GEOMETRIC AND ARITHMETIC SUBGROUPS OF THE GROTHENDIECK-TEICHMÜLLER GROUP

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Abstract. We compare two geometrically constructed subgroups $\mathbb{I}\Gamma$ and GTK of the Grothendieck-Teichmüller group $\widehat{GT}$ with an arithmetically constructed subgroup GTA. We show that the intersection of $\mathbb{I}\Gamma$ and GTK is contained in a certain modification of GTA.

0. Introduction

The Grothendieck-Teichmüller group $\widehat{GT}$ is a subgroup of the automorphism group $Aut\widehat{F_2}$ of the free pro-finite group of $\widehat{F_2}$ of rank 2. It is parameterized by elements of $\widehat{\mathbf{Z}}^\times \times [\widehat{F_2}, \widehat{F_2}]$ and is defined by three relations (I), (II) and (III) (see §1). It is a pro-finite group version of the pro-algebraic group $\underline{GT}$ [Dr]. In his study of Galois representations on fundamental groups, Y. Ihara showed that the absolute Galois group $G_{\mathbf{Q}} = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ can be embedded into $\widehat{GT}$ in [Ih94]. It is still open whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not. But recently there have been discovered other several new-type relations satisfied by $G_{\mathbf{Q}}$ in $\widehat{GT}$ (see [Ih00], [LNS] and [NT]). In this article, we show a certain relationship among them. In §1, we shall recall the definition of $\widehat{GT}$ and make a small remark (*Proposition* 1.3) which has been possibly unknown so far. §2 is a review of the definitions of three variants $\mathbb{I}\Gamma$ ([LNS]), GTK ([Ih00]) and GTA ([Ih00]) of $\widehat{GT}$. In §3, we introduce main results of the author's master's thesis [F], concerning on a relationship among defining equations of $\mathbb{I}\Gamma$, GTK and GTA. In §4, we give many lemmas to complete their proof.

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1. Review of the definition of $\widehat{GT}$

Let $\widehat{F_2}$ be the pro-finite free group of rank 2 with generators x and y. We define $\underline{\widehat{GT}}$ to be the set of pairs $(\lambda, f) \in \hat{\mathbf{Z}}^\times \times \widehat{F_2}'$ (where $\widehat{F_2}'$ means the topological commutator subgroup $[\widehat{F_2}, \widehat{F_2}]$ of $\widehat{F_2}$) satisfying the following three relations:

$$\begin{cases} \text{(I)} \ \ f(x,y)f(y,x) = 1 & \text{((2-cycle relation))} \\ \text{(II)} \ \ f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1 & \\ \qquad\qquad \text{for } m = \frac{1}{2}(\lambda-1), \ xyz = 1 & \text{((3-cycle relation))} \\ \text{(III)} \ f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51}) = 1 \quad \text{in } \hat{P}_5^* & \\ & \text{((5-cycle relation)).} \end{cases}$$

Here $\hat{P}_n^*$ is the pro-finite pure sphere braid group with n strings and $x_{i,j} := x_{i,j}^{(n)}$ $(1 \leqslant i, j \leqslant n)$ are its standard generators [Ih94]. For $f \in \widehat{F_2}$ and elements α, β of a pro-finite group H, $f(\alpha, \beta)$ stands for the image of f by the homomorphism $\phi : \widehat{F_2} \to H$ defined by $\phi(x) = \alpha$, $\phi(y) = \beta$. An element $\sigma = (\lambda, f) \in \underline{\widehat{GT}}$ induces an endomorphism of $\widehat{F_2}$ by $\sigma(x) = x^\lambda$, $\sigma(y) = f^{-1}y^\lambda f$, from which we get an embedding $\underline{\widehat{GT}} \hookrightarrow End\widehat{F_2}$ and in fact we can regard $\underline{\widehat{GT}}$ as a sub-monoid of $End\widehat{F_2}$ ([Dr]).

Definition 1.1 ([Dr])**.** The *Grothendieck-Teichmüller group* $\widehat{GT}$ is the group of invertible elements of $\underline{\widehat{GT}}$:

$$\widehat{GT} := \left\{ \sigma = (\lambda, f) \in Aut\ \widehat{F_2} \ \mid (\lambda, f) \text{ satisfies (I)} \sim \text{(III).} \right\}.$$

Remark 1.2. The above 5-cycle relation (III) is different from the original relation of $\widehat{GT}$ ([Dr])

$$\text{(III)}_{\text{DR}} \quad f(x_{12}^{(4)}, x_{23}^{(4)}x_{24}^{(4)})f(x_{13}^{(4)}x_{23}^{(4)}, x_{34}^{(4)}) \\ = f(x_{23}^{(4)}, x_{34}^{(4)})f(x_{12}^{(4)}x_{13}^{(4)}, x_{24}^{(4)}x_{34}^{(4)})f(x_{12}^{(4)}, x_{23}^{(4)}) \quad \text{in } \hat{B}_4$$

which appeared in [NT], where $\hat{B}_n$ $(n \in \mathbf{N})$ stands for the pro-finite braid group with n-strings. But we can easily show that (I)+(III) is equivalent to (I)+(III)$_{\text{DR}}$.

