law of thermodynamics. In this way, we see that a selection criterion suitably set up can give a characterization of certain class of states and, at the same time, provide associated relevant physical interpretations of the selected states.

4 Reformulation of DHR-DR superselection theory

According to the above viewpoint, we can now reformulate the essence of the DHR- and DR-superselection theory whose essence can be summarized as follows. Here the basic ingredients of the theory with localizable charges [2, 4] are a *net* $\mathcal{K} \ni \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ of local W*-subalgebras $\mathfrak{A}(\mathcal{O})$ of *local observables*, defined on the set, $\mathcal{K} := \{(a+V_+) \cap (b-V_+); a, b \in \mathbb{R}^4\}$, of all double cones in the Minkowski spacetime $\mathbb{R}^4$; it is assumed to satisfy the isotony $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$, allowing the global (or, quasi-local) algebra of observables $\mathfrak{A} := C^*\text{-}\varinjlim_{\mathcal{K} \ni \mathcal{O} \rightarrow \mathbb{R}^4} \mathfrak{A}(\mathcal{O})$ to be defined as the C*-inductive limit, to transform covariantly under the action of the Poincaré group $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes L_+^\uparrow \ni (a, \Lambda) \longmapsto \alpha_{(a,\Lambda)} \in Aut(\mathfrak{A})$, $\alpha_{(a,\Lambda)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda(\mathcal{O}) + a)$, and to satisfy the local commutativity, $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ spacelike separated: $\forall x \in \mathcal{O}_1, \forall y \in \mathcal{O}_2$, $(x-y)^2 < 0$.

- A physically relevant state $\omega \in E_{\mathfrak{A}}$ around a pure vacuum state $\omega_0 \in E_{\mathfrak{A}}$ is characterized by the Doplicher-Haag-Roberts (DHR) **selection criterion** requiring for ω the existence of $\exists \mathcal{O} \in \mathcal{K}$ s.t. $\pi_\omega \upharpoonright_{\mathfrak{A}(\mathcal{O}')} \cong \pi_{\omega_0} \upharpoonright_{\mathfrak{A}(\mathcal{O}')}$, or, $\omega \upharpoonright_{\mathfrak{A}(\mathcal{O}')} = \omega_0 \upharpoonright_{\mathfrak{A}(\mathcal{O}')}$, where $\mathcal{O}' := \{x \in \mathbb{R}^4; (x-y)^2 < 0 \text{ for } \forall y \in \mathcal{O}\}$ is the causal complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O}') := C^*\text{-}\varinjlim_{\mathcal{K} \ni \mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)$. We denote this criterion as

$$\omega \equiv \omega_0 \ \ (\text{mod. } \mathfrak{A}(\mathcal{O}')). \tag{9}$$

- In the GNS-vacuum representation $(\pi_0, \mathfrak{H}_0)$ corresponding to ω_0, the validity of Haag duality,

$$\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))'', \tag{10}$$

  is assumed. On the basis of the standard postulates [2], the selection criterion (9) can be shown to be equivalent to the existence of a *local endomorphism* $\rho \in End(\mathfrak{A})$ such that $\omega = \omega_0 \circ \rho$, localized in some $\mathcal{O} \in \mathcal{K}$ in the sense of

$$\rho(A) = A \quad \text{ for } \forall A \in \mathfrak{A}(\mathcal{O}'). \tag{11}$$

- **DR-category** [5]: Then, a **C*-tensor category** $\mathcal{T}$ which we call here a DR-category is defined as a full subcategory of $End(\mathfrak{A})$ consisting of such ρ's as *objects* together with the intertwiners $T \in \mathfrak{A}$ from a ρ to another σ s.t. $T\rho(A) = \sigma(A)T$ as *morphisms*.
  Up to the technical details, the essence of Doplicher-Roberts superselection theory can be summarized in the following basic results due to the structure of $\mathcal{T}$ as a C*-tensor category equipped with the permutation symmetry, the operations of taking direct sums, subobjects and determinants, and the dominance of objects with determinant 1:

  - Unique existence of an *internal symmetry group* G such that

$$\mathcal{T} \simeq Rep_G \overset{\text{Tannaka-Krein duality}}{\longleftrightarrow} G = End_\otimes(V), \tag{12}$$

    where $End_\otimes(V)$ is defined as the group of natural unitary transformations $g = (g_\rho)_{\rho \in \mathcal{T}} : V \overset{\cdot}{\to} V$ from the C*-tensor functor $V : \mathcal{T} \hookrightarrow Hilb$ to itself [5, 13]:

$$\begin{array}{ccccc} \rho_1 & V_{\rho_1} & \overset{g_{\rho_1}}{\to} & V_{\rho_1} \\ T\downarrow & T\downarrow & \circlearrowleft & \downarrow T \\ \rho_2 & V_{\rho_2} & \overset{g_{\rho_2}}{\to} & V_{\rho_2} \end{array} . \tag{13}$$

