Recursive Fermion System in Cuntz Algebra. II

— Endomorphism, Automorphism and Branching of Representation —

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Abstract

Based on an embedding formula of the CAR algebra into the Cuntz algebra $\mathcal{O}_{2^p}$, properties of the CAR algebra are studied in detail by restricting those of the Cuntz algebra. Various $*$-endomorphisms of the Cuntz algebra are explicitly constructed, and transcribed into those of the CAR algebra. In particular, a set of $*$-endomorphisms of the CAR algebra into its even subalgebra are constructed. According to branching formulae, which are obtained by composing representations and $*$-endomorphisms, it is shown that a KMS state of the CAR algebra is obtained through the above even-CAR endomorphisms from the Fock representation. A $U(2^p)$ action on $\mathcal{O}_{2^p}$ induces $*$-automorphisms of the CAR algebra, which are given by nonlinear transformations expressed in terms of polynomials in generators. It is shown that, among such $*$-automorphisms of the CAR algebra, there exists a family of one-parameter groups of $*$-automorphisms describing time evolutions of fermions, in which the particle number of the system changes by time while the Fock vacuum is kept invariant.

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§1. Introduction

In our previous papers,[1,2] we have presented a recursive construction of the CAR (canonical anticommutation relation) algebra[3] for fermions in terms of the Cuntz algebra[4] $\mathcal{O}_{2^p}$ ($p \in \mathbf{N}$), and shown that it may provide us a useful tool to study properties of fermion systems by using explicit expressions in terms of generators of the algebra. As a concrete example of applications, we have constructed an infinite-dimensional (outer) $*$-automorphism group of the CAR algebra, in which the transformations are expressed in terms of polynomials in creation/annihilation operators.[2] The basic ingredient necessary for this embedding is called a *recursive fermion system* and denoted by RFS_p, where a subscript p stands for $\mathcal{O}_{2^p}$. As a special example, the *standard* RFS_p, which describes an embedding of $\mathcal{O}_{2^p}$ onto its $U(1)$-invariant subalgebra $\mathcal{O}_2^{U(1)}$, has been introduced, and it has been shown that a certain permutation representation[5,6] of $\mathcal{O}_{2^p}$ reduces to the Fock representation of the CAR algebra. We have also shown[7] that it is possible to generalize this recursive construction to the algebra for the FP ghost fermions in string theory by introducing a $*$-algebra called the pseudo Cuntz algebra suitable for actions on an indefinite-metric state vector space. We have found that, according to embeddings of the FP ghost algebra into the pseudo Cuntz algebra with a special attention to the zero-mode operators, unitarily inequivalent representations for the FP ghost are obtained from a single representation of the pseudo Cuntz algebra.

The purpose of this paper is to develop the study of the recursive fermion system and to show concretely that it becomes to possible to manage some complicated properties of the CAR algebra as follows:

(1) Systematic construction of proper (i.e., not surjective) $*$-endomorphisms which are not necessarily expressed in terms of linear transformations: The existence of proper $*$-endomorphisms is characteristic for the infinite dimensionality of the algebra.

(2) Description of branchings of representations induced by proper $*$-endomorphisms: By using branchings, various reducible representations or mixed states for fermions are obtained.

(3) Systematic construction of outer $*$-automorphisms which are not necessarily expressed in terms of Bogoliubov (linear) transformations: Nonlinearity of transformations in one-parameter groups of (outer) $*$-automorphisms corresponding to time evolutions implies that the fermions under consideration are no longer (quasi-)free.

For this purpose, it is necessary to prepare beforehand some useful formulae for representations,[8] embeddings, and $*$-endomorphisms[9,10] of the Cuntz algebra. As for embeddings, from a fundamental formula[4] for embedding of $\mathcal{O}_3$ into $\mathcal{O}_2$, we can easily obtain some basic formulae for embeddings among the Cuntz algebras. Then, using an important relation between embeddings (of some $\mathcal{O}_{d'}$'s into $\mathcal{O}_d$) and $*$-endomorphisms (of $\mathcal{O}_d$), various $*$-endomorphisms of the Cuntz algebra are explicitly constructed. Conversely, from a set of given $*$-endomorphisms, we may also obtain new embeddings. By composing irreducible permutation representations and $*$-endomorphisms, some branching formulae are derived.[10] Based on these properties of the Cuntz algebra, we study the recursive fermion systems in detail. First, by restricting the irreducible permutation representations of the Cuntz algebra, the corresponding representations of the CAR algebra are obtained in the form of direct sums of irreducible ones. On the other hand, it is also shown that, for a certain type of irreducible permutation representation of $\mathcal{O}_2$, we can construct a RFS_1 such that the restricted representation is irreducible. Such a RFS_1 gives, in general, an embedding of the CAR algebra onto a subalgebra of $\mathcal{O}_2$ which is not $U(1)$-invariant. Furthermore, it is shown that a certain RFS_1 similar to the above one yields a direct sum of an infinite number of irreducible representations of the CAR algebra from any irreducible permutation representation of $\mathcal{O}_2$. Next, from some $*$-endomorphisms of $\mathcal{O}_2$, we explicitly construct $*$-endomorphisms of the CAR algebra, especially, a set of those giving $*$-homomorphism to its even subalgebra.[11,12] It is shown that, by composing the Fock representation and the above even-CAR endomorphisms, we may obtain a KMS state[13] of the CAR algebra with respect to a one-parameter group of $*$-automorphisms describing the time evolution of a (quasi-)free fermion system. In contrast with some KMS states of the Cuntz algebra,[14–16] the inverse temperature is not unique since the KMS condition is satisfied only by the induced state of the CAR algebra, but not by that of the Cuntz algebra. We also give some discussions on the relation to the Araki-Woods classification of factors for the CAR algebra.[17] Finally, we apply the induced $*$-automorphisms of the CAR algebra[2] to construct one-parameter groups of $*$-automorphisms describing nontrivial time evolutions of fermions. Since it is possible to describe nonlinear transformations of the CAR algebra by these $*$-automorphisms, the time evolutions are not restricted to those for (quasi-)free fermions. We explicitly construct some examples for such one-parameter groups of $*$-automorphisms of the CAR algebra, in which the particle number changes by time with keeping the Fock vacuum invariant.

The present paper is organized as follows. In Sec. 2 and Sec. 3, we summarize various properties of the Cuntz algebra and obtain some convenient formulae necessary for our discussions. In Sec. 4, after reviewing the construction of the recursive fermion system, we show the relation between RFS_1 and RFS_p ($p \geqq 2$). In Sec. 5, we study the restriction of the permutation representations of the Cuntz algebra. In Sec. 6, various $*$-endomorphisms of the CAR algebra are explicitly obtained from those of the Cuntz algebra. In Sec. 7, based on some formulae constructed in the previous sections, it is shown that a KMS state is obtained from the Fock representation through a certain $*$-endomorphism. In Sec. 8, we summarize $*$-automorphisms of the CAR algebra induced by a $U(2^p)$ action on $\mathcal{O}_{2^p}$, and obtain one-parameter groups of $*$-automorphisms describing nontrivial time evolutions for fermions. The final section is devoted to discussion.

§2. Properties of Cuntz Algebra: Embedding and Endomorphism

In this section and the next, we summarize some properties of the Cuntz algebra[c] necessary for our discussions in the succeeding sections.

First, let us recall that the Cuntz algebra[4] $\mathcal{O}_d$ ($d \geqq 2$) is a simple C^*-algebra generated by $s_1,\ s_2,\ \ldots,\ s_d$ satisfying the following relations:

$$s_i^*\, s_j = \delta_{i,j} I, \tag{2.1}$$

$$\sum_{i=1}^{d} s_i\, s_i^* = I, \tag{2.2}$$

where * is a $*$-involution (or an adjoint operation), I being the unit (or the identity

[c] Throughout this paper, we restrict ourself to consider the dense subset of the Cuntz algebra.

operator). We often use brief descriptions as follows:

$$s_{i_1,i_2,\ldots,i_m} \equiv s_{i_1}s_{i_2}\cdots s_{i_m}, \tag{2.3}$$

$$s^*_{i_1,i_2,\ldots,i_m} \equiv s^*_{i_m}\cdots s^*_{i_2}s^*_{i_1}, \tag{2.4}$$

$$s_{i_1,\ldots,i_m;\,j_n,\ldots,j_1} \equiv s_{i_1}\cdots s_{i_m}s^*_{j_n}\cdots s^*_{j_1}. \tag{2.5}$$

From the relation (2.1), $\mathcal{O}_d$ is a linear space generated by monomials of the form $s_{i_1,\ldots,i_m;\,j_n,\ldots,j_1}$ with $m+n \geqq 1$.

From (2.1) and (2.2), it is obvious that there is a $*$-automorphism α on $\mathcal{O}_d$ defined by a $U(d)$ action as follows:

$$\alpha_u(s_i) = \sum_{j=1}^{d} s_j\, u_{j,i}, \quad i = 1,\ldots,d\,; \quad u = (u_{j,i}) \in U(d). \tag{2.6}$$

Especially, we consider a $U(1)$ action γ defined by

$$\gamma_z(s_i) = z\, s_i, \quad i = 1,\ldots,d\,; \quad z \in \mathbf{C},\ |z| = 1. \tag{2.7}$$

Then, the $U(1)$ invariant subalgebra $\mathcal{O}_d^{U(1)}$ of $\mathcal{O}_d$ is a linear subspace generated by monomials of the form $s_{i_1,\ldots,i_m;\,j_m,\ldots,j_1}$ with $m \geqq 1$.

§§2-1. Embedding

If there exists an injective unital $*$-homomorphism ψ from $\mathcal{O}_{d'}$ to $\mathcal{O}_d$, which is defined by a mapping $\psi\,:\,\mathcal{O}_{d'} \to \mathcal{O}_d$ satisfying

$$\psi(\alpha X + \beta Y) = \alpha\psi(X) + \beta\psi(Y), \quad \alpha,\,\beta \in \mathbf{C},\ X,\,Y \in \mathcal{O}_{d'}, \tag{2.8}$$

$$\psi(XY) = \psi(X)\psi(Y) \qquad X,\,Y \in \mathcal{O}_{d'}, \tag{2.9}$$

$$\psi(X^*) = \psi(X)^*, \qquad X \in \mathcal{O}_{d'}, \tag{2.10}$$

$$\psi(I_{d'}) = I_d, \tag{2.11}$$

with $I_{d'}$ and I_d being unit of $\mathcal{O}_{d'}$ and that of $\mathcal{O}_d$, respectively, we say that $\mathcal{O}_{d'}$ is *embedded into* $\mathcal{O}_d$, and call ψ an *embedding of* $\mathcal{O}_{d'}$ *into* $\mathcal{O}_d$. We also denote an embedding as $\psi : \mathcal{O}_{d'} \hookrightarrow \mathcal{O}_d$. In this paper, we always assume the condition (2.11) for embeddings. To define an embedding of $\mathcal{O}_{d'}$ into $\mathcal{O}_d$, it is sufficient and necessary to give a correspondence of generators between these two Cuntz algebras because of the following reason. Let $\{s'_i \mid i = 1,\,\ldots,\,d'\}$ be generators of $\mathcal{O}_{d'}$. If $\mathcal{O}_{d'}$ is embedded into $\mathcal{O}_d$, define $S_i \equiv \psi(s'_i) \in \mathcal{O}_d$ ($i = 1,\,\ldots,\,d'$) by the above unital $*$-homomorphism ψ. Then, it is straightforward to show that $\{S_i \mid i = 1,\,\ldots,\,d'\}$ satisfy (2.1) and (2.2) by using (2.8)–(2.11). Conversely, if there exists a set of elements $\{S_i \in \mathcal{O}_d \mid i = 1,\,\ldots,\,d'\}$ satisfying (2.1) and (2.2), it is also straightforward to construct the $*$-homomorphism $\psi : \mathcal{O}_{d'} \to \mathcal{O}_d$ by defining $\psi(s'_i) \equiv S_i$ ($i = 1,\,\ldots,\,d'$) and by uniquely extending its domain to the whole $\mathcal{O}_{d'}$ in such a way that it satisfies (2.8)–(2.11). Therefore, we also denote an embedding by giving a set of generators as $\{S_1,\,\ldots,\,S_{d'}\} : \mathcal{O}_{d'} \hookrightarrow \mathcal{O}_d$.

In the following, we present some fundamental formulae for embeddings among the Cuntz algebras.

(1) Fundamental embedding by Cuntz: It is remarkable that $\mathcal{O}_d$ with arbitrary d ($d \geqq 2$) can be embedded into $\mathcal{O}_2$. For example, by setting[4)]

$$S_1 = s_1,\quad S_2 = s_2 s_1,\quad S_3 = (s_2)^2 s_1,\ \ldots,\quad S_{d-1} = (s_2)^{d-2}s_1,\quad S_d = (s_2)^{d-1}, \tag{2.12}$$

where s_1 and s_2 are the generators of $\mathcal{O}_2$, it is straightforward to show that S_i's satisfy the relations (2.1) and (2.2). This is called *Cuntz embedding*.

(2) Inductive construction: From any embedding $\{S_1, \ldots, S_d\} : \mathcal{O}_d \hookrightarrow \mathcal{O}_2$, we can obtain an embedding of $\mathcal{O}_{d+1}$ into $\mathcal{O}_2$ as follows:

$$\{S_1, \ldots, S_{d-1}, S_d\, s_1, S_d\, s_2\} : \mathcal{O}_{d+1} \hookrightarrow \mathcal{O}_2. \tag{2.13}$$

(3) Generalized Cuntz embedding: It is straightforward to generalize (2.12) for embedding of $\mathcal{O}_{(d-1)n+1}$ into $\mathcal{O}_d$ with $n \geqq 1$ as follows

$$\begin{aligned} S_i &= s_i && \text{for } 1 \leqq i \leqq d-1, \\ S_{(d-1)k+i} &= (s_d)^k s_i && \text{for } 1 \leqq k \leqq n-1,\ 1 \leqq i \leqq d-1, \\ S_{(d-1)n+1} &= (s_d)^n, \end{aligned} \tag{2.14}$$

where $\{s_i \mid i = 1, \ldots, d\}$ is the generators of $\mathcal{O}_d$.

(4) Generalized inductive construction: From any embedding $\{S_1, \ldots, S_{(d-1)n+1}\}$: $\mathcal{O}_{(d-1)n+1} \hookrightarrow \mathcal{O}_d$, we can obtain an embedding of $\mathcal{O}_{(d-1)(n+1)+1}$ into $\mathcal{O}_d$ as follows:

$$\{S_1, \ldots, S_{(d-1)n}, S_{(d-1)n+1}s_1, \ldots, S_{(d-1)n+1}s_d\} : \mathcal{O}_{(d-1)(n+1)+1} \hookrightarrow \mathcal{O}_d. \tag{2.15}$$

(5) Homogeneous embedding: For $\mathcal{O}_{d^p}$, we have its embedding into $\mathcal{O}_d$ in which all generators of $\mathcal{O}_{d^p}$ are mapped homogeneously to elements of $\mathcal{O}_d$ as follows:

$$\begin{gathered} \Psi_p : \mathcal{O}_{d^p} \hookrightarrow \mathcal{O}_d, \\ \Psi_p(s'_i) \equiv S_i^{(p)} \equiv s_{i_1,i_2,\ldots,i_p}, \qquad i-1 = \sum_{k=1}^{p} (i_k - 1)d^{k-1} \\ i = 1, 2, \ldots, d^p; \quad i_1, i_2, \ldots, i_p = 1, 2, \ldots, d, \end{gathered} \tag{2.16}$$

where the correspondence of $i-1$ and $(i_p - 1, \ldots, i_1 - 1)$ is the same as that of a decimal number and its d-ary expression. The embedding Ψ_p (2.16) for $\mathcal{O}_{d^p} \hookrightarrow \mathcal{O}_d$ is constructed inductively with respect to p as follows:

$$S^{(p+1)}_{(i-1)d^p+j} = S^{(p)}_j\, s_i \quad \text{for } i = 1, 2, \ldots, d;\ j = 1, 2, \ldots, d^p, \tag{2.17}$$

or

$$S^{(p+1)}_{(j-1)d+i} = s_i S^{(p)}_j \quad \text{for } i = 1, 2, \ldots, d;\ j = 1, 2, \ldots, d^p. \tag{2.18}$$

From (2.16), it is obvious that any monomial $s_{i_1,\ldots,i_n} \in \mathcal{O}_d$ is one of homogeneously embedded generators $\{S_1^{(n)}, \ldots, S_{d^n}^{(n)}\} : \mathcal{O}_{d^n} \hookrightarrow \mathcal{O}_d$. It should be noted that, from (2.14) and (2.15), there is also an embedding of $\mathcal{O}_{(d-1)n+1}$ into $\mathcal{O}_d$ for any monomial $s_{i_1,\ldots,i_n} \in \mathcal{O}_d$ such that $s_{i_1,\ldots,i_n}$ can be set on one of embedded generators $\{S_1, \ldots, S_{(d-1)n+1}\}$. Such an example is given by the following:

$$S_j = \begin{cases} s_j & \text{for } 1 \leqq j \leqq i_1 - 1, \\ s_{j+1} & \text{for } i_1 \leqq j \leqq d-1, \\ s_{i_1,\ldots,i_k\tilde{j}} & \text{for } (d-1)k+1 \leqq j \leqq (d-1)k + i_{k+1} - 1, \quad 1 \leqq k \leqq n-1, \\ s_{i_1,\ldots,i_k,\tilde{j}+1} & \text{for } (d-1)k + i_{k+1} \leqq j \leqq (d-1)(k+1), \qquad 1 \leqq k \leqq n-1, \\ s_{i_1,\ldots,i_{n-1},i_n} & \text{for } j = (d-1)n+1, \end{cases} \tag{2.19}$$

where $\tilde{j} \equiv j - (d-1)k$.