Proposition 1.3. *Relation* (III) *implies relation* (I) *.*

Proof . Recall that we have a basic projection $p : \hat{P}_5^* \to \hat{P}_4^*$ which sends $x_{i,j} = x_{i,j}^{(5)} \in \hat{P}_5^*$ to $x_{i,j}^{(4)} \in \hat{P}_4^*$ $(1 \leqslant i, j \leqslant 4)$ and $x_{i,5} \in \hat{P}_5^*$ to

$1 \in \hat{P}_4^*$ $(1 \leqslant i \leqslant 5)$. It is immediate to see that (III) implies (I) because

$$p\big(f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})\big)$$
$$= f(x_{12}^{(4)},x_{23}^{(4)})f(x_{23}^{(4)},x_{34}^{(4)}) = f(x_{12}^{(4)},x_{23}^{(4)})f(x_{23}^{(4)},x_{12}^{(4)})$$

and $\hat{P}_4^*$ is a free pro-finite group of rank 2 with generators $x_{12}^{(4)}$ and $x_{23}^{(4)}$. □

With the $G_{\mathbf{Q}}$-action on the geometric fundamental group $\pi_1(\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0,1,\infty\}, \overrightarrow{01})$ of the projective line minus 3 points (with respect to a certain tangential base point $\overrightarrow{01}$), we associate a pro-finite group homomorphism $\varphi : G_{\mathbf{Q}} \to Aut\widehat{F_2}$ (recall that $\widehat{F_2} \simeq \pi_1(\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0,1,\infty\}, \overrightarrow{01})$). It follows from Belyĭ's theorem [Be] that φ is injective. In [Ih90] and [Ih94], it was shown that $\varphi(G_{\mathbf{Q}})$ lies in $\widehat{GT}$. Now to determine whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not is a basic open problem, which is also related to a project posed by A. Grothendieck in [Gr]. Recently there have been discovered other new-type relations which $G_{\mathbf{Q}}$ satisfies in $\widehat{GT}$, for example (I′), (II′), (III′), (IV), (IV′), (V), (A$_n$), (K$_n$) for $n = 1, 2, 3, \ldots$ (see [Ih00], [LNS], [NS] and [NT]). Although they do not seem to be deduced from defining relations (I), (II) and (III) of $\widehat{GT}$, it has not been shown yet whether they are really new ones and whether they are enough to characterize $G_{\mathbf{Q}}$ as a subgroup of $\widehat{GT}$.

2. Review of the definitions of $\mathbb{I}\Gamma$, GTK and GTA

Three subgroups $\mathbb{I}\Gamma$, GTK and GTA of $\widehat{GT}$ were introduced in [Ih00] and [LNS]. The subgroup $\mathbb{I}\Gamma$ (resp. GTK) was geometrically constructed by P. Lochak, H. Nakamura, and L. Schneps in [LNS] (resp. by Y. Ihara in [Ih00]). On the other hand, GTA was arithmetically constructed by Y. Ihara in [Ih00]. They all contain $G_{\mathbf{Q}}$, but it has not been known whether they are really proper subgroups of $\widehat{GT}$ and whether they are equal to $G_{\mathbf{Q}}$.

2.1. $\mathbb{I}\Gamma$.

Definition 2.1 ([LNS]). The *new Grothendieck-Teichmüller group* $\mathbb{I}\Gamma$ is the subset of $\widehat{GT}$ defined as follows:

$$\mathbb{I}\Gamma := \Big\{\sigma = (\lambda, f) \in \widehat{GT} \mid (\lambda, f) \text{ satisfies (III}'\text{) and (IV) below.}\Big\}.$$

$$\begin{cases} (\mathrm{III}') \quad g(x_{45},x_{51})f(x_{12},x_{23})f(x_{34},x_{45}) = f(\sigma_1\sigma_3,\sigma_2^2) & \text{in } \hat{B}_5 \\ (\mathrm{IV}) \quad f(\sigma_1,\sigma_2^4) = \sigma_2^{-8\Psi_2^{(0)}(\sigma)} f(\sigma_1^2,\sigma_2^2)\sigma_1^{-4\Psi_2^{(0)}(\sigma)}(\sigma_2\sigma_1)^{6\Psi_2^{(0)}(\sigma)} & \text{in } \hat{B}_3. \end{cases}$$

Here $g(x,y)\in\widehat{F_2}$ is the auxiliary parameter (depending on $\sigma\in\widehat{GT}$) satisfying $f(x,y)=g(y,x)^{-1}g(x,y)$, which was introduced in [LS]. For $n\in\mathbf{N}$, σ_i $(1\leqslant i<n)$ stand for standard generators of $\hat{B}_n$ ([LNS]). For the definition of $\Psi_2^{(0)}(\sigma)^\dagger$, see §§2.2. In [LNS] and [NS], it was shown that actually $\mathbb{\Gamma}$ forms a subgroup of $\widehat{GT}$. Note that (III′) implies (III) ([NS]). These two relations (III′) and (IV) just describe the condition for elements of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on all types of pro-finite Teichmüller modular groups in a certain consistent way (for more details, see [LNS]).