Here, V embeds $\mathcal{T}$ into the category *Hilb* of Hilbert spaces and its image turns out to be just the category Rep_G of unitary representations (γ, V_γ) of a compact Lie group $G \subset SU(d)$, where the dimensionality d is intrinsically defined by $\mathcal{T}$ by the generating element $\rho \in \mathcal{T}$ [5]. In this formulation, the essence of the Tannaka-Krein duality [14] is found in the one-to-one correspondence,

$$\mathcal{T} \ni \rho = \rho_\gamma \longleftrightarrow \gamma = \gamma_\rho \in \hat{G}, \tag{14}$$

for $\rho \in \mathcal{T}$ satisfying $\rho(\mathfrak{A}) \cap \mathfrak{A} = \mathbb{C}\mathbf{1}$ (corresponding to the irreducibility of γ_ρ), and the identification $g_\rho = \gamma_\rho(g)$ for $g \in G$.

– Unique existence of a *field algebra*

$$\begin{aligned} \mathfrak{F} := \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d \qquad &\curvearrowleft G = Aut_{\mathfrak{A}}(\mathfrak{F}) = Gal(\mathfrak{F}/\mathfrak{A}) \\ &:= \{\tau \in Aut(\mathfrak{F}); \tau(A) = A, \ \forall A \in \mathfrak{A}\} \ (: \text{ Galois group}), \end{aligned} \tag{15}$$

with $\mathfrak{A} = \mathfrak{F}^G$ (fixed-point algebra), where $\mathcal{O}_d$ is the Cuntz algebra [15] defined as the unique simple C*-algebra consisting of d isometries ψ_i, $i = 1, 2, \cdots, d$,

$$\psi_i^* \psi_j = \delta_{ij}\mathbf{1}, \quad \sum_{i=1}^{d} \psi_i \psi_i^* = \mathbf{1}, \tag{16}$$

whose fixed-point subalgebra $\mathcal{O}_d^G$ is embedded into $\mathfrak{A}$, $\mu : \mathcal{O}_d^G \hookrightarrow \mathfrak{A}$, satisfying the relation $\mu \circ \sigma = \rho \circ \mu$ with respect to the canonical endomorphism σ of $\mathcal{O}_d$: $\sigma(C) := \sum_{i=1}^{d} \psi_i C \psi_i^*$ for $C \in \mathcal{O}_d$.

- The *superselection structure* in the irreducible vacuum representation $(\pi, \mathfrak{H})$ of the constructed field algebra $\mathfrak{F}$ is understood as follows: first, the group G of symmetry arising in this way is **unbroken** with a unitary implementer $U : G \to \mathcal{U}(\mathfrak{H})$, $\pi(\tau_g(F)) = U(g)\pi(F)U(g)^*$ and is *global* (i.e., gauge symmetry of the 1st kind) [due to the transportability in spacetime imposed on each $\rho \in \mathcal{T}$]. $\mathfrak{H}$ contains the starting Hilbert space $\mathfrak{H}_0$ of the vacuum representation π_0 of $\mathfrak{A}$ as a cyclic G-fixed point subspace, $\mathfrak{H}_0 = \mathfrak{H}^G = \{\xi \in \mathfrak{H}; U(g)\xi = \xi \text{ for } \forall g \in G\}$, $\overline{\pi(\mathfrak{F})\mathfrak{H}_0} = \mathfrak{H}$. Then, $\mathfrak{H}$ is decomposed into a direct sum in the following form [2],

$$\mathfrak{H} = \bigoplus_{\gamma \in \hat{G}} (\mathfrak{H}_\gamma \otimes V_\gamma), \tag{17}$$

$$\pi(\mathfrak{A}) = \bigoplus_{\gamma \in \hat{G}} (\pi_\gamma(\mathfrak{A}) \otimes \mathbf{1}_{V_\gamma}), \qquad U(G) = \bigoplus_{\gamma \in \hat{G}} (\mathbf{1}_{\mathfrak{H}_\gamma} \otimes \gamma(G)), \tag{18}$$

where **superselection sectors** defined as mutually disjoint irreducible representations $(\pi_\gamma, \mathfrak{H}_\gamma)$ of $\mathfrak{A}$ are in one-to-one correspondence, $\pi_\gamma = \pi_0 \circ \rho_\gamma \longleftrightarrow \rho_\gamma \in \mathcal{T} \longleftrightarrow (\gamma, V_\gamma)$, with mutually disjoint irreducible unitary representations (γ, V_γ) of G, the totality of which just constitutes the group dual $\hat{G}$.