§§2-2. Endomorphism

An embedding of $\mathcal{O}_d$ into itself is a *unital $*$-endomorphism* of $\mathcal{O}_d$. A typical $*$-endomorphism of $\mathcal{O}_d$ is the *canonical endomorphism* ρ defined by[d]

$$\rho(X) = \sum_{i=1}^{d} s_i X s_i^*, \quad X \in \mathcal{O}_d. \tag{2.20}$$

Indeed, from (2.1), ρ satisfies $\rho(X)\rho(Y) = \rho(XY)$ for X, $Y \in \mathcal{O}_d$. From (2.2), ρ is unital, that is, $\rho(I) = I$, hence $S_i \equiv \rho(s_i)$ satisfy the relations (2.1) and (2.2).

Let $U(k, \mathcal{O}_d)$ ($k \in \mathbf{N}$) be a set of all $k \times k$ unitary matrices in which each entry is an element of $\mathcal{O}_d$. Then, any unital $*$-endomorphism φ has a one-to-one correspondence with a unitary $u \in U(1,\, \mathcal{O}_d)$ given by

$$\varphi(s_i) = u\, s_i, \qquad i = 1, \ldots, d, \tag{2.21}$$

$$u = \sum_{i=1}^{d} \varphi(s_i)\, s_i^*. \tag{2.22}$$

Likewise, there is a one-to-one correspondence between any unital $*$-endomorphism φ and a $d \times d$ unitary $v = (v_{j,i}) \in U(d,\, \mathcal{O}_d)$ as follows:

$$\varphi(s_i) = \sum_{j=1}^{d} s_j\, v_{j,i}, \qquad i = 1, \ldots, d, \tag{2.23}$$

$$v_{j,i} = s_j^*\, \varphi(s_i), \qquad i, j = 1, \ldots, d. \tag{2.24}$$

It should be noted that, if it is possible to embed $\mathcal{O}_{d'}$ into $\mathcal{O}_d$ for certain d' and d, then any unitary $u \in U(1,\, \mathcal{O}_d)$ is expressed in the following form:[e]

$$u = \sum_{i=1}^{d'} S_i^{[2]} S_i^{[1]\,*}, \tag{2.25}$$

where $\{S_1^{[k]},\, \ldots,\, S_{d'}^{[k]}\}$ ($k = 1,\, 2$) are embeddings of $\mathcal{O}_{d'}$ into $\mathcal{O}_d$. Indeed, it is straightforward to show that u defined by (2.25) satisfies $u\,u^* = u^*\,u = I$ by using that $\{S_1^{[k]},\, \ldots,\, S_{d'}^{[k]}\}$ satisfy (2.1) and (2.2) for each $k = 1,\, 2$. Conversely, for an arbitrary unitary u and an arbitrary embedding $\{S_1^{[1]},\, \ldots,\, S_{d'}^{[1]}\} : \mathcal{O}_{d'} \hookrightarrow \mathcal{O}_d$, it is possible to obtain another embedding $\{S_1^{[2]},\, \ldots,\, S_{d'}^{[2]}\} : \mathcal{O}_{d'} \hookrightarrow \mathcal{O}_d$ in (2.25) as follows

$$S_i^{[2]} = u\, S_i^{[1]} \quad i = 1,\, \ldots,\, d'. \tag{2.26}$$

Using this formula, we obtain a new $*$-endomorphism from two known embeddings, and conversely, a new embedding from a known $*$-endomorphism and a known embedding.

[d]We always use the symbol ρ for the canonical endomorphism.

[e]The symbols $S_i^{[k]}$ and $S_i^{(k)}$ should not be confused. The former is used just for distinguishing one from some others, while the latter denotes the homogeneous embedding.

For example, for $d = 2,\ d' = 3,\ \varphi = \rho,\ \{S_1^{[1]} \equiv s_1,\, S_2^{[1]} \equiv s_{2,1},\, S_3^{[1]} \equiv s_{2,2}\}$, we obtain a new embedding of $\mathcal{O}_3$ into $\mathcal{O}_2$ by

$$S_1^{[2]} = \rho(s_1), \quad S_2^{[2]} = s_{1,2}, \quad S_3^{[2]} = s_{2,2}. \tag{2.27}$$

Although (2.21)–(2.25) are general formulae, they are not convenient to construct various $*$-endomorphisms explicitly. Next, we present a more effective way to express a generic $*$-endomorphism of $\mathcal{O}_d$ in terms of some embeddings of $\mathcal{O}_{d'}$ into $\mathcal{O}_d$ without recourse to unitaries. For this purpose, we need the following $d+1$ embeddings:

$$\{S_1^{[i]},\ \ldots,\ S_{d_i}^{[i]}\} :\ \mathcal{O}_{d_i} \hookrightarrow \mathcal{O}_d, \quad i = 1,\ \ldots,\ d, \tag{2.28}$$

$$\{S_1^{[d+1]},\ \ldots,\ S_D^{[d+1]}\} :\ \mathcal{O}_D \hookrightarrow \mathcal{O}_d, \quad D \equiv \sum_{i=1}^{d} d_i, \tag{2.29}$$

where $d_i \equiv (d-1)n_i + 1$ $(i = 1, \ldots, d)$ with $\{n_1, \ldots, n_d\}$ being nonnegative integers. For $d_i = d$, a trivial embedding (i.e., $S_j^{[i]} \equiv s_j$) is used, while for $d_i = 1$, we define $S_1^{[i]} \equiv I$. It should be noted that we have $D = (d-1)\big(\sum_{i=1}^{d} n_i + 1\big) + 1$. Thus, for any $\{d_1, \ldots, d_d\}$, there exists an embedding of $\mathcal{O}_D$ into $\mathcal{O}_d$ from (2.14). Given the above $d+1$ embeddings, we can define a $*$-endomorphism φ of $\mathcal{O}_d$ as follows:

$$\varphi(s_i) \equiv \sum_{j=1}^{d_i} S_{D_{i-1}+j}^{[d+1]}\, S_j^{[i]\,*}, \qquad D_i \equiv \sum_{j=1}^{i} d_j. \tag{2.30}$$

Indeed, it is straightforward to show that $\{\varphi(s_1),\ \ldots,\ \varphi(s_d)\}$ satisfy (2.1) and (2.2) by using (2.28) and (2.29). Conversely, for an arbitrary $*$-endomorphism φ of $\mathcal{O}_d$ and d arbitrary embeddings $\{S_1^{[i]},\ \ldots,\ S_{d_i}^{[i]}\} : \mathcal{O}_{d_i} \hookrightarrow \mathcal{O}_d$ $(i = 1,\ \ldots,\ d)$, we obtain an embedding $\{S_{D_{i-1}+j} \equiv \varphi(s_i)S_j^{[i]} \mid i = 1, \ldots, d;\ j = 1, \ldots, d_i\} : \mathcal{O}_D \hookrightarrow \mathcal{O}_d$ with $D_i = \sum_{j=1}^{i} d_j,\ D = D_d$, which reproduces $\varphi(s_i)$ itself when substituted into $S_{D_{i-1}+j}^{[d+1]}$ in (2.30). Therefore, any $*$-endomorphism of $\mathcal{O}_d$ is expressed in the form of (2.30).

From a $*$-endomorphism φ in the form of (2.30), we obtain various $*$-endomorphisms by using the $U(D)$ action on $\mathcal{O}_D$ given by (2.6) as follows:

$$\begin{aligned} \varphi_u(s_i) &\equiv \sum_{j=1}^{d_i} S_{D_{i-1}+j}^{[d+1]\,\prime}\, S_j^{[i]\,*}, \\ S_k^{[d+1]\,\prime} &\equiv \sum_{\ell=1}^{D} S_\ell^{[d+1]}\, u_{\ell,k}, \quad k = 1, \ldots, D, \quad u \in U(D). \end{aligned} \tag{2.31}$$

Especially, by using permutations given by $S_i^{[d+1]} \mapsto S_{\sigma(i)}^{[d+1]}$ $(\sigma \in \mathfrak{S}_D \subset U(D))$, we obtain $D!$ $*$-endomorphisms for a given set of $d+1$ embeddings. In the case $D = d^{p+1}$ $(p \in \mathbf{N})$, we may adopt the homogeneous embedding defined by (2.16) for the embedding of $\mathcal{O}_D$ in to $\mathcal{O}_d$, $S_i^{[d+1]} = s_{i_1,\ldots,i_{p+1}}$ $(i = 1, \ldots, D;\ i_1, \ldots, i_{p+1} = 1, \ldots, d)$. Then, each permutation of the indices $i \in \{1, \ldots, d^{p+1}\}$ induces a permutation of the multi indices $(i_1, \ldots, i_{p+1}) \in \{1, \ldots, d\}^{p+1}$ according to the one-to-one correspondence between them given by (2.16). For simplicity of description, we denote this induced permutation of the multi indices by the same symbol σ as for the single indices, that is, $S_{\sigma(i)}^{[d+1]} = s_{\sigma(i_1,\ldots,i_{p+1})} = s_{i_1^\sigma,\ldots,i_{p+1}^\sigma}$.

Hereafter, we assume that $*$-endomorphisms of $\mathcal{O}_d$ are expressed in terms of a finite sum of monomials. We, now, consider $*$-endomorphisms φ of $\mathcal{O}_d$ which commute with the $U(1)$ action γ defined by (2.7). Then, it satisfies the following:

$$\begin{aligned}\gamma_z\big(\varphi(s_i)\big) &= \varphi\big(\gamma_z(s_i)\big) \\ &= z\,\varphi(s_i), \quad i=1,\ldots,d, \quad z\in\mathbf{C}, \quad |z|=1. \end{aligned} \tag{2.32}$$

Hence, from $\gamma_z(s_{i_1,\ldots,i_m;\,j_n,\ldots,j_1}) = z^{m-n}\, s_{i_1,\ldots,i_m;\,j_n,\ldots,j_1}$, each term in $\varphi(s_i)$ ($i=1,\ldots,d$) is a monomial in the form of $s_{i_1,\ldots,i_{n+1};\,j_n,\ldots,j_1}$ with $0 \leqq n \leqq p_i$ ($i=1,\ldots,d$), where $P\equiv\{p_i \mid i=1,\ldots,d\}$ is a set of nonnegative integers. Let p be the maximum of P. By using (2.2), we can rewrite $\varphi(s_i)$ ($i=1,\ldots,d$) into a homogeneous polynomial of degree $(p+1,\,p)$, that is, a finite sum of monomials in the form of $s_{i_1,\ldots,i_{p+1};\,j_p,\ldots,j_1}$. Here, any monomial $s_{i_1,\ldots,i_p}$ (or $s_{i_1,\ldots,i_{p+1}}$) is one of the homogeneously embedded generators of $\mathcal{O}_{d^p}$ (or $\mathcal{O}_{d^{p+1}}$) into $\mathcal{O}_d$ defined by (2.16), hence φ is written as

$$\varphi(s_i) = \sum_{j=1}^{d^p}\sum_{k=1}^{d^{p+1}} c_{k,j;i}\, S_k^{(p+1)} S_j^{(p)\,*} \tag{2.33}$$

with an appropriate set of coefficients $c_{k,j;i}\in\mathbf{C}$. Since $\{\varphi(s_i)\mid i=1,\ldots,d\}$ satisfies (2.1) and (2.2), the relations among the coefficients $c_{k,j;i}$'s are obtained as follows:

$$\sum_{k=1}^{d^{p+1}} \bar{c}_{k,j;i}\, c_{k,j';i'} = \delta_{j,j'}\delta_{i,i'}, \qquad \sum_{i=1}^{d}\sum_{j=1}^{d^p} c_{k,j;i}\,\bar{c}_{k',j;i} = \delta_{k,k'}, \tag{2.34}$$

hence $u_{k,\ell}\equiv c_{k,j;i}$ with $\ell\equiv(j-1)d+i$ is an element of $U(d^{p+1})$. Therefore, any $*$-endomorphism φ of $\mathcal{O}_d$, which is expressed in terms of a finite sum of monomials, commuting with the U(1) action γ is written as follows:

$$\begin{aligned} \varphi(s_i) &\equiv \sum_{j=1}^{d^p} S_{(j-1)d+i}^{(p+1)\,\prime}\, S_j^{(p)\,*}, \quad i=1,\ldots,d, \\ S_\ell^{(p+1)\,\prime} &\equiv \sum_{k=1}^{d^{p+1}} S_k^{(p+1)}\, u_{k,\ell}, \quad \ell=1,\ldots,d^{p+1}, \quad u=(u_{k,\ell})\in U(d^{p+1}). \end{aligned} \tag{2.35}$$

We call this type of $*$-endomorphism the $(p+1)$*-th order homogeneous endomorphism*. Here, one should note that if $u_{k,\ell}=\delta_{k,\ell}$, (2.35) becomes the identity map $\varphi(s_i)=s_i$ from (2.18) as follows:

$$\begin{aligned} \varphi(s_i) &= \sum_{j=1}^{d^p} S_{(j-1)d+i}^{(p+1)}\, S_j^{(p)\,*} \\ &= s_i \sum_{j=1}^{d^p} S_j^{(p)}\, S_j^{(p)\,*} = s_i, \quad i=1,\ldots,d. \end{aligned} \tag{2.36}$$

Especially, as for the second order homogeneous endomorphism, by setting

$$u = \begin{pmatrix} v & 0 & \cdots & 0 \\ 0 & v & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & v \end{pmatrix} \in U(d^2), \quad v\in U(d), \tag{2.37}$$

so that we have

$$S^{(2)\,\prime}_{(j-1)d+i} = \sum_{k=1}^{d} S^{(2)}_{(j-1)d+k}\, v_{k,i}$$
$$= \sum_{k=1}^{d} s_{k,j}\, v_{k,i}, \qquad v = (v_{k,i}) \in U(d), \tag{2.38}$$

where use has been made of (2.18) for $p = 1$, the $*$-automorphism of $\mathcal{O}_d$ by the $U(d)$ action (2.6) is reproduced as follows:

$$\varphi(s_i) = \sum_{j=1}^{d}\sum_{k=1}^{d} s_{k,j}\, s_j^*\, v_{k,i}$$
$$= \sum_{k=1}^{d} s_k\, v_{k,i}, \quad i = 1, \ldots, d; \quad v = (v_{k,i}) \in U(d). \tag{2.39}$$

In the case that u in (2.35) is a permutation $\sigma \in \mathfrak{S}_{d^{p+1}} \subset U(2^{p+1})$, φ is called the $(p+1)$*-th order permutation endomorphism*[9)f] of $\mathcal{O}_d$. Explicitly, the permutation endomorphisms are written in the following:

(1) The second order permutation endomorphism:

$$\varphi(s_i) = \sum_{j=1}^{d} S^{(2)}_{\sigma((j-1)d+i)} s_j^* = \sum_{j=1}^{d} s_{\sigma(i,j)}\, s_j^*, \tag{2.40}$$

where $\sigma(i_1, i_2)$ denotes a permutation of multi indices $(i_1, i_2) \in \{1, \ldots, d\}^2$ induced from that of indices $i \in \{1, \ldots, d^2\}$ by the one-to-one correspondence defined by $i = \sum_{k=1}^{2}(i_k - 1)d^{k-1} + 1$.

(2) The $(p+1)$-th order permutation endomorphism:

$$\varphi(s_i) = \sum_{j=1}^{d^p} S^{(p+1)}_{\sigma((j-1)d+i)} S^{(p)\,*}_j = \sum_{j_1,\ldots,j_p=1}^{d} s_{\sigma(i,j_1,\ldots,j_p)}\, s^*_{j_1\cdots j_p}, \tag{2.41}$$

where $\sigma(i_1, \ldots, i_{p+1})$ denotes a permutation of multi indices $(i_1, \ldots, i_{p+1}) \in \{1, \ldots, d\}^{p+1}$ induced from that of indices $i \in \{1, \ldots, d^{p+1}\}$ by the one-to-one correspondence defined by $i = \sum_{k=1}^{p+1}(i_k - 1)d^{k-1} + 1$.

Let $\mathrm{PEnd}_{p+1}(\mathcal{O}_d)$ be a set of all $(p+1)$-th order permutation endomorphisms of $\mathcal{O}_d$. Then, we have

$$\mathrm{PEnd}_2(\mathcal{O}_d) \subset \mathrm{PEnd}_3(\mathcal{O}_d) \subset \cdots \subset \mathrm{PEnd}_{p+1}(\mathcal{O}_d) \subset \cdots\,, \tag{2.42}$$

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[f] As far as the present authors know, the first nontrivial example of the permutation endomorphisms other than the canonical endomorphism is the second order one in $\mathcal{O}_3$ presented by N. Nakanishi in private communication. The discussions in this subsection are based on the generalization of his result.

since a subset of $\mathrm{PEnd}_{p+1}(\mathcal{O}_d)$ with $\mathfrak{S}_{d^{p+1}} \supset \mathfrak{S}_{d^p} \ni \sigma$ preserving j_p in (2.41) ($j_p^\sigma = j_p$) is nothing but $\mathrm{PEnd}_p(\mathcal{O}_d)$ because of (2.2). The canonical endomorphism ρ given by (2.20) is the special case of the second order permutation endomorphism (2.40) with $\sigma \in \mathfrak{S}_{d^2}$ being a product of $d(d-1)/2$ transpositions $(i,j) \mapsto (j,i)$ with $i \neq j$. Likewise, ρ^p is a special case of the $(p+1)$-th order permutation endomorphism with $\sigma : (i, j_1, \ldots, j_n) \mapsto (j_1, \ldots, j_n, i)$.