2.2. *GTK***.** For a natural number n, let H_n be the index n normal subgroup of $\widehat{F_2}$ which is freely generated by $n+1$ elements x^n, y, $x^{-1}yx,\cdots,x^{-(n-1)}yx^{n-1}$. For $\sigma=(\lambda,f)\in\widehat{GT}$, f belongs to H_n, since $H_n\supset[\widehat{F_2},\widehat{F_2}]$. In [Ih99], Ihara constructed the extended Kummer 1-cocycle $\Psi_n^{(0)}(\sigma)$ for $\sigma\in\widehat{GT}$, which is the image of f by the continuous group homomorphism $H_n\to\widehat{\mathbf{Z}}$ defined by $x^n\mapsto 0$, $y\mapsto 1$, $x^{-j}yx^j\mapsto 0$ $(1\leqslant j<n)$. We remark that especially for $\sigma\in G_{\mathbf{Q}}$, $\Psi_n^{(0)}(\sigma)$ is the Kummer 1-cocycle which is characterized by $\sigma(\sqrt[k]{n})=\sqrt[k]{n}\zeta_k^{-\Psi_n^{(0)}(\sigma)}$ for $k\in\mathbf{N}$, where $\zeta_k=exp(\frac{2\pi i}{k})$. Suppose that $\varphi_n: H_n\to\widehat{F_2}$ is the continuous group homomorphism defined by $x^n\mapsto x$, $y\mapsto y$, $x^{-j}yx^j\mapsto 1$ $(1\leqslant j<n)$.

Definition 2.2 ([Ih00])**.** The *Grothendieck-Teichmüller-Kummer group* GTK is the subset of $\widehat{GT}$ defined as follows:

$$GTK_n:=\left\{\sigma=(\lambda,f)\in\widehat{GT}\ \mid (\lambda,f)\text{ satisfies }(\mathrm{K}_n)\text{ below.}\right\}$$

$$GTK:=\bigcap_{n\in\mathbf{N}}GTK_n$$

$$(\mathrm{K}_n)\qquad \varphi_n(f)=y^{\Psi_n^{(0)}(\sigma)}f\qquad .$$

It follows immediately from [Ih00] *Proposition 1* that GTK_n and GTK actually form subgroups of $\widehat{GT}$. Relation (K_n) just describes the condition for elements of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on H_n and $\widehat{F_2}$ consistently with two algebraic morphisms, the Kummer covering $\mathbf{P}^1_{\mathbf{Q}}-\{0,\mu_n,\infty\}\twoheadrightarrow\mathbf{P}^1_{\mathbf{Q}}-\{0,1,\infty\}$ defined by $t\mapsto t^n$ and the natural inclusion $\mathbf{P}^1_{\mathbf{Q}}-\{0,\mu_n,\infty\}\hookrightarrow\mathbf{P}^1_{\mathbf{Q}}-\{0,1,\infty\}$ defined by $t\mapsto t$ (for more details, see [Ih00]).

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†The 1-cocycles $\rho_2(\sigma)$ and $\rho_3(\sigma)$ studied in [LNS], [LS], [NS] and [NT] are equal to $-\Psi_2^{(0)}(\sigma)$ and $-\Psi_3^{(0)}(\sigma)$ respectively (see [NT]).

2.3. *GTA***.** By the $\widehat{F_2}^{ab}(:= \widehat{F_2}/\widehat{F_2}')$-action on $(\widehat{F_2}')^{ab} := \widehat{F_2}'/[\widehat{F_2}', \widehat{F_2}']$ induced from the conjugation $n \mapsto fnf^{-1}$ for $n \in \widehat{F_2}'$ and $f \in \widehat{F_2}$, we can regard $(\widehat{F_2}')^{ab}$ as a free A_2 $(:= \widehat{\mathbf{Z}}[[\widehat{F_2}ab]])$-module of rank 1, generated by the class $\overline{[x,y]} \in (\widehat{F_2}')^{ab}$ of $[x,y] := xyx^{-1}y^{-1} \in \widehat{F_2}'$ (for more details, see [Ih99]). Thus the action of $\sigma = (\lambda, f) \in \widehat{GT}$ on $(\widehat{F_2}')^{ab}$ induced from that on $\widehat{F_2}$ is expressed as $\sigma(\overline{[x,y]}) = B'_\sigma \cdot \overline{[x,y]}$, where $B'_\sigma \in A_2^\times$. The *adelic beta function* $B_\sigma \in A_2^\times$ was defined by $B'_\sigma = \frac{x^\lambda - 1}{x-1}\frac{y^\lambda-1}{y-1}B_\sigma$ in [A] (for $\sigma \in G_{\mathbf{Q}}$) and [Ih99] (for $\sigma \in \widehat{GT}$). By the embedding constructed by G. W. Anderson in [A], B_σ can be regarded as a function on $(\mathbf{Q}/\mathbf{Z})^{\oplus 2}$ valued in $\widehat{\mathbf{Z}} \otimes \mathbf{Q}^{ab}$, where $\mathbf{Q}^{ab}$ stands for the maximal abelian extension field over $\mathbf{Q}$. In [Ih99] *Proposition 1.6.1*, it was shown that the adelic beta function has much analogy with the classical beta function. Especially it is remarkable that by using the 5-cycle relation (III) Ihara showed that the adelic beta function $B_\sigma(s_1, s_2)$ ($\sigma \in \widehat{GT}$, $(s_1, s_2) \in (\mathbf{Q}/\mathbf{Z})^{\oplus 2}$) can be split (but not uniquely) into the following product: $B_\sigma(s_1, s_2) = \frac{\Gamma_\sigma(s_1)\Gamma_\sigma(s_2)}{\Gamma_\sigma(s_1+s_2)}$. Here the *adelic gamma function* Γ_σ is a function (uniquely determined up to a certain ambiguity) which is defined on $\mathbf{Q}/\mathbf{Z}$ and is valued in the product $\prod_{p:\text{prime}} \mathbb{W}(\overline{\mathbf{F}}_p)$ of the Witt vector ring $\mathbb{W}(\overline{\mathbf{F}}_p)$ of the algebraic closure of a finite field of characteristic p. In [A](i) *Corollary 8.6.3*, Anderson showed, as an analogy of Gauß' n-th multiplication formula of the classical gamma function, the n-th multiplication formula of the adelic gamma function Γ_σ (*for elements* σ *of* $G_{\mathbf{Q}}$):