Now, introducing the map W defined by

$$W : End(\mathfrak{A}) \ni \rho \longmapsto \omega_0 \circ \rho \in E_{\mathfrak{A}}, \tag{19}$$

we reformulate the essence of the above DHR-DR theory in a form parallel to the previous sections. First, the role of the DHR criterion is easily seen to make the map $W : End(\mathfrak{A}) \to E_{\mathfrak{A}}$ *invertible* on the subset E_{DHR} selected by it:

$$\begin{aligned} E_{DHR} &:= \{\omega \in E_{\mathfrak{A}}; \exists \mathcal{O} \in \mathcal{K} \text{ s.t. } \omega \equiv \omega_0 \ (\text{mod.}\mathfrak{A}(\mathcal{O}'))\} \\ &\ni \omega = \omega_0 \circ \rho \longmapsto \rho \in \mathcal{T} \subset End(\mathfrak{A}), \end{aligned} \tag{20}$$

which yields an alternative description of states $\omega \in E_{DHR}$ in terms of C*-tensor category $\mathcal{T}$ of local endomorphisms. What is important about (18) is that it means the existence of a non-trivial centre of $\pi(\mathfrak{A})''$,

$$\mathfrak{Z}(\pi(\mathfrak{A})'') = \bigoplus_{\gamma \in \hat{G}} \mathbb{C}(\mathbf{1}_{\mathfrak{H}_\gamma} \otimes \mathbf{1}_{V_\gamma}) = C(\hat{G}) = \mathfrak{Z}(U(G)''). \tag{21}$$

In view of the one-to-one correspondence (14) due to the isomorphism, $\mathcal{T} \simeq Rep_G$, we define a unital positive linear map $k : \mathfrak{A} \ni A \longmapsto k(A) \in \mathfrak{Z}(\pi(\mathfrak{A})'') = C(\hat{G})$ by

$$[k(A)](\gamma) := \omega_0(\rho_\gamma(A)) = W(\rho_\gamma)(A), \tag{22}$$

whose positivity and the normalization are evident: $[k(A^*A)](\gamma) =$
$= \omega_0(\rho_\gamma)(A^*A) \geq 0$, $[k(\mathbf{1})](\gamma) = \omega_0(\rho_\gamma)(\mathbf{1}) = 1$. Thus, similarly to the $c{\to}q$ channel Ξ^* in Sec.2, the dual of a completely positive (CP) map k defines a $c{\to}q$ channel $k^* : M_1(\hat{G}) \to E_{\mathfrak{A}}$, given for each probability weight $\nu = (\nu_\gamma)_{\gamma \in \hat{G}} \in M_1(\hat{G})$, $\nu_\gamma \geq 0$, $\sum_{\gamma \in \hat{G}} \nu_\gamma = 1$, by

$$\begin{aligned} k^*(\nu)(A) &= \nu(k(A)) = \sum_{\gamma \in \hat{G}} \nu_\gamma [k(A)](\gamma) \\ &= \sum_{\gamma \in \hat{G}} \nu_\gamma \omega_0(\rho_\gamma(A)) = (\sum_{\gamma \in \hat{G}} \nu_\gamma \omega_\gamma)(A) \\ &\Longrightarrow k^*(\nu) = \sum_{\gamma \in \hat{G}} \nu_\gamma \omega_\gamma = \sum_{\gamma \in \hat{G}} \nu_\gamma W(\rho_\gamma), \end{aligned} \tag{23}$$

where $\omega_\gamma := \omega_0 \circ \rho_\gamma$. Therefore, the map k^* extends W to "convex combinations" of ρ_γ's, and is a "*charging* map" to create from the vacuum ω_0 a state $k^*(\nu) = \sum_{\gamma\in\hat{G}} \nu_\gamma(\omega\circ\rho_\gamma)$ whose charge contents are described by the charge distribution $\nu = (\nu_\gamma)_{\gamma\in\hat{G}} \in M_1(\hat{G})$ over the group dual $\hat{G}$. While this picture clearly shows the parallelism with the previous discussion of the thermal interpretation based upon the $c\to q$ channel $\Xi : \mathcal{A} \to C(V_+)$, a different but equivalent formulation will be more familiar for describing the same situation in use of a reducible representation $\gamma_\nu := \oplus_{\gamma\in\hat{G},\nu_\gamma\neq 0}\gamma \in RepG$ as follows. For this purpose, we use the invariance of the vacuum state under $U(G)$ which implies the following relations in terms of the conditional expectation $m : \mathfrak{F} \ni F \longmapsto m(F) := \int_G dg\tau_g(F) \in \mathfrak{A}$ coming from the average over G by the Haar measure dg :