Of course, a generic $*$-endomorphism given by (2.30) does *not* necessarily commute with the $U(1)$ action γ. We call $*$-endomorphisms not commuting with γ the *inhomogeneous endomorphisms*. A typical example of the inhomogeneous endomorphism in the form of (2.30) is obtained by setting $d_i = 1$ ($i = 1, \ldots, d-1$) and $d_d = (d-1)(n-1)+1$ (hence $D = (d-1)n+1$) with $n-1 \in \mathbf{N}$ as follows:

$$\begin{aligned} \varphi(s_i) &= S_i^{[d+1]}, \quad i = 1, \ldots, d-1, \\ \varphi(s_d) &= \sum_{j=1}^{d_d} S_{j+d-1}^{[d+1]} S_j^{[d]\,*}, \end{aligned} \tag{2.43}$$

where $\{S_j^{[d]} \mid j = 1, \ldots, d_d\}$ and $\{S_j^{[d+1]} \mid j = 1, \ldots, D\}$ are embeddings of $\mathcal{O}_{d_d}$ and $\mathcal{O}_D$ into $\mathcal{O}_d$, respectively, in which the order of s_j's minus that of s_k^*'s appearing in at least one of $S_i^{[d+1]}$ ($i = 1, \ldots, d-1$) is not equal to 1.

In (2.43), we can assign an arbitrary n-th order monomial $s_{i_1,\ldots,i_n}$ ($i_1, \ldots, i_d = 1, \ldots, d$) to $\varphi(s_1)$ by adjusting the embedding (2.19) of $\mathcal{O}_{(d-1)n+1}$ into $\mathcal{O}_d$. This fact will be applied later.

§3. Properties of Cuntz Algebra: Representation and Branching

§§3-1. Permutation representation

A permutation representation[5,6] of $\mathcal{O}_d$ on a countable infinite-dimensional Hilbert space $\mathcal{H}$ is defined as follows. Let $\{e_n \mid n \in \mathbf{N}\}$ be a complete orthonormal basis of $\mathcal{H}$. A *branching function system* $\{\mu_i\}_{i=1}^d$ on $\mathbf{N}$ is defined by

$$\mu_i : \mathbf{N} \to \mathbf{N} \text{ is injective}, \qquad i = 1, 2, \ldots, d, \tag{3.1}$$

$$\mu_i(\mathbf{N}) \cap \mu_j(\mathbf{N}) = \emptyset \ \text{ for } i \neq j, \quad i, j = 1, 2, \ldots, d, \tag{3.2}$$

$$\bigcup_{i=1}^d \mu_i(\mathbf{N}) = \mathbf{N}. \tag{3.3}$$

Given a branching function system $\{\mu_i\}_{i=1}^d$ and a set of complex numbers $\{z_{i,n} \in \mathbf{C} \mid |z_{i,n}| = 1,\ i = 1, \ldots, d;\ n \in \mathbf{N}\}$, the permutation representation π of $\mathcal{O}_d$ on $\mathcal{H}$ is defined by[g]

$$\pi(s_i) e_n = z_{i,n} e_{\mu_i(n)}, \quad i = 1, \ldots, d, \quad n \in \mathbf{N}. \tag{3.4}$$

By this definition, $\pi(s_i)$ is defined on the whole $\mathcal{H}$ linearly as a bounded operator. Then, the action of $\pi(s_i^*) = \pi(s_i)^*$ on e_n is determined by the definition of the adjoint operation.

[g] The original definition of the permutation representation of the Cuntz algebra in Ref. 5) is the case of $z_{i,n} = 1$ ($i = 1, \ldots, d,\ n \in \mathbf{N}$). We have introduced a set of coefficients $z_{i,n}$ according to Ref. 6).

Since for any $n \in \mathbf{N}$ there exists a pair $\{j,\ m\}$ which satisfy $\mu_j(m) = n$, we consider $\pi(s_i)^*$ on $e_{\mu_j(m)}$:

$$\begin{aligned}\langle \pi(s_i)^* e_{\mu_j(m)} | e_\ell \rangle &= \langle e_{\mu_j(m)} | \pi(s_i) e_\ell \rangle = z_{i,\ell} \langle e_{\mu_j(m)} | e_{\mu_i(\ell)} \rangle = z_{i,\ell} \delta_{i,j} \delta_{m,\ell} \\ &= z_{i,m} \delta_{i,j} \langle e_m | e_\ell \rangle, \qquad \ell \in \mathbf{N},\end{aligned} \tag{3.5}$$

hence we obtain

$$\pi(s_i)^* e_{\mu_j(m)} = \delta_{i,j} \bar{z}_{i,m} e_m. \tag{3.6}$$

Here, $\langle \cdot | \cdot \rangle$ denotes the inner product on $\mathcal{H}$. It is, now, straightforward to show that $\pi(s_i)$ and $\pi(s_i)^*$ defined by (3.4) and (3.6) satisfy the relation (2.1) and (2.2) on any e_n.

We can classify permutation representations into two types as follows:[5,6,8)]

(1) Permutation representation with a *central cycle*: There exists a monomial $\pi(s_{i_0,\ldots,i_{\kappa-1}})$ having an eigenvalue z with $z \in \mathbf{C}$, $|z| = 1$. This representation is denoted by $\mathrm{Rep}(i_0, \ldots, i_{\kappa-1};\ z)$ and a positive integer κ is called the *length* of the central cycle. For the special case $z = 1$, we denote $\mathrm{Rep}(i_0, \ldots, i_{\kappa-1}) \equiv \mathrm{Rep}(i_0, \ldots, i_{\kappa-1};\ 1)$.

(2) Permutation representation with a *chain*: There is no eigenvector for any monomial in s_i's and there exists a vector $v \in \mathcal{H}$ satisfying $\|\pi(s^*_{i_0,\ldots,i_N})v\| = 1$, $(N \in \mathbf{N})$ for a certain sequence $\{i_k\}_{k=0}^{\infty}$ $(i_k = 1, \ldots, d)$. This representation is denoted by $\mathrm{Rep}(\{i_k\})$.

Any of other permutation representations is expressed as a direct sum and a direct integral of (1) and (2) with multiplicity. For $\mathrm{Rep}(i_0, \ldots, i_{\kappa-1};\ z)$, a *label* $(i_0, \ldots, i_{\kappa-1})$ is called to be *periodic*, if $i_k = i_{M+k}$ $(k = 0, 1, \ldots, \kappa-1)$ is satisfied for a certain positive integer $M (< \kappa)$ under understanding that the subscripts of i_k's take values in $\mathbf{Z}_\kappa$. The integer M (if there are more than one, the minimum of such M's) is called the *period* of the label $(i_0, \ldots, i_{\kappa-1})$. By definition, M is a divisor of κ smaller than κ. If $\pi(s_{i_0,\ldots,i_{\kappa-1}})$ has an eigenvalue z, so does any of its cyclic permutations $\pi(s_{i'_0,\ldots,i'_{\kappa-1}})$. Hence all of κ $\mathrm{Rep}(i'_0, \ldots, i'_{\kappa-1};\ z)$'s obtained by cyclic permutations of a label $(i_0, \ldots, i_{\kappa-1})$ are identified. Likewise for $\mathrm{Rep}(\{i_k\})$, a label $\{i_k\}_{k=0}^{\infty}$ is called to be *eventually periodic* if there exist a positive integer M satisfying $i_{k+M} = i_k$ for $k \geqq N$ with a nonnegative integer N. $\mathrm{Rep}(\{i_k\})$ and $\mathrm{Rep}(\{j_k\})$ are called to be *tail equivalent* if there exist nonnegative integers M and M' such that $i_{k+M} = j_{k+M'}$ $(k \in \mathbf{N})$. Two tail equivalent permutation representations with chains are unitarily equivalent to each other.[6)] It is known[5,6)] that a permutation representation of $\mathcal{O}_d$ is irreducible if and only if it is cyclic and its label is not (eventually) periodic.

In the following, we give explicit realizations of permutation representations. According to each type, it is convenient to rearrange the basis of $\mathcal{H}$ in an appropriate form.

$\mathrm{Rep}(i_0, \ldots, i_{\kappa-1};\ z)$ of $\mathcal{O}_d$: The complete orthonormal basis of $\mathcal{H}$ is denoted by $\{e_{\lambda,m} \mid \lambda \in \mathbf{Z}_\kappa,\ m \in \mathbf{N}\}$. The previous basis $\{e_n\}_{n=1}^{\infty}$ is recovered by such an identification as $e_{\kappa(m-1)+\lambda+1} \equiv e_{\lambda,m}$. We define the action of $\pi(s_i)$ on $\mathcal{H}$ by

$$\pi(s_i)\, e_{\lambda,1} = \begin{cases} e_{\lambda-1,\, i+1} & \text{for } 1 \leqq i \leqq i_{\lambda-1} - 1, \\ z^{1/\kappa}\, e_{\lambda-1,\, 1} & \text{for } i = i_{\lambda-1}, \\ e_{\lambda-1,\, i} & \text{for } i_{\lambda-1} + 1 \leqq i \leqq d, \end{cases} \tag{3.7}$$

$$\pi(s_i)\, e_{\lambda,m} = e_{\lambda-1,\, d(m-1)+i} \quad \text{for } m \geqq 2. \tag{3.8}$$

Then, $e_{\lambda,1}$'s become eigenvectors of operators $\{\pi(s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}}) \mid \lambda \in \mathbf{Z}_\kappa\}$ as follows:

$$\pi(s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}})\, e_{\lambda,1} = z\, e_{\lambda,1}, \quad \lambda \in \mathbf{Z}_\kappa. \tag{3.9}$$

The set of eigenvectors $\{e_{\lambda,1} \mid \lambda \in \mathbf{Z}_\kappa\}$ is called the *central cycle* of $\mathrm{Rep}(i_0,\ldots,i_{\kappa-1};\, z)$. Here, one should note that the subspace spanned by $\{e_{\lambda,m}\}_{m=1}^\infty$ for a fixed λ is generated by the action of $n\kappa$-th monomials $\{s_{j_1,\ldots,j_{n\kappa}} \mid n \geqq 0;\, j_1,\ldots,j_{n\kappa} = 1,\ldots,d\}$ on $e_{\lambda,1}$. The special case $\mathrm{Rep}(1) \equiv \mathrm{Rep}(1;\,1)$ is called the *standard representation* and denoted by π_s in Ref. 1):

$$\pi_s(s_i)\, e_n = e_{d(n-1)+i}, \qquad i = 1,2,\ldots,d\,;\; n \in \mathbf{N}, \tag{3.10}$$

where $e_n \equiv e_{0,n}$. From (3.10), it is straightforward to obtain the following formula:

$$\begin{aligned} &\pi_s(s_{i_1,\ldots,i_k})\, e_n = e_{N(i_1,\ldots,i_k;n)}, \\ &\quad N(i_1,\ldots,i_k;n) \equiv (n-1)d^k + \sum_{j=1}^{k}(i_j-1)d^{j-1} + 1, \end{aligned} \tag{3.11}$$

for $i_1,\cdots,i_k = 1,\ldots,d\,;\; n \in \mathbf{N}$.

$\mathrm{Rep}(\{i_k\})$ of $\mathcal{O}_d$: The complete orthonormal basis of $\mathcal{H}$ is denoted by $\{e_{\lambda,m} \mid \lambda \in \mathbf{Z},\, m \in \mathbf{N}\}$. We define the action of $\pi(s_i)$ on $\mathcal{H}$ by

$$\pi(s_i)\, e_{\lambda,1} = \begin{cases} e_{\lambda-1,\,i+1} & \text{for } 1 \leqq i \leqq i_{\lambda-1}-1, \\ e_{\lambda-1,\,1} & \text{for } i = i_{\lambda-1}, \\ e_{\lambda-1,\,i} & \text{for } i_{\lambda-1}+1 \leqq i \leqq d, \end{cases} \qquad \text{for } \lambda \geqq 1, \tag{3.12}$$

$$\pi(s_i)\, e_{\lambda,m} = e_{\lambda-1,\,d(m-1)+i} \qquad \text{for } \lambda \leqq 0 \;\text{ or }\; m \geqq 2. \tag{3.13}$$

Then, we obtain

$$\pi(s^*_{i_0,\ldots,i_N})\, e_{0,1} = e_{N+1,1}, \qquad N \in \mathbf{N}. \tag{3.14}$$

The set of vectors $\{e_{\lambda,1} \mid \lambda \in \mathbf{Z}\}$ is called the *chain* of $\mathrm{Rep}(\{i_k\})$. The subspace spanned by $\{e_{\lambda,m}\}_{m=1}^\infty$ for a fixed λ is generated by the action of $\mathcal{O}_d^{U(1)}$ on $e_{\lambda,1}$. It should be noted that from an equality

$$\pi(s^*_{i_M,\ldots,i_N})\, e_{M,1} = e_{N+1,1}, \qquad 0 \leqq M \leqq N, \tag{3.15}$$

it is obvious that $\mathrm{Rep}(\{j_k\})$ with $\{j_k \equiv i_{k+M}\}_{k=0}^\infty$, which is tail equivalent with $\{i_k\}$, is obtained from $\mathrm{Rep}(\{i_k\})$ by rearranging the basis of $\mathcal{H}$, hence $\mathrm{Rep}(\{j_k\})$ is unitarily equivalent with $\mathrm{Rep}(\{i_k\})$.

In concluding this subsection, we remark on an important property of the standard representation. By using the homogeneous embedding Ψ_q of $\mathcal{O}_{d^q}$ $(q \geqq 2)$ into $\mathcal{O}_d$ defined by (2.16), the standard representation $\pi_s^{(q)}$ of $\mathcal{O}_{d^q}$ is obtained from $\pi_s^{(1)}$ of $\mathcal{O}_d$ as follows:

$$\pi_s^{(q)} = \pi_s^{(1)} \circ \Psi_q. \tag{3.16}$$

Indeed, from (2.16), (3.10) and (3.11), we obtain

$$\begin{aligned} (\pi_s^{(1)} \circ \Psi_q)(s'_i)e_n &= \pi_s^{(1)}(s_{i_1,\ldots,i_q})e_n, \qquad i = \sum_{k=1}^{q}(i_k-1)d^{k-1} + 1 \\ &= e_{(n-1)d^q+i} \\ &= \pi_s^{(q)}(s'_i)e_n, \end{aligned} \tag{3.17}$$

where $\{s'_1, \ldots, s'_{d^q}\}$ and $\{s_1, \ldots, s_d\}$ are the generators of $\mathcal{O}_{d^q}$ and $\mathcal{O}_d$, respectively. On the other hand, as for other permutation representations with central cycles of length 1 and eigenvalue 1, we have

$$\pi^{(1)}_{i_0} \circ \Psi_q \cong \pi^{(q)}_{\tilde{i}_0}, \qquad \tilde{i}_0 \equiv \frac{d^q - 1}{d-1}(i_0 - 1) + 1, \quad i_0 = 2, \ldots, d, \tag{3.18}$$

where $\pi^{(1)}_{i_0}$ and $\pi^{(q)}_{\tilde{i}_0}$ stand for $\mathrm{Rep}(i_0)$ of $\mathcal{O}_d$ and $\mathrm{Rep}(\tilde{i}_0)$ of $\mathcal{O}_{d^q}$, respectively, and we have used the symbol "$\cong$" to denote the unitary equivalence with taking into account that realizations of the representations are different from those given by (3.7). Here, it should be noted that $\Psi_q(s'_{\tilde{i}_0}) = (s_{i_0})^d$, and (3.18) is obvious since there are no other monomials in s_i's having eigenvector except for those only in s_{i_0}. Generally, for an irreducible permutation representation with a label $L = (i_0, \ldots, i_{\kappa-1})$ $(2 \leqq \kappa < \infty)$, we obtain

$$\pi^{(1)}_L \circ \Psi_q \cong \bigoplus_{j=1}^{\kappa q/r} \pi^{(q)}_{\tilde{L}_j}, \tag{3.19}$$

where r is the least common multiple of κ and q, and $\{\tilde{L}_j\}_{j=1}^{\kappa q/r}$ is a certain set of nonperiodic labels with length r/q in $\mathcal{O}_{d^q}$ determined by L and q.