$$(\mathrm{A}_n) \qquad \prod_{nc=0} \frac{\Gamma_\sigma(s+c)}{\Gamma_\sigma(c)} \cdot \frac{1}{\Gamma_\sigma(ns)} = 1 \otimes exp\big[2\pi i \cdot n\Psi_n^{(0)}(\sigma)s\big]$$

by using Deligne's theory of absolute Hodge cycles. Ihara suggested that this arithmetic relation (A_n) could be a key condition to distinguish $G_{\mathbf{Q}}$ from $\widehat{GT}$ and considered the following new subgroup of $\widehat{GT}$ containing $G_{\mathbf{Q}}$.

Definition 2.3 ([Ih00])**.** The *Grothendieck-Teichmüller-Anderson group* GTA is the subset of $\widehat{GT}$ defined as follows:

$$GTA_n := \Big\{\sigma = (\lambda, f) \in \widehat{GT} \ \mid \sigma \text{ satisfies } (\mathrm{A}_n). \Big\}$$
$$GTA := \bigcap_{n \in \mathbf{N}} GTA_n \ .$$

Remark 2.4. To state (A_n) independently from (I)∼(III), it is better to re-formulate it as follows:

$$(\mathrm{A}_n^0) \qquad \prod_{0\leqslant k\leqslant n-1} B_\sigma(s,ks) \Big/ \prod_{nc=0} B_\sigma(c,s) = 1\otimes exp\big[2\pi i\cdot n\Psi_n^{(0)}(\sigma)s\big]$$

because the existence of Γ_σ depends on (I)∼(III). Here we use $\Gamma_\sigma(0)=1\otimes 1$ ([Ih99]Proposition 1.7.1.(i)).

It can be checked directly from the definitions of $\Psi_n^{(0)}(\sigma)$ and $\Gamma_\sigma(s)$ that GTA and GTA_n actually form subgroups of $\widehat{GT}$.

The relationship among the above three subgroups $\mathbb{I}\Gamma$, GTK and GTA has not been fully understood yet. But we remark that it was shown in [Ih99] that relation (K_n) implies $(Dlog\mathrm{A}_n)$, the logarithmic derivative of equation (A_n).

3. Main results

Theorem 3.1. *The combination of relations* (I) , (II), (IV) *and* (K_2) *imply relation* (A_2^0).

Proof . At first, we introduce a new parameter f_+ in the rank 3 free group $\widehat{F_3}$ with generators W, X and Y. Recall that the index 2 subgroup H_2 (§§2.2) of $\widehat{F_2}$ generated by xyx^{-1}, x^2 and y can be identified with $\widehat{F_3}$ by sending xyx^{-1}, x^2, y into W, X, Y respectively. Since $y^{-2\Psi_2^{(0)}(\sigma)}f$ lies on H_2, we obtain a unique element $f_+(W,X,Y)$ of $\widehat{F_3}$ such that

$$y^{-2\Psi_2^{(0)}(\sigma)}f(x,y)=f_+(xyx^{-1},x^2,y). \tag{1}$$

We note that $W^{\Psi_2^{(0)}(\sigma)}Y^{\Psi_2^{(0)}(\sigma)}f_+\in[\widehat{F_3},\widehat{F_3}]$. It is immediate that relation (K_2) is equivalent to

$$y^{\Psi_2^{(0)}(\sigma)}f_+(xyx^{-1},x^2,y)=f_+(1,x,y)\ . \tag{2}$$

Here 1 stands for the unit element of $\widehat{F_2}$. In Lemma 4.1, we shall see that (IV) is re-expressed in terms of $f_+\in\widehat{F_3}$ as follows:

$$f_+(W,X,Y)f_+(X^{-1}W^{-1}Y^{-1},Y,X)=1\ . \tag{3}$$

In Lemma 4.2, we deduce the following equation from (I), (II) and (3):

$$\begin{aligned} Y^m f_+(W,X,Y)X^m f_+(Z,W,X)W^m f_+(Y,Z,W)\\ Z^m f_+(X,Y,Z)=1 \qquad \text{for} \quad WXYZ=1 \end{aligned} \tag{4}$$