$$\omega_\gamma(m(F)) = (\omega_0\circ\rho_\gamma)(m(F))) = \langle\Omega_0 \mid \sum_i \psi_i^\gamma m(F)\psi_i^{\gamma*}\Omega_0\rangle, \tag{24}$$

where the last expression is understood in the representation space $\mathfrak{H}$ of $\mathfrak{F}$ and $\psi_i^\gamma \in \mathfrak{F}$ are such that $\psi_i^\gamma\pi(A) = \pi\circ\rho_\gamma(A)\psi_i^\gamma$ for $\forall A\in\mathfrak{A}$. Then, owing to the disjointness among different sectors, the state $k^*(\nu)$ can be rewritten as an induced state $k^*(\nu)\circ m$ of $\mathfrak{F}$ by

$$\begin{aligned} k^*(\nu)(m(F)) &= \sum_{\gamma\in\hat{G}} \nu_\gamma\omega_\gamma(m(F)) = \sum_{\gamma\in\hat{G}}\sum_i \langle\sqrt{\nu_\gamma}\psi_i^{\gamma*}\Omega_0 \mid m(F)\sqrt{\nu_\gamma}\psi_i^{\gamma*}\Omega_0\rangle \\ &= \langle\Psi \mid m(F)\Psi\rangle = \langle\Psi \mid F\Psi\rangle, \end{aligned} \tag{25}$$

with a vector $\Psi := \sum_{\gamma\in\hat{G}}\sum_i \sqrt{\nu_\gamma}\psi_i^{\gamma*}\Omega_0 \in \mathfrak{H}$ belonging to the above mentioned reducible representation $\gamma_\nu := \oplus_{\gamma\in\hat{G},\nu_\gamma\neq 0}\gamma$ of G. Thus, on the basis of the basic results of DHR-DR theory, we have the inverse map $(k^*)^{-1}$ on E_{DHR} as a $q\to c$ channel, which provides the interpretation of a state $\omega \in E_{DHR}$ selected out by the DHR criterion, with respect to their internal-symmetry aspects, specifying its ***G-charge contents*** understood as the G-representation contents. Since the spacetime behaviours of quantum fields are expressed by the observable net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ and since the internal symmetry aspects are described in the above machinery also encoded in $\mathfrak{A}$, the role of the field algebra $\mathfrak{F}$ and the internal symmetry group G seems to be quite subsidiary, simply providing comprehensible vocabulary based on the covariant objects under the symmetry transformaions.

According to the naive physical picture, contents of the sectors parametrized by $\gamma\in\hat{G}$, $\gamma\neq\iota$(: the trivial representation corresponding to the vacuum sector) are *excited states* above the vacuum. What is lacking in the stardard theory of superselection sectors is this kind of problems concerning the ***energy contents of sectors***, namely, the mutual relations between the energy-momentum spectrum and the sectors as internal-symmetry spectrum: since only the sector of the trivial representation $\iota\in\hat{G}$ contains the vacuum state with the *minimum energy* 0 and since all other sectors consist of the excited states, the above picture suggests the following situations to be valid:

$$\min\{Spec(\hat{P}_0 \restriction_{\mathfrak{H}_0})\} = 0, \qquad \inf\{Spec(\hat{P}_0 \restriction_{\mathfrak{H}_0^{\perp}})\} > 0. \tag{26}$$

In the treatment of thermal functions in Sec.2, it is easily seen that, while the entropy density $s(\beta)$ is not contained in the image set $\Xi(\mathcal{T}_x)$ due to the absence of such a quantum observable $\hat{s}(x) \in \mathcal{T}_x$ that $\omega_\beta(\hat{s}(x)) = s(\beta)$, it can be approximated by the thermal functions belonging to $\Xi(\mathcal{T}_x)$. In order to facilitate the discussions of problems of this sort, it is important to have those observables freely at hand which detect the G-charge contents in $\mathfrak{Z}(\pi(\mathfrak{A})'') = C(\hat{G})$, and, for this purpose, we need also here to consider the problem as to how such observables can be supplied from the *local* observables belonging to $\mathfrak{A}(\mathcal{O})$, i.e., the approximation of order parameters by order fields or central sequences. For this purpose, the analyses of point-like fields and the rigorous method of their operator-product expansions developed by Bostelmann in [12] would be quite useful in these contexts. All the above sort of considerations (with the modifications of the DHR selection criterion necessitated by the possible presence of the long range forces, such as of Buchholz-Fredenhagen type) will be crucially relevant to the approach to the colour confinement problem, and, especially the latter one seems to be quite a non-trivial issue there.