§§3-2. Branching of permutation representations

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras on $\mathbf{C}$. From a representation π of $\mathcal{A}$ and a homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$, we have a representation $\pi \circ \varphi$ of $\mathcal{B}$ by composing π and φ. Even if π is irreducible (or indecomposable), $\pi \circ \varphi$ is not necessarily so. If it is possible to decompose $\pi \circ \varphi$ into a direct sum of a family $\{\pi_\lambda\}_{\lambda \in \Lambda}$ of representations of $\mathcal{B}$, which are representatives of the unitary equivalence class of representations of $\mathcal{B}$, we write

$$\pi \circ \varphi \cong \bigoplus_{\lambda \in \Lambda} \pi_\lambda, \tag{3.20}$$

and call it the *branching* of π by φ. It should be noted that the symbol "$\cong$" denotes unitary equivalence. Likewise, for an endomorphism $\varphi : \mathcal{A} \to \mathcal{A}$, we have a similar branching.

In general, for a given irreducible permutation representation π of the Cuntz algebra, a branching of π by a kind of $*$-endomorphism φ is given by[10)]

$$\pi \circ \varphi \cong \bigoplus_{L \in \mathcal{S}} \pi_L, \tag{3.21}$$

where $\mathcal{S}$ denotes a certain set of irreducible permutation representations determined by π and φ. Here, we give a few examples in $\mathcal{O}_2$ necessary in later discussions. For more examples and detailed discussions, see Ref. 10).

We consider a family of particular $(p+1)$-th order permutation endomorphisms $\{\varphi_{\sigma_p}\}_{p \geqq 1}$ defined by (2.41) with $\sigma_p \in \mathfrak{S}_{2^{p+1}}$ being the transposition $\sigma_p(1, j_1, \ldots, j_p) \equiv (1, j_1, \ldots, j_p)$, $\sigma_p(2, j_1, \ldots, j_{p-1}, j_p) \equiv (2, j_1, \ldots, j_{p-1}, \hat{j}_p)$, $\hat{j}_p \equiv 3 - j_p$:

$$\begin{cases} \varphi_{\sigma_p}(s_1) = s_1, \\ \varphi_{\sigma_p}(s_2) = s_2\, \rho^{p-1}(J), \quad J \equiv s_{2;1} + s_{1;2} = J^* = J^{-1}, \end{cases} \tag{3.22}$$

where ρ is the canonical endomorphism of $\mathcal{O}_2$ and $\rho^0(X) \equiv X$. Then, it is shown that

$$\varphi_{\sigma_p} \circ \varphi_{\sigma_q} = \varphi_{\sigma_q} \circ \varphi_{\sigma_p}, \quad p, q \in \mathbf{N}, \tag{3.23}$$

$$(\varphi_{\sigma_p})^2 = \varphi_{\sigma_{2p}}, \qquad p \in \mathbf{N}. \tag{3.24}$$

Indeed, from the equality

$$J\rho(X) = \rho(X)J, \quad X \in \mathcal{O}_2, \tag{3.25}$$

we have

$$\varphi_{\sigma_p}(\rho^n(J)) = \rho^n(J)\,\rho^{p+n}(J), \qquad n+1 \in \mathbf{N}. \tag{3.26}$$

Hence we obtain

$$\begin{aligned}(\varphi_{\sigma_p}\circ\varphi_{\sigma_q})(s_1) &= s_1\\ &= (\varphi_{\sigma_q}\circ\varphi_{\sigma_p})(s_1), \qquad (3.27)\\ (\varphi_{\sigma_p}\circ\varphi_{\sigma_q})(s_2) &= s_2\,\rho^{p-1}(J)\,\rho^{q-1}(J)\,\rho^{p+q-1}(J)\\ &= (\varphi_{\sigma_q}\circ\varphi_{\sigma_p})(s_2). \qquad (3.28)\end{aligned}$$

Now, we consider eigenvectors of operators in the range of $\pi_s\circ\varphi_{\sigma_p}$. First, one should note that (3.22) is rewritten as follows:

$$\begin{aligned}&\varphi_{\sigma_p}(s_i) = s_i\,u_i, \quad i = 1,\, 2,\\ &u_1 \equiv I, \quad u_2 \equiv \sum_{j=1}^{2^{p-1}}(S^{(p)}_{j+2^{p-1};\,j} + S^{(p)}_{j;\,j+2^{p-1}}) = u_2^* = u_2^{-1},\end{aligned} \tag{3.29}$$

where $\{S^{(p)}_j \mid j = 1,2,\ldots,2^p\}$ denote the homogeneously embedded generators of $\mathcal{O}_{2^p}$ into $\mathcal{O}_2$ defined by (2.16). By using equalities

$$u_2\,S^{(p)}_j = \begin{cases} S^{(p)}_{j+2^{p-1}} & \text{for } 1 \leqq j \leqq 2^{p-1},\\ S^{(p)}_{j-2^{p-1}} & \text{for } 2^{p-1}+1 \leqq j \leqq 2^p,\end{cases} \tag{3.30}$$

$$u_{j_1}s_{j_2,\ldots,j_p,1} = s_{j_2,\ldots,j_p,j_1}, \qquad j_1,\ldots,j_p = 1,2 \tag{3.31}$$

and $\pi_s(s_1)\,e_1 = e_1$, we obtain

$$\begin{aligned}\pi_s\bigl(\varphi_{\sigma_p}(S^{(p)\,*}_k)\,S^{(p)}_j\bigr)\,e_1 &= \pi_s(u_{k_p}s^*_{k_p}\cdots u_{k_1}s^*_{k_1}\,s_{j_1,\ldots,j_p})\,e_1\\ &= \delta_{k_1,j_1}\pi_s(u_{k_p}s^*_{k_p}\cdots u_{j_1}\,s_{j_2,\ldots,j_p,1})\,e_1\\ &= \delta_{k_1,j_1}\pi_s(u_{k_p}s^*_{k_p}\cdots u_{k_2}s^*_{k_2}\,s_{j_2,\ldots,j_p,j_1})\,e_1\\ &= \cdots\\ &= \delta_{k_1,j_1}\cdots\delta_{k_p,j_p}\pi_s(s_{j_1,\ldots,j_p})\,e_1\\ &= \delta_{k,j}\pi_s\bigl(S^{(p)}_j\bigr)\,e_1, \qquad j,k = 1,2,\ldots,2^p.\end{aligned} \tag{3.32}$$

Making $\pi_s(\varphi_{\sigma_p}(S^{(p)}_k))$ act on (3.32) and summing up with respect to $k = 1,\ldots,2^p$, we obtain

$$\pi_s\bigl(\varphi_{\sigma_p}(S^{(p)}_j)\,S^{(p)}_j\bigr)\,e_1 = \pi_s(S^{(p)}_j)\,e_1, \quad j = 1,2,\ldots,2^p, \tag{3.33}$$

hence $\pi_s(S^{(p)}_j)\,e_1$ is the eigenvector of $(\pi_s\circ\varphi_{\sigma_p})(S^{(p)}_j)$. Furthermore, from (3.30) and (3.31), for any set of indices $j_1,\ldots,j_{mp+k} = 1,2$ with $m \in \mathbf{N}$ and $1 \leqq k \leqq p$, we can show that there is a unique set of indices $j'_1(=j_1), j'_2, \ldots, j'_{(m-1)p+k} = 1,2$ and $j = 1,\ldots,2^p$ such that

$$s_{j_1,j_2,\ldots,j_{mp+k}} = \varphi_{\sigma_p}(s_{j'_1,j'_2,\ldots,j'_{(m-1)p+k}})\,S^{(p)}_j, \quad m \in \mathbf{N},\ k = 1,\ldots,p. \tag{3.34}$$

As for $s_{j_1,\ldots,j_k}$ with $k=1,\ldots,p$, from $\pi_s(s_1)\,e_1=e_1$, we have

$$\begin{aligned}\pi_s(s_{j_1,\ldots,j_k})\,e_1 &= \pi_s(s_{j_1,\ldots,j_k})\,\pi_s\big((s_1)^{p-k}\big)\,e_1\\ &= \pi_s\big(s_{j_1,\ldots,j_k}(s_1)^{p-k}\big)\,e_1\\ &= \pi_s\big(S^{(p)}_j\big)\,e_1, \qquad j\equiv\sum_{\ell=1}^{k}(j_\ell-1)2^{\ell-1}+1. \end{aligned} \tag{3.35}$$

Therefore, any of the basis $\{e_n\}_{n=1}^{\infty}$, which satisfies (3.11) with $d=2$ and $n=1$, is given by an action of $(\pi_s\circ\varphi_{\sigma_p})(s_{j_1,\ldots,j_m})$ $(j_1,\ldots,j_m=1,2,\ m\in\mathbf{N})$ on one of the 2^p eigenvectors in (3.33) .

For any divisor κ of p, we can rewrite u_2 as

$$u_2=\rho^{(\kappa)}{}^{\frac{p}{\kappa}-1}\Big(\sum_{i=1}^{2^{\kappa-1}}\big(S^{(\kappa)}_{i+2^{\kappa-1};\,i}+S^{(\kappa)}_{i;\,i+2^{\kappa-1}}\big)\Big), \tag{3.36}$$

$$\rho^{(\kappa)}(X)\equiv\sum_{k=1}^{2^\kappa}S^{(\kappa)}_k\,X\,S^{(\kappa)\,*}_k=\rho^{\kappa}(X),\quad X\in\mathcal{O}_2, \tag{3.37}$$

where $\{S^{(\kappa)}_i\mid i=1,2,\ldots,2^\kappa\}$ denote the homogeneously embedded generators of $\mathcal{O}_{2^\kappa}$ into $\mathcal{O}_2$. Then, in the same way as above, it is shown that

$$\pi_s\big(\varphi_{\sigma_p}(S^{(\kappa)}_i)\,(S^{(\kappa)}_i)^{\frac{p}{\kappa}}\big)\,e_1=\pi_s\big((S^{(\kappa)}_i)^{\frac{p}{\kappa}}\big)\,e_1,\quad i=1,2,\ldots,2^\kappa. \tag{3.38}$$

From (2.16), we have

$$(S^{(\kappa)}_i)^{\frac{p}{\kappa}}=S^{(p)}_{\tilde{i}},\quad \tilde{i}\equiv\frac{2^p-1}{2^\kappa-1}(i-1)+1,\quad i=1,\ldots,2^\kappa, \tag{3.39}$$

hence some eigenvectors in (3.33) are reduced to those in (3.38). By writing $S^{(\kappa)}_i$ $(i=1,\ldots,2^\kappa)$ explicitly as

$$S^{(\kappa)}_i=s_{i_0,\ldots,i_{\kappa-1}},\quad i=\sum_{k=1}^{\kappa}(i_{k-1}-1)2^{k-1}+1 \tag{3.40}$$

with $i_0,\ldots,i_{\kappa-1}=1,2$, we obtain

$$\pi_s\big(\varphi_{\sigma_p}(s_{i_{\kappa-1}})\,(S^{(\kappa)}_i)^{\frac{p}{\kappa}}\big)\,e_1=\pi_s\big((S^{(\kappa)}_{i'})^{\frac{p}{\kappa}}\big)\,e_1,\qquad S^{(\kappa)}_{i'}\equiv s_{i_{\kappa-1},i_0,\ldots,i_{\kappa-2}}. \tag{3.41}$$

Therefore, if the set of indices $(i_0,\ldots,i_{\kappa-1})$ is not periodic in the sense stated in Sec. 3-1, we can see that $(\pi_s\circ\varphi_{\sigma_p})(s_i)$ $(i=1,2)$ act on a set of κ vectors

$$\Big\{\pi_s\big((s_{i_0,\ldots,i_{\kappa-1}})^{\frac{p}{\kappa}}\big)\,e_1,\ \pi_s\big((s_{i_1,\ldots,i_{\kappa-1},i_0})^{\frac{p}{\kappa}}\big)\,e_1,\ \ldots,\ \pi_s\big((s_{i_{\kappa-1},i_0,\ldots,i_{\kappa-2}})^{\frac{p}{\kappa}}\big)\,e_1\Big\} \tag{3.42}$$

in such way that they constitute a central cycle of length κ of the irreducible permutation representation $\mathrm{Rep}(i_0,i_1,\ldots,i_{\kappa-1})$ of $\mathcal{O}_2$.

From the above discussions, we can, now, show the branching formula of the standard representation π_s by φ_{σ_p} as follows:

$$\pi_s\circ\varphi_{\sigma_p}\cong\bigoplus_{L\in IPR_p}\pi_L, \tag{3.43}$$

where IPR_p denotes a set of all irreducible permutation representations with central cycles and with eigenvalue 1, in which each length κ of central cycles is a divisor of p. Here, κ eigenvectors in Rep(L) with a nonperiodic label $L = (i_0, \ldots, i_{\kappa-1})$ $(1 \leqq \kappa \leqq p)$ is given by

$$(\pi_s \circ \varphi_{\sigma_p})(s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}})\, e_{N(L,\lambda)} = e_{N(L,\lambda)}, \quad \lambda \in \mathbf{Z}_\kappa, \tag{3.44}$$

$$N(L,\lambda) \equiv \frac{2^p - 1}{2^\kappa - 1} \sum_{\ell=1}^{\kappa} (i_{\lambda+\ell-1} - 1) 2^{\ell-1} + 1, \tag{3.45}$$

For better understanding, we explicitly write (3.43) for $p = 1, 2, 3, 4$ as follows:

$$\pi_s \circ \varphi_{\sigma_1} \cong \pi_s \oplus \pi_2, \tag{3.46}$$

$$\pi_s \circ \varphi_{\sigma_2} \cong \pi_s \oplus \pi_2 \oplus \pi_{1,2}, \tag{3.47}$$

$$\pi_s \circ \varphi_{\sigma_3} \cong \pi_s \oplus \pi_2 \oplus \pi_{1,1,2} \oplus \pi_{1,2,2}, \tag{3.48}$$

$$\pi_s \circ \varphi_{\sigma_4} \cong \pi_s \oplus \pi_2 \oplus \pi_{1,2} \oplus \pi_{1,1,1,2} \oplus \pi_{1,1,2,2} \oplus \pi_{1,2,2,2}, \tag{3.49}$$

where $\pi_{i_0,\ldots,i_{\kappa-1}}$ $(\kappa = 1, 2, 3, 4)$ denotes Rep$(i_0, \ldots, i_{\kappa-1})$. Since it is easy to reconfirm (3.46), we show it concretely in the following.

From $\pi_s(s_i)e_1 = e_i$ $(i = 1, 2)$, which is obtained from (3.10) with $d = 2$, and

$$J s_j = s_{\hat{j}}, \quad \hat{j} \equiv 3 - j, \quad j = 1, 2, \tag{3.50}$$

we have

$$(\pi_s \circ \varphi_{\sigma_1})(s_1)\, e_1 = e_1, \tag{3.51}$$

$$(\pi_s \circ \varphi_{\sigma_1})(s_2)\, e_2 = e_2. \tag{3.52}$$

Furthermore, from (3.50) and $u_2 = J$, we obtain

$$s_{j_1,j_2,\ldots,j_{k+1}} = \varphi_{\sigma_1}(s_{j'_1,j'_2,\ldots,j'_k})\, s_{j'_{k+1}}, \quad j_1, j_2, \ldots, j_{k+1} = 1,\ 2, \tag{3.53}$$

$$s_{j'_1} \equiv s_{j_1}, \qquad s_{j'_\ell} \equiv u_{j'_{\ell-1}} s_{j_\ell} = \begin{cases} s_{j_\ell} & \text{for } j'_{\ell-1} = 1, \\ s_{\hat{j}_\ell} & \text{for } j'_{\ell-1} = 2, \end{cases} \qquad \ell = 2, \ldots, k+1, \tag{3.54}$$

hence we have

$$(-1)^{j'_\ell - 1} = (-1)^{j_\ell - 1}(-1)^{j'_{\ell-1} - 1}, \quad \ell = 2, \ldots, k+1, \tag{3.55}$$

$$(-1)^{j'_{k+1} - 1} = \prod_{\ell=1}^{k+1} (-1)^{j_\ell - 1}, \tag{3.56}$$

that is, we have $j'_{k+1} = 1$ if the number of 2 in $\{j_1, \ldots, j_{k+1}\}$ is even, and $j'_{k+1} = 2$ otherwise. Therefore, any of $\{e_n\}_{n=1}^{\infty}$, which is expressed as $\pi_s(s_{i_1,\ldots,i_{k+1}})\, e_1$ with $i_1, \ldots, i_{k+1} = 1, 2$, is uniquely given by an action of $(\pi_s \circ \varphi_{\sigma_1})(s_{j_1,\ldots,j_k})$ on either e_1 or e_2 with an appropriate set of indices $\{j_1, \ldots, j_k = 1,\ 2\}$. Thus, we obtain (3.46).