Translating (A_2^0) into the terms of $A_2 = \widehat{\mathbf{Z}}[[\widehat{F_2}^{ab}]]$ via Anderson's embedding (see §2.3), we get

$$(\mathrm{A}_2^0) \quad \frac{B_\sigma(s,s)}{B_\sigma(\frac{1}{2},s)} = 1 \otimes exp\big[2\pi i \cdot \Psi_2^{(0)}(\sigma) s\big]$$
$$\Longleftrightarrow B_\sigma\{\mathbf{x},\mathbf{x}\} = \mathbf{x}^{-2\Psi_2^{(0)}(\sigma)} B_\sigma\{-1,\mathbf{x}\}.$$

Here $B_\sigma\{\mathbf{x},\mathbf{x}\}$ (resp. $B_\sigma\{-1,\mathbf{x}\}$) stands for the image of $B_\sigma \in A_2$ by the map $A_2 \to A_2$ induced from $\mathbf{x} \to \mathbf{x}$, $\mathbf{y} \to \mathbf{x}$ (resp. $\mathbf{x} \to -1$, $\mathbf{y} \to \mathbf{x}$), where boldfaces $\mathbf{x}$, $\mathbf{y}$ stand for the images in A_2 of standard generators x, y of $\widehat{F_2}$. By properties [II]∼[IV] of [Ih99]§2.3 which were deduced from (I) and (II), we get

$$(\mathrm{A}_2^0) \Longleftrightarrow \frac{\mathbf{x}^\lambda + 1}{\mathbf{x}^m(\mathbf{x}+1)} B_\sigma\{\mathbf{x}, \frac{1}{\mathbf{x}^2}\} = \mathbf{x}^{-2\Psi_2^{(0)}(\sigma)} B_\sigma\{-1,\mathbf{x}\}.$$

By [Ih99]Proposition 2.2.2,

$$(\mathrm{A}_2^0) \Longleftrightarrow \frac{\mathbf{x}^\lambda + 1}{\mathbf{x}^m(\mathbf{x}+1)} \Big\{1 - (\mathbf{x}-1)\frac{df}{dx}(\mathbf{x}, \frac{1}{\mathbf{x}^2})\Big\} = \mathbf{x}^{-2\Psi_2^{(0)}(\sigma)} \Big\{1 + 2\frac{df}{dx}(-1,\mathbf{x})\Big\}.$$

Here we denote the image of $\frac{df}{dx} \in \Lambda_2 := \widehat{\mathbf{Z}}[[\widehat{F_2}]]$ by the map $\Lambda_2 \to A_2$ induced from $\mathbf{x} \mapsto -1$, $\mathbf{y} \mapsto \mathbf{x}$ by $\frac{df}{dx}(-1,\mathbf{x})$ (for the definition of $\frac{df}{dx} \in \Lambda_2$, see §4). For $\alpha \in \Lambda_2$ and $\mathbf{u}, \mathbf{v} \in \widehat{F_2}^{ab}$, we denote the image of α by the map $\Lambda_2 \to A_2$ induced from $x \to \mathbf{u}$, $y \to \mathbf{v}$ by $\alpha(\mathbf{u},\mathbf{v}) \in A_2$. By Lemma 4.3, which will be deduced from (1), we get

$$\begin{aligned}(\mathrm{A}_2^0) \Longleftrightarrow & \frac{\mathbf{x}^\lambda + 1}{\mathbf{x}^m(\mathbf{x}+1)} \Big[1 - (\mathbf{x}-1)\Big\{\mathbf{x}^{-2\Psi_2^{(0)}(\sigma)} \frac{df_+}{dW}(\frac{1}{\mathbf{x}^2}, \mathbf{x}^2, \frac{1}{\mathbf{x}^2}) \cdot (1 - \frac{1}{\mathbf{x}^2}) \\ & + \frac{df_+}{dX}(\frac{1}{\mathbf{x}^2}, \mathbf{x}^2, \frac{1}{\mathbf{x}^2}) \cdot 2\mathbf{x}\Big\}\Big] \\ & - \mathbf{x}^{-2\Psi_2^{(0)}(\sigma)} \Big[1 + 2\mathbf{x}^{2\Psi_2^{(0)}(\sigma)} \Big\{(1-\mathbf{x})\frac{df_+}{dW}(\mathbf{x},1,\mathbf{x}) - 2\frac{df_+}{dX}(\mathbf{x},1,\mathbf{x})\Big\}\Big] = 0\ .\end{aligned}$$

For $\frac{df_+}{dW}$ and $\frac{df_+}{dX}$, see §4. By Lemma 4.4, which will be deduced from (2) (namely (K_2)), we get

$(\mathrm{A}_2^0) \Longleftrightarrow$

(5)

$$\frac{\mathbf{x}^\lambda+1}{\mathbf{x}^m(\mathbf{x}+1)}\Big\{1-\mathbf{x}^{-2\Psi_2^{(0)}(\sigma)}\cdot(\mathbf{x}-1)\frac{df_+}{dX}(1,\mathbf{x},\frac{1}{\mathbf{x}^2})\Big\}$$
$$-\mathbf{x}^{-2\Psi_2^{(0)}(\sigma)}\Big[1+2\mathbf{x}^{2\Psi_2^{(0)}(\sigma)}\Big\{(1-\mathbf{x})\frac{df_+}{dW}(\mathbf{x},1,\mathbf{x})-2\frac{df_+}{dX}(\mathbf{x},1,\mathbf{x})\Big\}\Big]=0\ .$$