5 Selection criteria as categorical adjunctions

Here we emphasize the important roles played by the *categorical adjunctions* underlying our discussions so far, in achieving the systematic organizations of various different theories in physics, controlled by the selection criteria to characterize specific physical domains so that the physical interpretations are automatically afforded. While the two formal formulae (5) and (8) mentioned at the end of Sec.2 and Sec.3 were just for the sake of suggesting such relevance of the adjunctions, our selection criteria, expressed in terms of states, may not directly look like a familiar form of categorical adjunctions [13]:

$$A(F(x), a) \simeq X(x, G(a)), \tag{27}$$

involving two functors, a forgetful functor $G : A \to X$, and its left adjoint $F : X \to A$. This is mainly because the category consisting of states of (C*-)algebras is a rather rigid one, allowing only few meaningful morphisms among different objects, requiring a strict equality, or, at most, some closeness w.r.t. a norm or seminorms. For this reason, the essence of adjunctions in our context mostly is found in such a quantitative form as

$$\begin{array}{ccc} X\ (=\mathrm{q})\text{: to be classified} & q\leftarrow c & A\ (=\mathrm{c})\text{: to classify} \\ x\equiv G(a)\ (\mathrm{mod}.X) & \underset{F}{\overset{G}{\leftrightarrows}} & F(x)\equiv a\ (\mathrm{mod}.A) \\ \text{as selection criterion} & q\rightarrow c & \text{interpretation} \end{array} \quad , \qquad (28)$$

with A a classical classifying space as the spectrum of centre, X a quantum domain (of states) to be classified, G the *c→q channel* and with F the *q→c channel* to provide the interpretation of X in terms of the vocabulary in A.

As we have seen above, once the contents of imposed selection criteria are paraphrased into different languages, such as thermal functions in Sec.2 and local endomorphisms in Sec.4, then the standard machinery stored in the category theory starts to work, in such a form as intertwiners among local endomorphisms or among group representations, which provides us with powerful tools to analyze the given structures, as seen in Sec.4 and II; Sec.3.

For decoding the deep messages encoded in a selection criterion, what plays the decisive roles at the first stage is the identification of the *centre* of a representation containing universally all the selected quantum states; its spectrum provides us with the information on the structure of the associated superselection sectors, which serves as the vocabulary to be used when the interpretations of a given quantum state are presented. The necessary bridge between the selected generic quantum states and the classical familar objects living on the above centre is provided, in one direction, by the *c→q channel* which embeds all the known classical states (=probability measures) into the form of quantum states constituting the totality of the selected states by the starting selection criterion. The achieved identification between what is selected and what is embedded from the known world is nothing but the most important consequence of the categorical adjunction formulated in the form of selection criterion. This automatically enables us to take the inverse of the *c→q channel* which brings in another most important ingredient, the *q→c channel* to decode the physical contents of selected states from the viewpoint of those aspects selected out by the starting criterion. Mathematically speaking, the spectrum of the above centre is nothing but the *classifying space* universally appearing in the geometrical contexts, for instance, in Sec.4 of DR superselection theory of unbroken symmetry described by a compact Lie group G, its dual $\hat{G}$ (of all the irreducible unitary representations) is such a case, $\hat{G} = B_{\mathcal{T}}$ for $\mathcal{T}$ the DR category of local endomorphisms of the observable net, where our *q→c channel* $(k^*)^{-1}$ plays the role of the *classifying map* by embedding the G-representation contents of a given quantum state into the subset of $\hat{G}$ consisting of its irreducible components:

$$\begin{array}{ccc}\coprod_{\gamma\in M} V_\gamma = E & & (\pi_u, \bigoplus_{\gamma\in\hat{G}} V_\gamma)\text{: all the sectors} \\ \downarrow_{RepG} & & \downarrow_{RepG} \\ (RepG\supset) M & \overset{(k^*)^{-1}}{\hookrightarrow} & \hat{G} = B_{\mathcal{T}} \end{array} . \tag{29}$$

From this viewpoint, the present scheme can easily be related with many current topics concerning the geometric and classification aspects of commutative as well as non-commutative geometry based upon the (homotopical) notions of classifying spaces, K-theory and so on.

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