Next, we consider the branching number B_p of π_s by φ_{σ_p}, which is defined by the number of irreducible permutation representations appearing in the rhs of (3.43). We can obtain B_p in the following way. First, let C_n be the number of irreducible permutation representations Rep$(i_0, \ldots, i_{n-1})$. One should note the following: (1) Rep$(i_0, \ldots, i_{n-1})$ is defined up to cyclic permutations of the label; (2) if Rep$(i_0, \ldots, i_{n-1})$ is periodic, its periodicity is a divisor of n except for n itself; (3) the total number of $\mathrm{Rep}(i_0, \ldots, i_{n-1})$'s involving reducible or redundant ones is given by 2^n. Then, the recurrence formula for C_n is given by

$$\begin{cases} C_1 = 2, \\ \displaystyle\sum_{k \in D_n} k\, C_k = 2^n, \end{cases} \tag{3.57}$$

with D_n being the whole set of divisors of n. In terms of C_n, B_p is given by

$$B_p = \sum_{n \in D_p} C_n \tag{3.58}$$

with D_p being the whole set of divisors of p. Since it is an elementary problem to solve (3.57), we give here its solution without proof. Let $n = n_1^{m_1} \cdots n_r^{m_r}$ be the factorization of n in prime numbers. Then, C_n $(n \geqq 2)$ is given by

$$C_n = \frac{1}{n} \left[2^n + \sum_{\ell=1}^{r} (-1)^\ell \sum_{k_1 < \cdots < k_\ell} 2^{n/(n_{k_1} \cdots n_{k_\ell})} \right]. \tag{3.59}$$

In particular, for a prime number $n (\geqq 2)$, we have

$$C_n = \frac{2^n - 2}{n}. \tag{3.60}$$

In concluding this subsection, we show that, for any irreducible permutation representation with a central cycle, π_L, we can explicitly construct a $*$-endomorphism φ so that it yields only the standard representation π_s as follows:

$$\pi_L \circ \varphi \cong \pi_s. \tag{3.61}$$

For example, in $\mathcal{O}_2$, (3.61) for $L = (1, 2)$ is satisfied by the $*$-endomorphism φ as follows:

$$\varphi(s_1) \equiv s_{1,2}, \qquad \varphi(s_2) \equiv s_{2;1} + s_{1,1;2}, \tag{3.62}$$

which is one of the inhomogeneous endomorphisms defined by (2.43). Indeed, it is straightforward to show that $\varphi(s_{i_1,\ldots,i_n})$ for any index $(i_1, \ldots, i_n)$ involves none of monomials in the form of $\{(s_{2,1})^m,\ s_{j_1,\ldots,j_k}(s_{2,1})^{m-1}s^*_{j_1,\ldots,j_k},\ s_{j_1,\ldots,j_k}(s_{1,2})^{m-1}s^*_{j_1,\ldots,j_k}\}$ with $j_1, \ldots, j_k = 1, 2;\ k \geqq 1$, and $m \geqq 1$, hence there is no eigenvector except for $(\pi_{1,2} \circ \varphi)(s_1)$. On the other hand, from (3.62), we have

$$\left.\begin{aligned} \varphi\big((s_2)^{2m-1}\big) &= s_2(s_1)^{m-1}s_1^* + (s_1)^{m+1}s_2^*, \\ \varphi\big((s_2)^{2m}\big) &= s_2(s_1)^m s_2^* + (s_1)^{m+1}s_1^*, \end{aligned}\right. \qquad m \geqq 1, \tag{3.63}$$

hence we obtain

$$\left.\begin{aligned} \varphi\big((s_2)^{2m+1}s_1\big) &= s_2(s_1)^m s_2, \\ \varphi\big((s_2)^{2m}s_1\big) &= (s_1)^m s_{1,2}, \end{aligned}\right. \qquad m \geqq 0. \tag{3.64}$$

Since any monomial $s_{j_1,\ldots,j_k,1,2}$ is uniquely written as a product of the monomials appearing in the rhs of (3.64), it is rewritten into $\varphi(s_{j'_1,\ldots,j'_\ell})$ with an appropriate set of indices $\{j'_1, \ldots, j'_\ell = 1, 2\}$. Therefore, any of the basis of $\mathrm{Rep}(1, 2)$, $\{e_{\lambda,m} \mid \lambda = 0, 1;\ m \in \mathbf{N}\}$, which is expressed as $\pi_{1,2}(s_{j_1,\ldots j_k})\, e_{0,1} = \pi_{1,2}(s_{j_1,\ldots j_k,1,2})\, e_{0,1}$, is rewritten into $(\pi_{1,2} \circ$

$\varphi)(s_{j'_1,\ldots,j'_\ell})\, e_{0,1}$. Thus, from $(\pi_{1,2}\circ\varphi)(s_1)\, e_{0,1} = e_{0,1}$, we obtain (3.61) for $L=(1,2)$. Besides (3.62), (3.61) is satisfied also by $\varphi'(s_1)\equiv s_{2,1}$ and $\varphi'(s_2)\equiv s_{2,2;1}+s_{1;2}$. However, it should be noted that the $*$-endomorphism defined by $\varphi''(s_1)\equiv s_{1,2}$ and $\varphi''(s_2)\equiv s_{1,1;1}+s_{2;2}$ (or $\varphi''(s_1)\equiv s_{2,1}$ and $\varphi''(s_2)\equiv s_{1;1}+s_{2,2;2}$) does not satisfy (3.61), that is, $\pi_{1,2}\circ\varphi''$ yields a direct sum of π_s and an infinite number of π_2.

In general, in $\mathcal{O}_2$, for any π_L with a nonperiodic label $L=(i_0,\ldots,i_{\kappa-1};\, z)$, (3.61) is satisfied by the $*$-endomorphism φ defined by

$$\varphi(s_1)\equiv S_{\kappa+1}, \qquad \varphi(s_2)\equiv \sum_{j=1}^{\kappa} S_j T_j^*, \tag{3.65}$$

$$\begin{cases} S_1 \equiv s_{\hat{i}_0}, & \\ S_j \equiv s_{i_0,\ldots,i_{j-2},\hat{i}_{j-1}}, & 2\leqq j\leqq \kappa, \\ S_{\kappa+1}\equiv \bar{z}\, s_{i_0,\ldots,i_{\kappa-2},i_{\kappa-1}}, & \end{cases} \tag{3.66}$$

$$\begin{cases} T_1 \equiv s_{i_0}, & \\ T_j \equiv s_{\hat{i}_0,\ldots,\hat{i}_{j-2},i_{j-1}}, & 2\leqq j\leqq \kappa-1 \\ T_\kappa \equiv s_{\hat{i}_0,\ldots,\hat{i}_{\kappa-3},\hat{i}_{\kappa-2}}, & \end{cases} \tag{3.67}$$

with $\hat{i}\equiv 3-i$, where $\{S_1,\ldots,S_{\kappa+1}\}$ and $\{T_1,\ldots,T_\kappa\}$ are specific generators of $\mathcal{O}_{\kappa+1}$ and $\mathcal{O}_\kappa$ embedded into $\mathcal{O}_2$, respectively.

It is possible to obtain such a $*$-endomorphism also for $\mathcal{O}_d$ ($d\geqq 3$), but it is rather complicated to construct a general formula similar to (3.65)–(3.67). We give here an example in $\mathcal{O}_3$: (3.61) for π_L with $L=(1,2,1,3)$ is satisfied by the $*$-endomorphism φ defined by

$$\begin{aligned} &\varphi(s_1)\equiv s_{1,2,1,3},\\ &\varphi(s_2)\equiv s_2,\\ &\varphi(s_3)\equiv s_{3;1}+s_{1,3;2,2}+s_{1,1;3,2}+s_{1,2,2;1,2}+s_{1,2,3;2,3}+s_{1,2,1,1;3,3}+s_{1,2,1,2;1,3}. \end{aligned} \tag{3.68}$$

§4. Recursive Fermion System

In this section, we summarize the construction of the recursive fermion system (RFS_p),[1] which gives embeddings of the CAR algebra into $\mathcal{O}_{2^p}$ ($p\in\mathbf{N}$). We denote the generators of the CAR algebra by $\{a_n \mid n\in\mathbf{N}\}$ which satisfy

$$\{a_m,\, a_n\}=0,\quad \{a_m,\, a_n^*\}=\delta_{m,n}I,\quad m,\, n\in\mathbf{N}. \tag{4.1}$$

§§4-1. Definition of RFS_p in $\mathcal{O}_{2^p}$

Let $\boldsymbol{a}_1,\, \boldsymbol{a}_2,\, \ldots,\, \boldsymbol{a}_p\in\mathcal{O}_{2^p}$, $\zeta_p:\mathcal{O}_{2^p}\to\mathcal{O}_{2^p}$ be a linear mapping, and φ_p a unital $*$-endomorphism of $\mathcal{O}_{2^p}$, respectively. A set $R_p=(\boldsymbol{a}_1,\, \boldsymbol{a}_2,\, \ldots,\, \boldsymbol{a}_p\,;\, \zeta_p,\, \varphi_p)$ is called a *recursive fermion system of order p (RFS_p) in $\mathcal{O}_{2^p}$*, if it satisfies the following conditions

(i) seed condition: $\{\boldsymbol{a}_j,\, \boldsymbol{a}_k\}=0,\quad \{\boldsymbol{a}_j,\, \boldsymbol{a}_k^*\}=\delta_{j,k}I,\qquad j,k=1,\ldots,p,$ (4.2)

(ii) recursive condition: $\{\boldsymbol{a}_i,\, \zeta_p(X)\}=0,\quad \zeta_p(X)^*=\zeta_p(X^*),\quad X\in\mathcal{O}_{2^p},$ (4.3)

(iii) normalization condition: $\zeta_p(X)\zeta_p(Y)=\varphi_p(XY),\qquad X,\, Y\in\mathcal{O}_{2^p}$ (4.4) and none of $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\}$ is expressed as $\zeta_p(X)$ with $X \in \mathcal{O}_{2^p}$. We call $\boldsymbol{a}_j$ $(j = 1, \ldots, p)$ and ζ_p the *seeds* and the *recursive map* of RFS_p, respectively. The embedding Φ_{R_p} of the CAR algebra into $\mathcal{O}_{2^p}$ associated with R_p is defined by

$$\begin{aligned} &\Phi_{R_p} : \mathrm{CAR} \hookrightarrow \mathcal{O}_{2^p}, \\ &\Phi_{R_p}(a_{p(m-1)+j}) \equiv \zeta_p^{n-1}(\boldsymbol{a}_j) \quad j = 1, \ldots, p\,;\ m \in \mathbf{N}. \end{aligned} \tag{4.5}$$

We denote $\mathcal{A}_{R_p} \equiv \Phi_{R_p}(\mathrm{CAR})$ and call it the *CAR subalgebra of $\mathcal{O}_{2^p}$ associated with R_p*.

The simplest example of RFS_p is given by the *standard* RFS_p $SR_p = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\,;\, \zeta_p, \varphi_p)$, which is defined by

$$\boldsymbol{a}_j = \sum_{k=1}^{2^{p-j}} \sum_{\ell=1}^{2^{j-1}} (-1)^{\sum_{m=1}^{j-1}\left[\frac{\ell-1}{2^{m-1}}\right]} s_{2^j(k-1)+\ell}\, s^*_{2^{j-1}(2k-1)+\ell}, \quad j = 1, \ldots, p, \tag{4.6}$$

$$\zeta_p(X) = \sum_{i=1}^{2^p} (-1)^{\sum_{m=1}^{p}\left[\frac{i-1}{2^{m-1}}\right]} s_i X s_i^*, \quad X \in \mathcal{O}_{2^p}, \tag{4.7}$$

$$\varphi_p(X) = \rho_{2^p}(X) \equiv \sum_{i=1}^{2^p} s_i X s_i^*, \quad X \in \mathcal{O}_{2^p}, \tag{4.8}$$

where $[x]$ denotes the largest integer not greater than x, and ρ_{2^p} being the canonical endomorphism (2.20) of $\mathcal{O}_{2^p}$. It is shown that $\mathcal{A}_{SR_p} = \mathcal{O}_{2^p}^{U(1)}$ by mathematical induction, that is, any $s_{i_1,\ldots,i_k;\,j_k,\ldots,j_1} \in \mathcal{O}_{2^p}^{U(1)}$ is expressed in terms of a_n $(n \leqq kp)$.

Especially, SR_p for $p = 1, 2, 3, 4$ are given by

$$SR_1 \quad \begin{cases} \boldsymbol{a}_1 \equiv s_{1;2}, \\ \zeta_1(X) \equiv s_1 X s_1^* - s_2 X s_2^*, \quad X \in \mathcal{O}_2, \\ \varphi_1(X) \equiv \rho_2(X) = \sum_{i=1}^{2} s_i X s_i^*, \quad X \in \mathcal{O}_2. \end{cases} \tag{4.9}$$

$$SR_2 \quad \begin{cases} \boldsymbol{a}_1 \equiv s_{1;2} + s_{3;4}, \\ \boldsymbol{a}_2 \equiv s_{1;3} - s_{2;4}, \\ \zeta_2(X) \equiv s_1 X s_1^* - s_2 X x_2^* - s_3 X s_3^* + s_4 X s_4^*, \quad X \in \mathcal{O}_4, \\ \varphi_2(X) \equiv \rho_4(X) = \sum_{i=1}^{4} s_i X s_i^*, \quad X \in \mathcal{O}_4. \end{cases} \tag{4.10}$$

$$SR_3 \quad \begin{cases} \boldsymbol{a}_1 \equiv s_{1;2} + s_{3;4} + s_{5;6} + s_{7;8}, \\ \boldsymbol{a}_2 \equiv s_{1;3} - s_{2;4} + s_{5;7} - s_{6;8}, \\ \boldsymbol{a}_3 \equiv s_{1;5} - s_{2;6} - s_{3;7} + s_{4;8}, \\ \zeta_3(X) \equiv s_1 X s_1^* - s_2 X x_2^* - s_3 X s_3^* + s_4 X s_4^* \\ \qquad\qquad - s_5 X s_5^* + s_6 X x_6^* + s_7 X s_7^* - s_8 X s_8^*, \quad X \in \mathcal{O}_8, \\ \varphi_3(X) \equiv \rho_8(X) = \sum_{i=1}^{8} s_i X s_i^*, \quad X \in \mathcal{O}_8. \end{cases} \tag{4.11}$$

$$SR_4\quad\begin{cases} \boldsymbol{a}_1 \equiv s_{1;2}+s_{3;4}+s_{5;6}+s_{7;8}+s_{9;10}+s_{11;12}+s_{13;14}+s_{15;16},\\ \boldsymbol{a}_2 \equiv s_{1;3}-s_{2;4}+s_{5;7}-s_{6;8}+s_{9;11}-s_{10;12}+s_{13;15}-s_{14;16},\\ \boldsymbol{a}_3 \equiv s_{1;5}-s_{2;6}-s_{3;7}+s_{4;8}+s_{9;13}-s_{10;14}-s_{11;15}+s_{12;16},\\ \boldsymbol{a}_4 \equiv s_{1;9}-s_{2;10}-s_{3;11}+s_{4;12}-s_{5;13}+s_{6;14}+s_{7;15}-s_{8;16},\\ \zeta_3(X) \equiv s_1Xs_1^*-s_2Xx_2^*-s_3Xs_3^*+s_4Xs_4^* \\ \qquad\qquad -s_5Xs_5^*+s_6Xx_6^*+s_7Xs_7^*-s_8Xs_8^* \\ \qquad\qquad -s_9Xs_9^*+s_{10}Xx_{10}^*+s_{11}Xs_{11}^*-s_{12}Xs_{12}^* \\ \qquad\qquad +s_{13}Xs_{13}^*-s_{14}Xx_{14}^*-s_{15}Xs_{15}^*+s_{16}Xs_{16}^*, \quad X\in\mathcal{O}_{16},\\ \varphi_3(X) \equiv \rho_{16}(X)=\sum_{i=1}^{16} s_iXs_i^*, \quad X\in\mathcal{O}_8. \end{cases} \tag{4.12}$$

As for the standard RFS$_1$, it is easy to write down $s_{i_1,\dots,i_k;\,j_k,\dots,j_1}\in\mathcal{O}_2^{U(1)}$, $k\geqq 1$, in terms of $\Phi_{SR_1}(a_n)$ $(n\leqq k)$ explicitly as follows:

$$s_{i_1,\dots,i_k;\,j_k,\dots,j_1}=(-1)^{\sum_{m=1}^{k-1}(j_m-1)N_m}A_1A_2\cdots A_k, \tag{4.13}$$

$$A_m\equiv\begin{cases} \Phi_{SR_1}(a_m)\Phi_{SR_1}(a_m)^* & \text{for } (i_m,\,j_m)=(1,\,1),\\ \Phi_{SR_1}(a_m) & \text{for } (i_m,\,j_m)=(1,\,2),\\ \Phi_{SR_1}(a_m)^* & \text{for } (i_m,\,j_m)=(2,\,1),\\ \Phi_{SR_1}(a_m)^*\Phi_{SR_1}(a_m) & \text{for } (i_m,\,j_m)=(2,\,2), \end{cases} \qquad m=1,\,2,\,\dots,\,k, \tag{4.14}$$

$$N_m\equiv\sum_{\ell=m+1}^{k}(i_\ell+j_\ell-2)=\sharp\Big\{i\in\{i_{m+1},\dots,i_k,j_{m+1},\dots,j_k\}\;\Big|\;i=2\Big\}. \tag{4.15}$$

Besides the above standard RFS SR_p satisfying $\mathcal{A}_{SR_p}=\mathcal{O}_{2^p}^{U(1)}$, we can construct a RFS R_p so as to obtain $\mathcal{A}_{R_p}\not\subset\mathcal{O}_{2^p}^{U(1)}$. Let φ be an arbitrary inhomogeneous endomorphism of $\mathcal{O}_{2^p}$ in the form of (2.43), and define a RFS$_p$ by $R_p=(\boldsymbol{a}_1,\dots,\boldsymbol{a}_p;\ \zeta_p,\,\varphi_p)$ which is obtained from (4.6)–(4.8) by replacing s_i by $\varphi(s_i)$ $(i=1,\dots,2^p)$. Then, the embedding Φ_{R_p} of the CAR algebra into $\mathcal{O}_{2^p}$ associated with R_p is given by

$$\Phi_{R_p}\equiv\varphi\circ\Phi_{SR_p}. \tag{4.16}$$

Since φ does not commute with the $U(1)$ action γ defined by (2.7), we have indeed $\Phi_{R_p}(\mathrm{CAR})=(\varphi\circ\Phi_{SR_p})(\mathrm{CAR})=\varphi(\mathcal{O}_{2^p}^{U(1)})\not\subset\mathcal{O}_{2^p}^{U(1)}$. For later use, we give two such examples in the case of $p=1$:

$$\begin{cases} \varphi(s_1)\equiv s_{1,2}, \qquad \varphi(s_2)\equiv s_{2;\,1}+s_{1,1;\,2},\\ \quad \boldsymbol{a}_1\equiv s_{1,2,1;\,2}+s_{1,2,2;\,1,1},\\ \quad \zeta_1(X)\equiv s_{1,2}Xs_{1,2}^*-(s_{2;1}+s_{1,1;\,2})X(s_{1;2}+s_{2;\,1,1}), \end{cases} \tag{4.17}$$

$$\begin{cases} \varphi(s_1)\equiv s_{1;\,1}+s_{2,1;\,2}, \qquad \varphi(s_2)\equiv s_{2,2},\\ \quad \boldsymbol{a}_1\equiv s_{1;\,1,2,2}+s_{2,1;\,2,2,2},\\ \quad \zeta_1(X)\equiv (s_{1;1}+s_{2,1;\,2})X(s_{1;1}+s_{2;\,1,2})-s_{2,2}Xs_{2,2}^*. \end{cases} \tag{4.18}$$

Here, the $*$-endomorphism φ in (4.17) is the same as (3.62).