By formulae (6)∼(14) in §4, which will be deduced from (I), (II), (III) and (K_2), we get the following

$$\frac{dF_+}{dW}(\mathbf{x},1,\mathbf{x}) = \mathbf{x}^{-2\Psi_2^{(0)}-1}\cdot\frac{dF_+}{dW}(\frac{1}{\mathbf{x}^2},\mathbf{x},1)$$
$$\frac{dF_+}{dX}(\mathbf{x},1,\mathbf{x}) = 0$$
$$\frac{dF_+}{dX}(\frac{1}{\mathbf{x}^2},\mathbf{x},1) = \frac{\mathbf{x}+1}{\mathbf{x}^2}\frac{dF_+}{dW}(\frac{1}{\mathbf{x}^2},\mathbf{x},1)+\frac{\mathbf{x}^{2\Psi_2^{(0)}(\sigma)}-1}{\mathbf{x}-1}$$
$$\frac{dF_+}{dY}(\mathbf{x},\frac{1}{\mathbf{x}^2},\mathbf{x}) = -\mathbf{x}^{1-2\Psi_2^{(0)}(\sigma)}\frac{dF_+}{dW}(1,\mathbf{x},\frac{1}{\mathbf{x}^2})-\frac{\mathbf{x}^{2\Psi_2^{(0)}(\sigma)}-1}{\mathbf{x}^{2\Psi_2^{(0)}(\sigma)}(\mathbf{x}-1)}$$
$$\frac{dF_+}{dY}(\frac{1}{\mathbf{x}^2},\mathbf{x},1) = \frac{dF_+}{dW}(\frac{1}{\mathbf{x}^2},\mathbf{x},1)$$

By substituting $(W,X,Y)=(1,\mathbf{x},\frac{1}{\mathbf{x}^2})$ and above formulae to (15) in Lemma 4.6, we obtain equation (5), which implies the validity of the Anderson duplication formulae (A_2^0). □

Therefore the combination of geometric relations (I), (II), (IV) and (K_2) implies arithmetic relation (A_2^0). Next we will consider three subgroups $\mathbb{I}\Gamma$, GTK and GTA.

Lemma 3.2. *For $n,m\in\mathbf{N}$, the combination of relations* (A_n) *and* (A_m) *implies* (A_{nm}).

Proof . Let $n\in\mathbf{N}$. By the slightly more detailed discussion on [Ih00], we find that relation (A_n) implies

$$\Psi_{na}^{(0)}(\sigma)=\Psi_n^{(0)}(\sigma)+\Psi_a^{(0)}(\sigma)\qquad\text{for }\ \forall a\in\mathbf{N}.$$

Therefore the combination of (A_n) and (A_m) implies the following:

$$\frac{\prod_{k=0}^{nm-1}\Gamma_\sigma(s+\frac{k}{nm})}{\Gamma_\sigma(nms)\prod_{k=0}^{nm-1}\Gamma_\sigma(\frac{k}{nm})} = 1\otimes exp\big[2\pi i\cdot nm\{\Psi_n^{(0)}(\sigma)+\Psi_m^{(0)}(\sigma)\}s\big]$$

$$=1\otimes exp\big[2\pi i\cdot nm\Psi_{nm}^{(0)}(\sigma)s\big] \quad \square$$

We also remark the following formulae, but it is not required to prove Theorem 3.4 below.

Lemma 3.3. *For $n, m \in \mathbf{N}$, the combination of relations* (K_n) *and* (K_m) *implies* (K_{nm}).

Proof . Let $n \in \mathbf{N}$. As in the same way of the case of Lemma 3.2, we get that relation (K_n) implies

$$\Psi_{na}^{(0)}(\sigma) = \Psi_n^{(0)}(\sigma) + \Psi_a^{(0)}(\sigma) \qquad \text{for } \ \forall a \in \mathbf{N}.$$

Therefore the combination of (K_n) and (K_m) implies the following:

$$\varphi_{nm}(f) = \varphi_n(y^{\Psi_m^{(0)}(\sigma)})\varphi_n(f) = y^{\Psi_n^{(0)}(\sigma)+\Psi_m^{(0)}(\sigma)}f = y^{\Psi_{nm}^{(0)}(\sigma)}f. \quad \square$$

Theorem 3.4. *$GTK \cap \mathbb{\Gamma} \subseteq GTA_{2^\infty}$, where $GTA_{2^\infty} := \bigcap_{n\in\mathbf{N}} GTA_{2^n}$.*

Proof . We note that one relation (A_2) implies infinite ones (A_{2^n}) for $n = 1, 2, 3, \cdots$. Therefore, by combining Lemma 3.2 with Theorem 3.1, we get the claim. $\square$