§§4-2. Reduction of the standard RFS_p ($p \geqq 2$) to the standard RFS_1

Using the homogeneous embedding Ψ_p of $\mathcal{O}_{2^p}$ ($p \geqq 2$) into $\mathcal{O}_2$ defined by (2.16) with $d = 2$, it is shown that the standard RFS_p reduces to the standard RFS_1. In this subsection, we denote the seeds of the standard RFS_p ($p \geqq 1$) in $\mathcal{O}_{2^p}$ by $\boldsymbol{a}_j^{(p)}$ ($j = 1, \ldots, p$).

First, we show that the following equality is satisfied:

$$\Psi_p(\boldsymbol{a}_j^{(p)}) = \zeta_1^{j-1}(\boldsymbol{a}_1^{(1)}), \qquad j = 1,\, 2,\, \ldots,\, p. \tag{4.19}$$

Indeed, from (4.6) and (2.18) with $d = 2$, we have

$$\Psi_p(\boldsymbol{a}_1^{(p)}) = \sum_{k=1}^{2^{p-1}} S_{2k-1}^{(p)} S_{2k}^{(p)\,*} = \sum_{k=1}^{2^{p-1}} s_1 S_k^{(p-1)} S_k^{(p-1)\,*} s_2^* = s_{1;2} = \boldsymbol{a}_1^{(1)}, \tag{4.20}$$

$$\begin{aligned}
\Psi_p(\boldsymbol{a}_j^{(p)}) &= \sum_{k=1}^{2^{p-j}} \sum_{\ell=1}^{2^{j-1}} (-1)^{\sum_{m=1}^{j-1}\left[\frac{\ell-1}{2^{m-1}}\right]} S_{2^j(k-1)+\ell}^{(p)} S_{2^{j-1}(2k-1)+\ell}^{(p)\,*} \\
&= \sum_{k=1}^{2^{p-j}} \sum_{\ell'=1}^{2^{j-2}} \Big[(-1)^{\sum_{m=1}^{j-1}\left[\frac{2\ell'-2}{2^{m-1}}\right]} S_{2^j(k-1)+2\ell'-1}^{(p)} S_{2^{j-1}(2k-1)+2\ell'-1}^{(p)\,\prime} \\
&\qquad\qquad + (-1)^{\sum_{m=1}^{j-1}\left[\frac{2\ell'-1}{2^{m-1}}\right]} S_{2^j(k-1)+2\ell'}^{(p)} S_{2^{j-1}(2k-1)+2\ell'}^{(p)\,\prime} \Big] \\
&= \sum_{k=1}^{2^{p-j}} \sum_{\ell'=1}^{2^{j-2}} (-1)^{\sum_{m=1}^{j-2}\left[\frac{\ell'-1}{2^{m-1}}\right]} \Big[s_1 S_{2^{j-1}(k-1)+\ell'}^{(p-1)} S_{2^{j-2}(2k-1)+\ell'}^{(p-1)\,*} s_1^* \\
&\qquad\qquad - s_2 S_{2^{j-1}(k-1)+\ell'}^{(p-1)} S_{2^{j-2}(2k-1)+\ell'}^{(p-1)\,*} s_2^* \Big] \\
&= \zeta_1(\Psi_{p-1}(\boldsymbol{a}_{j-1}^{(p-1)})) = \cdots = \zeta_1^{j-1}(\Psi_{p-j+1}(\boldsymbol{a}_1^{(p-j+1)})) \\
&= \zeta_1^{j-1}(\boldsymbol{a}_1^{(1)}), \qquad j = 2,\, 3, \ldots,\, p,
\end{aligned} \tag{4.21}$$

where (2.2) for $\mathcal{O}_{2^{p-1}}$ is used, and Ψ_1 should be understood as the identity map on $\mathcal{O}_2$. Likewise, for the recursive map ζ_p, we have

$$\begin{aligned}
\Psi_p(\zeta_p(X)) &= \sum_{i=1}^{2^p} (-1)^{\sum_{m=1}^{p}\left[\frac{i-1}{2^{m-1}}\right]} S_i^{(p)} \Psi_p(X) S_i^{(p)\,*} \\
&= \sum_{i'=1}^{2^{p-1}} \Big[(-1)^{\sum_{m=1}^{p}\left[\frac{2i'-2}{2^{m-1}}\right]} S_{2i'-1}^{(p)} \Psi_p(X) S_{2i'-1}^{(p)\,*} + (-1)^{\sum_{m=1}^{p}\left[\frac{2i'-1}{2^{m-1}}\right]} S_{2i'}^{(p)} \Psi_p(X) S_{2i'}^{(p)\,*} \Big] \\
&= \sum_{i'=1}^{2^{p-1}} (-1)^{\sum_{m=1}^{p-1}\left[\frac{i'-1}{2^{m-1}}\right]} \Big[s_1 S_{i'}^{(p-1)} \Psi_p(X) S_{i'}^{(p-1)\,*} s_1^* - s_2 S_{i'}^{(p-1)} \Psi_p(X) S_{i'}^{(p-1)\,*} s_2^* \Big] \\
&= \zeta_1\Big(\sum_{i=1}^{2^{p-1}} (-1)^{\sum_{m=1}^{p-1}\left[\frac{i-1}{2^{m-1}}\right]} S_i^{(p-1)} \Psi_p(X) S_i^{(p-1)\,*} \Big) \\
&= \cdots = \zeta_1^p(\Psi_p(X)), \qquad X \in \mathcal{O}_{2^p}.
\end{aligned} \tag{4.22}$$

Therefore, we obtain

$$
\begin{aligned}
(\Psi_p \circ \Phi_{SR_p})(a_{p(m-1)+j}) &= \Psi_p(\zeta_p^{m-1}(\boldsymbol{a}_j^{(p)})) \\
&= \zeta_1^{p(m-1)+j-1}(\boldsymbol{a}_1^{(1)}) \\
&= \Phi_{SR_1}(a_{p(m-1)+j}), \quad j = 1, 2, \ldots, p; \quad m \in \mathbf{N}, \qquad (4.23)
\end{aligned}
$$

hence

$$
(\Psi_p \circ \Phi_{SR_p})(a_n) = \Phi_{SR_1}(a_n), \qquad n \in \mathbf{N}. \qquad (4.24)
$$

From the above calculations, it is straightforward to generalize (4.24) to the following form:

$$
\Psi_{r,p} \circ \Phi_{SR_p} = \Phi_{SR_r}, \qquad (4.25)
$$

where $\Psi_{r,p}$ denotes the homogeneous embedding of $\mathcal{O}_{2^p}$ into $\mathcal{O}_{2^r}$ with r being an arbitrary divisor of p.

§5. Restriction of Permutation Representations to CAR Subalgebra

In the previous sections, we have discussed on some properties of embeddings, $*$-endomorphisms, the permutation representations and branchings in the Cuntz algebra, and introduced the construction of the recursive fermion system. Hereafter, we discuss on properties of the CAR algebra by restricting those of the Cuntz algebra through the recursive fermion system.

§§5-1. Fock(-like) representation

As shown in Ref. 1), the restriction of the standard representation $\pi_s^{(p)}$ of $\mathcal{O}_{2^p}$ to $\mathcal{A}_{SR_p}$ for an arbitrary p gives the Fock representation, which is denoted by Rep[1], as follows:

$$
\pi_s^{(p)}(a_n)\, e_1 = 0, \quad n \in \mathbf{N}, \qquad (5.1)
$$

$$
\pi_s^{(p)}(a_{n_1}^* a_{n_2}^* \cdots a_{n_k}^*)\, e_1 = e_{N(n_1,\ldots,n_k)}, \quad 1 \leqq n_1 < n_2 < \cdots < n_k, \qquad (5.2)
$$

$$
N(n_1, \ldots, n_k) \equiv 1 + 2^{n_1-1} + \cdots + 2^{n_k-1}, \qquad (5.3)
$$

where we make an identification of $\Phi_{SR_p}(a_n)$ with a_n for simplicity of description. Since it is obvious that any $n \in \mathbf{N}$ is expressible in the form of $N(n_1, \ldots, n_k) - 1$, e_n ($n \in \mathbf{N}$) is uniquely given in the form of the lhs of (5.2), that is, e_1 is the unique vacuum and a cyclic vector of the representation. We can, now, see the fact that the above Fock representation is strictly common to all p is nothing but a direct consequence of (3.16) and (4.24):

$$
\begin{aligned}
\pi_s^{(p)} \circ \Phi_{SR_p} &= (\pi_s \circ \Psi_p) \circ \Phi_{SR_p} \\
&= \pi_s \circ (\Psi_p \circ \Phi_{SR_p}) = \pi_s \circ \Phi_{SR_1}, \qquad (5.4)
\end{aligned}
$$

where π_s is the standard representation of $\mathcal{O}_2$.

As a straightforward generalization of the above, we consider the restriction of Rep(i_0; z) ($i_0 = 1, \ldots, 2^p$) of $\mathcal{O}_{2^p}$ to $\mathcal{A}_{SR_p}$. From (4.5)–(4.7) and (3.7) with $e_{0,m} \equiv e_m$, we have

$$
\pi_{i_0}^{(p)}(a_{p(m-1)+j})\, e_1 = \delta_{i_{0,j},2}\, (-1)^{N_{i_0,j,m}}\, z^{-m}\, \pi_{i_0}^{(p)}((s_{i_0})^{m-1} s_{i_0-2^{j-1}})\, e_1, \qquad (5.5)
$$

$$
\pi_{i_0}^{(p)}(a_{p(m-1)+j}^*)\, e_1 = \delta_{i_{0,j},1}\, (-1)^{N'_{i_0,j,m}}\, z^{-m}\, \pi_{i_0}^{(p)}((s_{i_0})^{m-1} s_{i_0+2^{j-1}})\, e_1. \qquad (5.6)
$$

where $m \in \mathbf{N}$, $j = 1, 2, \ldots, p$, and $i_{0,j}$ is obtained from $i_0 \equiv \sum_{j=1}^{p}(i_{0,j} - 1)2^{j-1} + 1$ with $i_{0,1}, \ldots, i_{0,p} = 1, 2$; $N_{i_0,j,m}$ and $N'_{i_0,j,m}$ are certain integers determined by i_0, j, m. Therefore, using a Bogoliubov transformation $\phi_{i_0}(= \phi_{i_0}^{-1})$ defined by

$$\phi_{i_0}(a_{p(m-1)+j}) \equiv \begin{cases} a_{p(m-1)+j} & \text{for } i_{0,j} = 1, \\ a^*_{p(m-1)+j} & \text{for } i_{0,j} = 2, \end{cases} \quad m \in \mathbf{N}, \quad j = 1, \ldots, p, \tag{5.7}$$

we obtain

$$\pi^{(p)}_{i_0}(a^{(i_0)}_n)\, e_1 = 0, \quad a^{(i_0)}_n \equiv \phi_{i_0}(a_n), \qquad n \in \mathbf{N}. \tag{5.8}$$

It is shown in the same way as (5.2) that any of $\{e_n\}_{n=1}^{\infty}$ is given by an action of $\pi^{(p)}_{i_0}(a^{(i_0)\,*}_{n_1} \cdots a^{(i_0)\,*}_{n_k})$ $(n_1 < \cdots < n_k)$ on e_1. We call this (irreducible) Fock-like representation of the CAR algebra the ϕ_{i_0}-*Fock representation*, and denote it by $\mathrm{Rep}^{(p)}[i_0]$. We have the following relations:

$$\begin{aligned} \mathrm{Rep}^{(p)}[i_0] &\equiv \mathrm{Rep}^{(p)}(i_0;\, z) \circ \Phi_{SR_p} \\ &= \mathrm{Rep}^{(p)}(1) \circ \Phi_{SR_p} \circ \phi_{i_0} \\ &= \mathrm{Rep}[1] \circ \phi_{i_0} \\ &= \mathrm{Fock} \circ \phi_{i_0}, \quad i_0 = 1, 2, \ldots, 2^p. \end{aligned} \tag{5.9}$$

§§5-2. Restriction of permutation representation with central cycle to CAR

Now, we consider a generic irreducible permutation representation with a central cycle and with an eigenvalue z $(|z| = 1)$, $\mathrm{Rep}(L;\, z)$ of $\mathcal{O}_2$, where $L = (i_0, i_1, \ldots, i_{\kappa-1})$ denotes the label of the representation. First, let us recall the κ eigenvectors $e_{\lambda,1}$ $(\lambda = 0, 1, \ldots, \kappa - 1)$ in $\mathrm{Rep}(L;\, z)$ given by (3.9). Then, for $n = \kappa(m-1) + \ell$ with $m \in \mathbf{N}$ and $\ell = 1, 2, \ldots, \kappa$, we have

$$\begin{aligned} &\pi_L(a_n)\, e_{\lambda,1} \\ &\quad= \pi_L\big(\zeta_1^{\kappa(m-1)+\ell-1}(s_{1;2})\big)\, e_{\lambda,1} \\ &\quad= \bar{z}^m \pi_L\big(\zeta_1^{\kappa(m-1)+\ell-1}(s_{1;2})\big)\, \pi_L\big((s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}})^m\big)\, e_{\lambda,1} \\ &\quad= (-1)^{(m-1)N_{\lambda,\kappa}+N_{\lambda,\ell-1}}\, \bar{z}^m \pi_L\big((s_{i_\lambda,\ldots,i_{\lambda-1}})^{m-1} s_{i_\lambda,\ldots,i_{\lambda+\ell-2},1;\,2} s_{i_{\lambda+\ell-1},i_{\lambda+\ell},\ldots,i_{\lambda-1}}\big)\, e_{\lambda,1} \\ &\quad= \delta_{i_{\lambda+\ell-1},2}\, (-1)^{N_{\lambda,n-1}}\, \bar{z}^m \pi_L\big((s_{i_\lambda,\ldots,i_{\lambda-1}})^{m-1} s_{i_\lambda,\ldots,i_{\lambda+\ell-2},1,i_{\lambda+\ell},\ldots,i_{\lambda-1}}\big)\, e_{\lambda,1}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\pi_L(a^*_n)\, e_{\lambda,1} \\ &\quad= \pi_L\big(\zeta_1^{\kappa(m-1)+\ell-1}(s_{2;1})\big)\, e_{\lambda,1} \\ &\quad= \bar{z}^m \pi_L\big(\zeta_1^{(m-1)N+\ell-1}(s_{2;1})\big)\, \pi_L\big((s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}})^m\big)\, e_{\lambda,1} \\ &\quad= (-1)^{(m-1)N_{\lambda,\kappa}+N_{\lambda,\ell-1}}\, \bar{z}^m \pi_L\big((s_{i_\lambda,\ldots,i_{\lambda-1}})^{m-1} s_{i_\lambda,\ldots,i_{\lambda+\ell-2},2;\,1} s_{i_{\lambda+\ell-1},i_{\lambda+\ell},\ldots,i_{\lambda-1}}\big)\, e_{\lambda,1} \\ &\quad= \delta_{i_{\lambda+\ell-1},1}\, (-1)^{N_{\lambda,n-1}}\, \bar{z}^m \pi_L\big((s_{i_\lambda,\ldots,i_{\lambda-1}})^{m-1} s_{i_\lambda,\ldots,i_{\lambda+\ell-2},2,i_{\lambda+\ell},\ldots,i_{\lambda-1}}\big)\, e_{\lambda,1}. \end{aligned} \tag{5.11}$$

where $N_{\lambda,j} \equiv \sum_{r=0}^{j-1}(i_{\lambda+r} - 1)$ $(N_{\lambda,0} \equiv 0)$ is the number of 2 in $\{i_\lambda, \ldots, i_{\lambda+j-1}\}$. One should note that the subscripts of indices i_k's take values in $\mathbf{Z}_\kappa$. Therefore, using a Bogoliubov transformation $\phi_{L,\lambda}$ defined by

$$\phi_{L,\lambda}(a_{\kappa(m-1)+\ell}) \equiv \begin{cases} (-1)^{N_{\lambda,n-1}} a_{\kappa(m-1)+\ell} & \text{for } i_{\lambda+\ell-1} = 1, \\ (-1)^{N_{\lambda,n-1}} a^*_{\kappa(m-1)+\ell} & \text{for } i_{\lambda+\ell-1} = 2, \end{cases} \qquad m \in \mathbf{N}, \quad \ell = 1, \ldots, \kappa, \tag{5.12}$$

we obtain

$$\pi_L(a_n^{(\lambda)})\, e_{\lambda,1} = 0, \quad a_n^{(\lambda)} \equiv \phi_{L,\lambda}(a_n), \qquad n \in \mathbf{N}, \tag{5.13}$$

hence $e_{\lambda,1}$ $(\lambda = 0, 1, \ldots, \kappa-1)$ is a vacuum for the annihilation operators $\{a_n^{(\lambda)} \mid n \in \mathbf{N}\}$, and the corresponding Fock space $\mathcal{H}^{[\lambda]}$ is generated by $\pi_L(a_{n_1}^{(\lambda)*} a_{n_2}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*})\, e_{\lambda,1}$ with $n_1 < n_2 < \cdots < n_r$, $r \in \mathbf{N}$. In the special case $n_i = \kappa(m-1) + \ell_i$ $(m \in \mathbf{N},\ 1 \leqq \ell_1 < \cdots < \ell_r \leqq \kappa,\ 1 \leqq r \leqq k)$, we have

$$\begin{aligned} &\pi_L\big(a_{n_1}^{(\lambda)*} a_{n_2}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1} = \bar{z}^m\, \pi_L\big((s_{i_{\lambda+1},\ldots,i_\lambda})^{m-1} s_J\big)\, e_{\lambda,1}, \\ &s_J \equiv s_{i_\lambda,\ldots,i_{\lambda+\ell_1-2},\hat{i}_{\lambda+\ell_1-1},i_{\lambda+\ell_1},\ldots,i_{\lambda+\ell_2-2},\hat{i}_{\lambda+\ell_2-1},i_{\lambda+\ell_2},\ldots,i_{\lambda+\ell_r-2},\hat{i}_{\lambda+\ell_r-1},i_{\lambda+\ell_r},\ldots,i_{\lambda-1}} \end{aligned} \tag{5.14}$$

with $\hat{i}_\ell \equiv 3 - i_\ell$. Here, one should note that s_J in (5.14) takes any κ-th order monomial of s_i $(i = 1, 2)$ other than $s_{i_\lambda,\ldots,i_{\lambda-1}}$. On the other hand, in the case $n_i = \kappa(m_i - 1) + \ell_i$ $(m_1 < \cdots < m_r,\ 1 \leqq \ell_i \leqq \kappa)$, we have

$$\begin{aligned} \pi_L\big(a_{n_1}^{(\lambda)*} a_{n_2}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1} = \bar{z}^{m_r}\, \pi_L\big(&(s_{i_\lambda,\ldots,i_{\lambda-1}})^{m_1-1} s_{J_1} (s_{i_\lambda,\ldots,i_{\lambda-1}})^{m_2-m_1-1} s_{J_2} \cdots \\ &\times \cdots (s_{i_\lambda,\ldots,i_{\lambda-1}})^{m_r-m_{r-1}-1} s_{J_r}\big)\, e_{\lambda,1}, \\ s_{J_k} \equiv s_{i_\lambda,\ldots,i_{\lambda+\ell_k-2},\hat{i}_{\lambda+\ell_k-1},i_{\lambda+\ell_k},\ldots,i_{\lambda-1}}, &\qquad k = 1, \ldots, r. \end{aligned} \tag{5.15}$$

Taking (3.9) into account, it is, now, easy to infer that any $n\kappa$-th $(n = 0,\ 1,\ \ldots)$ order monomial of s_i $(i = 1,\ 2)$ acting on $e_{\lambda,1}$ is uniquely given by $\pi_L\big(a_{n_1}^{(\lambda)*} a_{n_2}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1}$ up to a $U(1)$ factor with a suitable set of $\{n_1 < n_2 < \cdots < n_r\}$. Therefore, we have[h]

$$\begin{aligned} \mathcal{H}^{[\lambda]} &= \mathrm{Lin}\langle\, \{\ e_{\lambda,1},\ \pi_L\big(a_{n_1}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1},\ 1 \leqq n_1 < \cdots < n_r;\ r \geqq 1\ \}\,\rangle \\ &= \mathrm{Lin}\langle\, \{\ \pi_L(s_{j_1,\ldots,j_{n\kappa}})\, e_{\lambda,1} \ \mid\ j_i = 1,\ 2;\ i = 1, \ldots, n\kappa;\ n \geqq 0\ \}\,\rangle \\ &= \mathrm{Lin}\langle\, \{\ e_{\lambda,m} \ \mid\ m \in \mathbf{N}\ \}\,\rangle. \end{aligned} \tag{5.16}$$

It is obvious that a direct sum of $\mathcal{H}^{[\lambda]}$ $(\lambda = 0, 1, \ldots, \kappa-1)$ gives the total Hilbert space $\mathcal{H}$:

$$\bigoplus_{\lambda=0}^{\kappa-1} \mathcal{H}^{[\lambda]} = \mathrm{Lin}\langle\, \{\ e_{\lambda,m} \ \mid\ \lambda \in \mathbf{Z}_\kappa,\ m \in \mathbf{N}\ \}\,\rangle = \mathcal{H}. \tag{5.17}$$

Therefore, the restriction of the irreducible permutation representation $\mathrm{Rep}(L;\, z)$ of $\mathcal{O}_2$ to $\mathcal{A}_{SR_1}$ gives a direct sum of κ $\phi_{L,\lambda}$-Fock representations as follows:

$$\mathrm{Rep}(L;\, z)\Big|_{\mathcal{A}_{SR_1}} \cong \bigoplus_{\lambda=0}^{\kappa-1} \Big(\mathrm{Fock} \circ \phi_{L,\lambda}\Big). \tag{5.18}$$

This result is nothing but an explicit realization of the general theory for restriction of the permutation representation with a central cycle of $\mathcal{O}_d$ to $\mathcal{O}_d^{U(1)}$ discussed in Ref. 5).

---

[h]It should be understood that the completion of the Hilbert space is carried out.

It should be noted that it is also possible to derive the above formula (5.18) by using the homogeneous embedding Ψ_κ of $\mathcal{O}_{2^\kappa}$ into $\mathcal{O}_2$. From (2.16), we have

$$\Psi_\kappa^{-1}(s_{i_\lambda,\ldots,i_{\kappa-1},i_0,\ldots,i_{\lambda-1}}) = s'_{i(\lambda)}, \quad i(\lambda) \equiv \sum_{\ell=1}^{\kappa}(i_{\lambda+\ell-1}-1)2^{\ell-1}+1, \quad \lambda=0,1,\ldots,\kappa-1, \tag{5.19}$$

where the generators of $\mathcal{O}_{2^\kappa}$ are denoted by $\{s'_i \mid i = 1, 2, \ldots, 2^\kappa\}$. Hence we can rewrite (3.9) as

$$(\pi_L \circ \Psi_\kappa)(s'_{i(\lambda)})\, e_{\lambda,1} = z\, e_{\lambda,1}, \quad \lambda = 0, 1, \ldots, \kappa-1, \tag{5.20}$$

which shows that $\pi_L \circ \Psi_\kappa$ is a reducible permutation representation of $\mathcal{O}_{2^\kappa}$ consisting of a direct sum of κ irreducible ones, i.e., $\mathrm{Rep}^{(\kappa)}(i(\lambda);\, z)$ $(\lambda = 0, 1, \ldots, \kappa-1)$. Therefore, we obtain

$$\begin{aligned}\mathrm{Rep}(L;\, z)\Big|_{\mathcal{A}_{SR_1}} &= \pi_L \circ \Phi_{SR_1} = \pi_L \circ \Psi_\kappa \circ \Phi_{SR_\kappa} \\ &\cong \bigoplus_{\lambda=0}^{\kappa-1} \mathrm{Rep}^{(\kappa)}(i(\lambda);\, z) \circ \Phi_{SR_\kappa} \cong \bigoplus_{\lambda=0}^{\kappa-1} \Big(\mathrm{Fock} \circ \phi_{i(\lambda)}\Big)\end{aligned} \tag{5.21}$$

where use has been made of (5.9) with $p = \kappa$. Here, from (5.7) with $p = \kappa$ and (5.12), $\phi_{i(\lambda)}(a_n)$ for each $n \in \mathbf{N}$ is identical with $\phi_{L,\lambda}(a_n)$ up to sign. Hence the rhs of (5.21) is unitarily equivalent with the rhs of (5.18).

§§5-3. Restriction of permutation representation with chain to CAR

In the same way as above, it is straightforward to obtain the restriction of the permutation representation with a chain of $\mathcal{O}_2$, $\mathrm{Rep}(L_\infty)$ with $L_\infty = \{i_k\}_{k=1}^\infty$, to $\mathcal{A}_{SR_1}$. By direct calculations using (3.12)–(3.15), we have

$$\pi_{L_\infty}(a_n)\, e_{\lambda,1} = \delta_{i_{\lambda+n-1},2}\, (-1)^{N_{\lambda,n}} s_{i_\lambda,\ldots,i_{\lambda+n-2},1}\, e_{\lambda+n,1}, \tag{5.22}$$

$$\pi_{L_\infty}(a_n^*)\, e_{\lambda,1} = \delta_{i_{\lambda+n-1},1}\, (-1)^{N_{\lambda,n}} s_{i_\lambda,\ldots,i_{\lambda+n-2},2}\, e_{\lambda+n,1}, \tag{5.23}$$

where $N_{\lambda,n} \equiv \sum_{j=0}^{n-2}(i_{\lambda+j}-1)$ $(n \geqq 2,\ N_{\lambda,1} \equiv 0)$ is the number of 2 in $\{i_\lambda,\, i_{\lambda+1},\, \ldots,\, i_{\lambda+n-2}\}$, and we set $i_k \equiv 1$ for $k < 0$. By using a Bogoliubov transformation $\phi_{L_\infty,\lambda}$ defined by

$$\phi_{L_\infty,\lambda}(a_n) \equiv \begin{cases}(-1)^{N_{\lambda,n}} a_n & \text{for } i_{\lambda+n-1} = 1, \\ (-1)^{N_{\lambda,n}} a_n^* & \text{for } i_{\lambda+n-1} = 2,\end{cases} \tag{5.24}$$

we obtain

$$\pi_{L_\infty}(a_n^{(\lambda)})\, e_{\lambda,1} = 0, \quad a_n^{(\lambda)} \equiv \phi_{L_\infty,\lambda}(a_n), \quad n \in \mathbf{N}, \tag{5.25}$$

hence $e_{\lambda,1}$ $(\lambda \in \mathbf{Z})$ is a vacuum for the annihilation operators $\{a_n^{(\lambda)} \mid n \in \mathbf{N}\}$, and the corresponding Fock space $\mathcal{H}^{[\lambda]}$ is generated by $\pi_{L_\infty}\big(a_{n_1}^{(\lambda)*} a_{n_2}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1}$ with $n_1 < n_2 < \cdots < n_r$, $r \in \mathbf{N}$. From

$$\begin{gathered}\pi_{L_\infty}\big(a_{n_1}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big)\, e_{\lambda,1} = \pi_{L_\infty}(s_J)\, e_{\lambda+n_r,1}, \\ s_J \equiv s_{i_\lambda,\ldots,i_{\lambda+n_1-2},\hat{i}_{\lambda+n_1-1},i_{\lambda+n_1},\ldots,i_{\lambda+n_2-2},\hat{i}_{\lambda+n_2-1},i_{\lambda+n_2},\ldots,i_{\lambda+n_r-2},\hat{i}_{\lambda+n_r-1}}\end{gathered} \tag{5.26}$$

with $\hat{i}_j \equiv 3 - i_j$, and noting that s_J takes any n_r-th monomial except for $s_{i_\lambda,\ldots,i_{\lambda+n_r-1}}$, we obtain

$$\begin{aligned} \mathcal{H}^{[\lambda]} &= \mathrm{Lin}\langle \{ \, e_{\lambda,1}, \; \pi_L\big(a_{n_1}^{(\lambda)*} \cdots a_{n_r}^{(\lambda)*}\big) \, e_{\lambda,1}, \; 1 \leqq n_1 < \cdots < n_r; \, r \geqq 1 \, \} \rangle \\ &= \mathrm{Lin}\langle \{ \, e_{\lambda,m} \mid m \in \mathbf{N} \, \} \rangle. \end{aligned} \tag{5.27}$$

Hence the total Hilbert space $\mathcal{H}$ is a direct sum of an infinite number of the above Fock-like spaces:

$$\mathcal{H} = \bigoplus_{\lambda \in \mathbf{Z}} \mathcal{H}^{[\lambda]}. \tag{5.28}$$

Thus, the restriction of the permutation representation with a chain of $\mathcal{O}_2$ to $\mathcal{A}_{SR_1}$ gives a direct sum of an infinite number of $\phi_{L_\infty,\lambda}$-Fock representations.

§§5-4. $U(1)$-variant RFS

So far, we have studied the restriction of the permutation representations of $\mathcal{O}_2$ (or $\mathcal{O}_{2^p}$ with $p \geqq 2$) to $\mathcal{A}_{SR_1} = \mathcal{O}_2^{U(1)}$ (or $\mathcal{A}_{SR_p} = \mathcal{O}_{2^p}^{U(1)}$), and found that the resultant representations are reducible in general except for the case of the permutation representation with a central cycle of length 1. However, for a RFS$_1$ R_1 with $\mathcal{A}_{R_1} \not\subset \mathcal{O}_2^{U(1)}$, the situation changes drastically. In the following, we briefly describe this feature.

In $\mathcal{O}_2$, for any irreducible permutation representation with a central cycle, π_L, there exists a RFS R_1 such that $\pi_L \circ \Phi_{R_1}$ is an irreducible representation of the CAR algebra. This fact is nothing but the result of the existence of the $*$-endomorphism φ satisfying (3.61), which is explicitly given by (3.65)–(3.67). Indeed, if we define R_1 by (4.16), then we have

$$\begin{aligned} \pi_L \circ \Phi_{R_1} &= \pi_L \circ (\varphi \circ \Phi_{SR_1}) \\ &= (\pi_L \circ \varphi) \circ \Phi_{SR_1} \\ &\cong \pi_s \circ \Phi_{SR_1} \\ &= \mathrm{Fock}. \end{aligned} \tag{5.29}$$

An example for the case $L = (1, 2)$ is given by (4.17). In this case, the eigenvector $e_{0,1}$ of $\pi_{1,2}(s_{1,2})$ satisfies

$$(\pi_{1,2} \circ \Phi_{R_1})(a_n) \, e_{0,1} = 0, \quad n \in \mathbf{N}, \tag{5.30}$$

and any vector in $\{e_{0,m}, \; e_{1,n}\}$ $(m \geqq 2, \; n \geqq 1)$ is uniquely given by

$$(\pi_{1,2} \circ \Phi_{R_1})(a_{n_1}^* \cdots a_{n_k}^*) \, e_{0,1} \tag{5.31}$$

with an appropriate set of positive integers $n_1 < \cdots < n_k$, $k \geqq 1$.

On the other hand, there also exists a RFS$_1$ R_1 with $\mathcal{A}_{R_1} \not\subset \mathcal{O}_1^{U(1)}$ such that $\pi_s \circ \Phi_{R_1}$ (more generally, $\pi \circ \Phi_{R_1}$ with an arbitrary irreducible representation π) gives an infinitely decomposable representation just like $\pi_{L_\infty} \circ \Phi_{SR_1}$. An example is given by (4.18). Since the term involving $s_{1;1}$ at the right of X in $\zeta_1(X)$ defined by (4.18) vanishes if X includes s_2^* at its right end, it is obvious that any of $\Phi_{R_1}(a_n)$ $(n \in \mathbf{N})$ involves s_2^* at its right end. Therefore, from $\pi_s(s_1) \, e_m = e_{2m-1}$ $(m \in \mathbf{N})$, we obtain

$$(\pi_s \circ \Phi_{R_1})(a_n) e_{2m-1} = 0, \quad n \in \mathbf{N} \tag{5.32}$$

for each $m \in \mathbf{N}$, which means that there exist an infinite number of vacuums $\{e_{2m-1}\}$ $(m \in \mathbf{N})$. Hence the restriction of Rep(1) to the above RFS is a direct sum of an infinite number of Fock representations as follows:

$$\begin{aligned}\pi_s \circ \Phi_{R_1} &\cong (\pi_s \circ \Phi_{SR_1})^{\oplus\infty} \\ &= (\mathrm{Fock})^{\oplus\infty}.\end{aligned} \tag{5.33}$$

§6. Restriction of Permutation Endomorphisms of $\mathcal{O}_2$ to $\mathcal{A}_{SR_1}$

We consider the restriction of the permutation endomorphism φ_σ of $\mathcal{O}_2$ defined by (2.40) and (2.41) to the CAR subalgebra $\mathcal{A}_{SR_1}$ associated with the standard RFS_1 defined by (4.6)–(4.8). Since φ_σ commutes with the $U(1)$ action γ defined by (2.7), we have

$$\varphi_\sigma(\mathcal{O}_2^{U(1)}) \subset \mathcal{O}_2^{U(1)}. \tag{6.1}$$

Thus, the restriction of φ_σ to $\mathcal{O}_2^{U(1)} = \mathcal{A}_{SR_1}$ yields a $*$-endomorphism

$$\begin{aligned}\tilde{\varphi}_\sigma &: \mathrm{CAR} \to \mathrm{CAR}, \\ \tilde{\varphi}_\sigma &\equiv \Phi_{SR_1}^{-1} \circ \varphi_\sigma \circ \Phi_{SR_1}\end{aligned} \tag{6.2}$$

of the CAR algebra. In this section, we identify $\Phi_{SR_1}(a_n)$ with a_n, hence $\tilde{\varphi}_\sigma$ with φ_σ, for simplicity of description. First, we study all the second order permutation endomorphisms and after that we consider some higher order ones.