Thus we get a relationship among the arithmetic subgroup GTA_{2^∞} and the geometric subgroups GTK and $\mathbb{\Gamma}$. Relations (3) and (4) describe the D_4 (not D_2) symmetry of $\mathbf{P}^1 - \{0, \pm 1, \infty\}$. Since they were essential in the proof of Theorem 3.1, it does not look possible, at least to the author, to deduce Anderson's n-th multiplication formula ($n \geqslant 5$) from the D_n-symmetry of $\mathbf{P}^1 - \{0, \mu_n, \infty\}$. H. Tsunogai suggest to the author the possibility of deducing (A_3) from the $\mathfrak{S}_4$-symmetry of $\mathbf{P}^1 - \{0, \mu_3, \infty\}$. Still we would expect the geometric interpretation of arithmetic Anderson's multiplication formulae which is originally of arithmetic nature to be a key to distinguish $G_{\mathbf{Q}}$ from $\widehat{GT}$.

4. Miscellaneous lemmas

We present auxiliary lemmas which were required in the proof of Theorem 3.1. Here we follow the notation in [Ih99].

Lemma 4.1. *Relation* (IV) *is equivalent to* (3).

Proof . Let $\widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle$ denote the quotient pro-finite group of $\widehat{F_2}$ by the normal closure $\langle\langle z^2\rangle\rangle$ of $z^2=(xy)^{-2}$. By [NS] Theorem 2.2.(22),

$$\begin{aligned}
\text{(IV)} \Longleftrightarrow & y^{4\Psi_2^{(0)}(\sigma)} f(x,y^2) x^{4\Psi_2^{(0)}(\sigma)} f(y,x^2) \equiv 1 && \text{in} \quad \widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle \\
\Longleftrightarrow & f_+(xy^2x^{-1},x^2,y^2) f_+(yx^2y^{-1},y^2,x^2) \equiv 1 && \text{in} \quad \widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle \\
\Longleftrightarrow & f_+(xy^2x^{-1},x^2,y^2) f_+(x^{-1}y^{-2}x^{-1}y^{-2},y^2,x^2) \equiv 1 && \text{in} \quad \widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle.
\end{aligned}$$

On the other hand, since xy^2x^{-1},x^2 and y^2 generate a free pro-finite subgroup of rank 3 in $\widehat{F_2}$,

$$(3) \Longleftrightarrow f_+(xy^2x^{-1},x^2,y^2) f_+(x^{-1}y^{-2}x^{-1}y^{-2},y^2,x^2) = 1 \qquad \text{in} \quad \widehat{F_2}.$$

From the argument in the proof of [NS] Theorem 2.2, the quotient classes of x, y^2 in $\widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle$ generate a free pro-finite subgroup of rank 2. Thus the images of xy^2x^{-1}, x^2, y^2 in $\widehat{F_2} \,/\, \langle\langle z^2\rangle\rangle$ generate a free pro-finite subgroup of rank 3, from which it follows that (IV) is equivalent to (3). □

Lemma 4.2. *The combination of relations* (I), (II) *and* (3) *implies* (4).

Proof . By the permutation $x\mapsto y$, $y\mapsto x$, $z\mapsto x^{-1}y^{-1}$, (II) is re-expressed as follows:

$$f(y,x)y^{m+1}f(z,y)z^{m+1}f(x,z)x^{m+1}=1.$$

By combining this with (II) and applying (I), we get

$$x^{2m+1}f(x,z)^{-1}z^m f(y,z)y^{2m+1}f(y,z)^{-1}z^{m+1}f(x,z)=1.$$

Taking the image of the last equation by the continuous homomorphism induced from $x\mapsto y^{-1}x^{-1}$, $y\mapsto x$, $z\mapsto y$, we get

$$\begin{aligned}
&(y^{-1}x^{-1})^{2m+1} f_+(y^{-1}x^{-1}yxy, y^{-1}x^{-1}y^{-1}x^{-1}, y)^{-1} y^m f_+(xyx^{-1},x^2,y) x^{2m+1} \\
&\quad f_+(xyx^{-1},x^2,y)^{-1} y^{m+1} f_+(y^{-1}x^{-1}yxy, y^{-1}x^{-1}y^{-1}x^{-1}, y) = 1.
\end{aligned}$$

By (3),

$$\begin{aligned}
&(y^{-1}x^{-1}y^{-1}x^{-1})^m \cdot f_+(x^2,y,y^{-1}x^{-1}y^{-1}x^{-1}) \cdot y^m \cdot f_+(xyx^{-1},x^2,y)\cdot (x^2)^m \\
&f_+(y^{-1}x^{-1}y^{-1}x^{-1}, xyx^{-1}, x^2)\cdot (xyx^{-1})^m \cdot f_+(y, y^{-1}x^{-1}y^{-1}x^{-1}, xyx^{-1}) = 1.
\end{aligned}$$

Since the subgroup generated by x^2, y and $y^{-1}x^{-1}y^{-1}x^{-1}$ is equal to the one generated by x^2, y and xyx^{-1}, it is a free pro-finite subgroup of rank 3 in $\widehat{F_2}$, which implies (4). □

By Anderson's theorem (see [Ih99] Theorem A.1.), the element $f \in \widehat{F_2}$ can be written uniquely in $\Lambda_2 := \widehat{\mathbf{Z}}[[\widehat{F_2}]]$ as follows:

$$f(x,y) = 1 + \frac{df}{dx}\cdot(x-1) + \frac{df}{dy}\cdot(y-1). \tag{6}$$

Put $\Lambda_3 := \widehat{\mathbf{Z}}[[\widehat{F_3}]]$. Similarly the element $f_+ \in \widehat{F_3} = \langle W, X, Y\rangle^\wedge$ in §3 can be written uniquely in Λ_3 as follows:

$$f_+(W,X,Y) = 1 + \frac{df_+}{dW}\cdot(W-1) + \frac{df_+}{dX}\cdot(X-1) + \frac{df_+}{dY}\cdot(Y-1), \tag{7}$$

where $\frac{df_+}{dW}, \frac{df_+}{dX}, \frac{df_+}{dY} \in \Lambda_3$.

Lemma 4.3.

$$\begin{aligned} \frac{df}{dx}(x,y) =& y^{2\Psi_2^{(0)}(\sigma)}\frac{df_+}{dW}(xyx^{-1},x^2,y)\cdot(1-xyx^{-1}) \\ &+ \frac{df_+}{dX}(xyx^{-1},x^2,y)\cdot 2x. \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{df}{dy}(x,y) =& \frac{y^{2\Psi_2^{(0)}(\sigma)}-1}{y-1} + y^{2\Psi_2^{(0)}(\sigma)}\Big\{\frac{df_+}{dW}(xyx^{-1},x^2,y)\cdot x \\ &+ \frac{df_+}{dY}(xyx^{-1},x^2,y)\Big\}. \end{aligned} \tag{9}$$

Here for $\beta \in \Lambda_3$ and $a,b,c \in \widehat{F_2}$, we denote the image of β by the map $\Lambda_3 \to \Lambda_2$ induced from $W \mapsto a$, $X \mapsto b$, $Y \mapsto c$ by $\beta(a,b,c) \in \Lambda_2$.

Proof . It follows from (1) by a direct calculation. □

Lemma 4.4.

$$\begin{aligned} \frac{df_+}{dX}(1,x,y) = & y^{\Psi_2^{(0)}(\sigma)}\frac{df_+}{dW}(xyx^{-1},x^2,y)\cdot(1-xyx^{-1}) \\ &+ y^{-\Psi_2^{(0)}(\sigma)}\frac{df_+}{dX}(xyx^{-1},x^2,y)\cdot 2x \quad . \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{df_+}{dY}(1,x,y) = & \frac{y^{\Psi_2^{(0)}(\sigma)}-1}{y-1} + y^{\Psi_2^{(0)}(\sigma)}\Big\{\frac{df_+}{dW}(xyx^{-1},x^2,y)\cdot x \\ &+ \frac{df_+}{dY}(xyx^{-1},x^2,y)\Big\}. \end{aligned} \tag{11}$$

Proof . It follows from (2) by a direct calculation. □

Lemma 4.5.

$$(12)\quad \frac{df_+}{dW}(W,X,Y) = f_+(W,X,Y)\frac{df_+}{dW}(X^{-1}W^{-1}Y^{-1},Y,X)X^{-1}W^{-1}.$$

$$(13)\quad \frac{df_+}{dX}(W,X,Y) = f_+(W,X,Y)\Big\{\frac{df_+}{dW}(X^{-1}W^{-1}Y^{-1},Y,X)\cdot X^{-1} - \frac{df_+}{dY}(X^{-1}W^{-1}Y^{-1},Y,X)\Big\}.$$

$$(14)\quad \frac{df_+}{dY}(W,X,Y) = f_+(W,X,Y)\Big\{\frac{df_+}{dW}(X^{-1}W^{-1}Y^{-1},Y,X)\cdot X^{-1}W^{-1}Y^{-1} + \frac{df_+}{dX}(X^{-1}W^{-1}Y^{-1},Y,X)\Big\}.$$

Proof . It follows from (3) by a direct calculation. □

Lemma 4.6.

(15)

$$\begin{aligned}
&Y^m\frac{df_+}{dW}(W,X,Y)\\
&+Y^m f_+(W,X,Y)X^m\Big\{\frac{df_+}{dW}(Z,W,X)\cdot(-Z)+\frac{df_+}{dX}(Z,W,X)\Big\}\\
&+Y^m f_+(W,X,Y)X^m f_+(Z,W,X)\frac{W^m-1}{W-1}\\
&+Y^m f_+(W,X,Y)X^m f_+(Z,W,X)W^m\Big\{\frac{df_+}{dX}(Y,Z,W)\cdot(-Z)+\frac{df_+}{dY}(Y,Z,W)\Big\}\\
&+Y^m f_+(W,X,Y)X^m f_+(Z,W,X)W^m f_+(Y,Z,W)\frac{Z^m-1}{Z-1}\cdot(-Z)\\
&+Y^m f_+(W,X,Y)X^m f_+(Z,W,X)W^m f_+(Y,Z,W)Z^m\cdot\\
&\Big\{\frac{df_+}{dY}(X,Y,Z)\cdot(-Z)\Big\}=0 \qquad \textit{for} \qquad WXYZ=1.
\end{aligned}$$

Proof . It follows from (4) by a direct calculation. □

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