§§6-1. The second order permutation endomorphisms

To specify each of the second order permutation endomorphisms of $\mathcal{O}_2$ defined by (2.40), we denote it as follows:

$$\varphi_\sigma(s_i) \equiv \sum_{j=1}^{2} S^{(2)}_{\sigma((j-1)2+i)} s_j^* = \sum_{j=1}^{2} s_{\sigma(i,j)}\, s_j^*, \quad i = 1, 2, \quad \sigma \in \mathfrak{S}_4, \tag{6.3}$$

where $S_1^{(2)} = s_{1,1}$, $S_1^{(2)} = s_{2,1}$, $S_3^{(2)} = s_{1,2}$, $S_4^{(2)} = s_{2,2}$. Here, from the one-to-one correspondence between $S_i^{(2)}$ and s_{i_1,i_2}, a natural action of $\sigma \in \mathfrak{S}_4$ on $(i_1, i_2) \in \{1,2\}^2$ is induced. For example, $\sigma = [1,3]$ and $\sigma = [1,2,4]$ denote the transposition of $(1,1) \leftrightarrow (1,2)$, and the cyclic permutation of $(1,1) \to (2,1) \to (2,2) \to (1,1)$, respectively.

Let α be the $*$-automorphism of $\mathcal{O}_2$ defined by $\alpha(s_1) = s_2$, $\alpha(s_2) = s_1$. Then, all the second order permutation endomorphisms of $\mathcal{O}_2$ are given by[9)]

$$\varphi_{id} = id, \tag{6.4}$$

$$\varphi_{[1,2]}(s_1) = s_{2,1;1} + s_{1,2;2} \qquad \varphi_{[1,2]}(s_2) = s_{1,1;1} + s_{2,2;2}, \tag{6.5}$$

$$\varphi_{[1,3]}(s_1) = s_{1,2;1} + s_{1,1;2}, \qquad \varphi_{[1,3]}(s_2) = s_2, \qquad \varphi_{[1,3]} = \alpha \circ \varphi_{[2,4]} \circ \alpha, \tag{6.6}$$

$$\varphi_{[1,4]}(s_1) = s_{2,2;1} + s_{1,2;2} \qquad \varphi_{[1,4]}(s_2) = s_{2,1;1} + s_{1,1;2}, \tag{6.7}$$

$$\varphi_{[2,3]}(s_1) = s_{1,1;1} + s_{2,1;2}, \qquad \varphi_{[2,3]}(s_2) = s_{1,2;1} + s_{2,2;2}, \qquad \varphi_{[2,3]} = \rho, \tag{6.8}$$

$$\varphi_{[2,4]}(s_1) = s_1, \qquad \varphi_{[2,4]}(s_2) = s_{2,2;1} + s_{2,1;2}, \tag{6.9}$$

$$\varphi_{[3,4]}(s_1) = s_{1,1;1} + s_{2,2;2} \qquad \varphi_{[3,4]}(s_2) = s_{2,1;1} + s_{1,2;2}, \qquad \varphi_{[3,4]} = \varphi_{[1,2]} \circ \alpha, \tag{6.10}$$

$$\varphi_{[1,2][3,4]}(s_1) = s_2, \qquad \varphi_{[1,2][3,4]}(s_2) = s_1, \qquad \varphi_{[1,2][3,4]} = \alpha, \tag{6.11}$$

$$\varphi_{[1,3][2,4]}(s_1)=s_{1,2;1}+s_{1,1;2},\quad \varphi_{[1,3][2,4]}(s_2)=s_{2,2;1}+s_{2,1;2},\quad \varphi_{[1,3][2,4]}=\varphi_{[1,4][2,3]}\circ\alpha, \tag{6.12}$$
$$\varphi_{[1,4][2,3]}(s_1)=s_{2,2;1}+s_{2,1;2},\quad \varphi_{[1,4][2,3]}(s_2)=s_{1,2;1}+s_{1,1;2}, \tag{6.13}$$
$$\varphi_{[1,2,3]}(s_1)=s_{2,1;1}+s_{1,1;2},\quad \varphi_{[1,2,3]}(s_2)=s_{1,2;1}+s_{2,2;2}, \tag{6.14}$$
$$\varphi_{[1,2,4]}(s_1)=s_{2,1;1}+s_{1,2;2},\quad \varphi_{[1,2,4]}(s_2)=s_{2,2;1}+s_{1,1;2}, \tag{6.15}$$
$$\varphi_{[1,3,2]}(s_1)=s_{1,2;1}+s_{2,1;2},\quad \varphi_{[1,3,2]}(s_2)=s_{1,1;1}+s_{2,2;2},\quad \varphi_{[1,3,2]}=\varphi_{[2,3,4]}\circ\alpha, \tag{6.16}$$
$$\varphi_{[1,3,4]}(s_1)=s_{1,2;1}+s_{2,2;2},\quad \varphi_{[1,3,4]}(s_2)=s_{2,1;1}+s_{1,1;2},\quad \varphi_{[1,3,4]}=\varphi_{[1,2,3]}\circ\alpha, \tag{6.17}$$
$$\varphi_{[1,4,2]}(s_1)=s_{2,2;1}+s_{1,2;2},\quad \varphi_{[1,4,2]}(s_2)=s_{1,1;1}+s_{2,1;2},\quad \varphi_{[1,4,2]}=\varphi_{[2,4,3]}\circ\alpha, \tag{6.18}$$
$$\varphi_{[1,4,3]}(s_1)=s_{2,2;1}+s_{1,1;2},\quad \varphi_{[1,4,3]}(s_2)=s_{2,1;1}+s_{1,2;2},\quad \varphi_{[1,4,3]}=\varphi_{[1,2,4]}\circ\alpha, \tag{6.19}$$
$$\varphi_{[2,3,4]}(s_1)=s_{1,1;1}+s_{2,2;2},\quad \varphi_{[2,3,4]}(s_2)=s_{1,2;1}+s_{2,1;2}, \tag{6.20}$$
$$\varphi_{[2,4,3]}(s_1)=s_{1,1;1}+s_{2,1;2},\quad \varphi_{[2,4,3]}(s_2)=s_{2,2;1}+s_{1,2;2},\quad \varphi_{[2,4,3]}=\alpha\circ\varphi_{[1,2,3]}\circ\alpha, \tag{6.21}$$
$$\varphi_{[1,2,3,4]}(s_1)=s_2,\quad \varphi_{[1,2,3,4]}(s_2)=s_{1,2;1}+s_{1,1;2},\quad \varphi_{[1,2,3,4]}=\varphi_{[1,3]}\circ\alpha, \tag{6.22}$$
$$\varphi_{[1,2,4,3]}(s_1)=s_{2,1;1}+s_{1,1;2},\quad \varphi_{[1,2,4,3]}(s_2)=s_{2,2;1}+s_{1,2;2},\quad \varphi_{[1,2,4,3]}=\varphi_{[1,4]}\circ\alpha, \tag{6.23}$$
$$\varphi_{[1,3,2,4]}(s_1)=s_{1,2;1}+s_{2,1;2},\quad \varphi_{[1,3,2,4]}(s_2)=s_{2,2;1}+s_{1,1;2}, \tag{6.24}$$
$$\varphi_{[1,3,4,2]}(s_1)=s_{1,2;1}+s_{2,2;2},\quad \varphi_{[1,3,4,2]}(s_2)=s_{1,1;1}+s_{2,1;2},\quad \varphi_{[1,3,4,2]}=\varphi_{[2,3]}\circ\alpha, \tag{6.25}$$
$$\varphi_{[1,4,2,3]}(s_1)=s_{2,2;1}+s_{1,1;2},\quad \varphi_{[1,4,2,3]}(s_2)=s_{1,2;1}+s_{2,1;2},\quad \varphi_{[1,4,2,3]}=\varphi_{[1,3,2,4]}\circ\alpha, \tag{6.26}$$
$$\varphi_{[1,4,3,2]}(s_1)=s_{2,2;1}+s_{2,1;2},\quad \varphi_{[1,4,3,2]}(s_2)=s_1,\quad \varphi_{[1,4,3,2]}=\varphi_{[2,4]}\circ\alpha, \tag{6.27}$$

where ρ is the canonical endomorphism of $\mathcal{O}_2$.

First, we note that there are four $*$-automorphisms in the above $*$-endomorphisms: $\varphi_{id}=id$, $\varphi_{[1,2][3,4]}=\alpha$, $\varphi_{[1,3][2,4]}$ and $\varphi_{[1,4][2,3]}$. As for $\varphi_{[1,4][2,3]}$, we can rewrite it as follows:

$$\varphi_{[1,4][2,3]}(s_1)=s_2J=Js_1J^*, \tag{6.28}$$
$$\varphi_{[1,4][2,3]}(s_2)=s_1J=Js_2J^*, \tag{6.29}$$
$$J\equiv s_{2;1}+s_{1;2},\quad J^*=J,\quad J^2=I, \tag{6.30}$$

where J satisfies $Js_1=s_2$ and $Js_2=s_1$. Thus, $\varphi_{[1,4][2,3]}$ is an inner $*$-automorphism: s_1 and s_2 are expressed in terms of $t_i\equiv\varphi_{[1,4][2,3]}(s_i)$, $(i=1,\,2)$ owing to the identity $t_{2;1}+t_{1;2}=JJJ^*=J$. On the other hand, since α is an outer $*$-automorphism, so is $\varphi_{[1,3][2,4]}(=\varphi_{[1,4][2,3]}\circ\alpha)$.

It should be noted that the restriction of the $*$-automorphism $\alpha=\varphi_{[1,2][3,4]}$ to $\mathcal{A}_{SR_1}$ gives the following Bogoliubov transformation:

$$\alpha(a_n)=\alpha(\zeta_1^{n-1}(s_{1;2}))=(-1)^{n-1}a_n^*,\qquad n\in\mathbf{N}. \tag{6.31}$$

Therefore, we obtain $(\varphi_\sigma\circ\alpha)(a_n)=(-1)^{n-1}\varphi_\sigma(a_n)^*$. Hence we hereafter restrict ourselves to consider only the eleven $*$-endomorphisms in the following: $\varphi_{[1,2]}$, $\varphi_{[1,3]}$, $\varphi_{[1,4]}$, $\varphi_{[2,3]}(=\rho)$, $\varphi_{[2,4]}$, $\varphi_{[1,4][2,3]}$, $\varphi_{[1,2,3]}$, $\varphi_{[1,2,4]}$, $\varphi_{[2,3,4]}$, $\varphi_{[2,4,3]}$, $\varphi_{[1,3,2,4]}$.

Since $\varphi_{[1,4][2,3]}$ is an inner $*$-automorphism, it is easy to obtain its restriction to $\mathcal{A}_{SR_1}$:

$$\varphi_{[1,4][2,3]}(a_n)=Ja_nJ^*=(a_1+a_1^*)a_n(a_1+a_1^*)=\begin{cases}a_1^* & \text{for } n=1,\\ -a_n & \text{for } n\geqq 2.\end{cases} \tag{6.32}$$

Hence this $*$-automorphism of $\mathcal{A}_{SR_1}$ is nothing but a Bogoliubov transformation up to sign for a specific mode a_1.

Next, we consider the restrictions of $\varphi_{[1,2,3]}$ and $\varphi_{[2,4,3]}$. As for $\varphi_{[1,2,3]}$, we have

$$\varphi_{[1,2,3]}(a_1) = s_{2,1;\,2,1} + s_{1,1;\,2,2} = s_2\, a_1\, s_1^* + s_1\, a_1\, s_2^*. \tag{6.33}$$

If $X \in \mathcal{O}_2$ satisfies $\varphi_{[1,2,3]}(X) = s_2\, X\, s_1^* + s_1\, X\, s_2^*$, we have

$$\begin{aligned}\varphi_{[1,2,3]}(\zeta_1(X)) =& (s_{2,1;\,1} + s_{1,1;\,2})(s_2\, X\, s_1^* + s_1\, X\, s_2^*)(s_{1;\,1,2} + s_{2;\,1,1}) \\ &- (s_{1,2;\,1} + s_{2,2;\,2})(s_2\, X\, s_1^* + s_1\, X\, s_2^*)(s_{1;\,2,1} + s_{2;\,2,2}) \\ =& s_2\zeta_1(X)s_1^* + s_1\zeta_1(X)s_2^*. \end{aligned}\tag{6.34}$$

Hence we obtain

$$\begin{aligned}\varphi_{[1,2,3]}(a_n) &= s_2\, a_n\, s_1^* + s_1\, a_n\, s_2^* = (s_{2;1} - s_{1;2})\zeta_1(a_n) \\ &= (a_1^* - a_1)a_{n+1}, \quad n \in \mathbf{N}. \end{aligned}\tag{6.35}$$

Then, from (6.21) and (6.31), we have

$$\begin{aligned}\varphi_{[2,4,3]}(a_n) &= (\alpha \circ \varphi_{[1,2,3]} \circ \alpha)(a_n) = (-1)^{n-1}(\alpha \circ \varphi_{[1,2,3]})(a_n^*) \\ &= (-1)^{n-1}\alpha(a_{n+1}^*(a_1 - a_1^*)) = (-1)^{n-1}(-1)^n a_{n+1}(a_1^* - a_1) \\ &= (a_1^* - a_1)a_{n+1} = \varphi_{[1,2,3]}(a_n). \end{aligned}\tag{6.36}$$

Therefore, $\varphi_{[1,2,3]}$ and $\varphi_{[2,4,3]}$ induce the $*$-endomorphisms of the CAR algebra which are expressed in terms of the second order binomials.

As for $\varphi_{[2,3,4]}$, we have

$$\varphi_{[2,3,4]}(a_1) = s_{1,1;\,2,1} + s_{2,2;\,1,2} = s_1\, a_1\, s_1^* + s_2\, a_1^*\, s_2^*. \tag{6.37}$$

If $X \in \mathcal{O}_2$ satisfies $\varphi_{[2,3,4]}(X) = s_1\, X\, s_1^* \pm s_2\, X^*\, s_2^*$, we have

$$\begin{aligned}\varphi_{[2,3,4]}(\zeta_1(X)) =& (s_{1,1;\,1} + s_{2,2;\,2})(s_1\, X\, s_1^* \pm s_2\, X^*\, s_2^*)(s_{1;\,1,1} + s_{2;\,2,2}) \\ &- (s_{1,2;\,1} + s_{2,1;\,2})(s_1\, X\, s_1^* \pm s_2\, X^*\, s_2^*)(s_{1;\,2,1} + s_{2;\,1,2}) \\ =& s_1\zeta_1(X)s_1^* \mp s_2\zeta_1(X)^* s_2^*. \end{aligned}\tag{6.38}$$

Hence we obtain

$$\begin{aligned}\varphi_{[2,3,4]}(a_n) &= s_1\, a_n\, s_1^* + (-1)^{n-1}\, s_2\, a_n^* s_2^* = s_{1;1}\zeta_1(a_n) + (-1)^n s_{2;2}\zeta_1(a_n)^* \\ &= a_1 a_1^* a_{n+1} + (-1)^n a_1^* a_1 a_{n+1}^*, \quad n \in \mathbf{N}. \end{aligned}\tag{6.39}$$

In a similar way, we obtain

$$\varphi_{[1,2,4]}(a_n) = \begin{cases} (-1)^{\frac{n-1}{2}}\varphi_{[2,3,4]}(a_n)^* & \text{for odd } n, \\ (-1)^{\frac{n}{2}}\varphi_{[2,3,4]}(a_n) & \text{for even } n. \end{cases}\tag{6.40}$$

For the canonical endomorphism $\rho = \varphi_{[2,3]}$, by simple calculations, we obtain

$$\rho(a_n) = \zeta_1(I)\zeta_1(a_n) = (s_{1,1} - s_{2,2})a_{n+1} = K_1 a_{n+1}, \qquad n \in \mathbf{N}, \tag{6.41}$$

$$K_1 \equiv a_1 a_1^* - a_1^* a_1 = I - 2a_1^* a_1 = \exp(\sqrt{-1}\,\pi\, a_1^* a_1), \tag{6.42}$$

where use has been made of (4.4) and an identity $(a_1^*a_1)^2 = a_1^*a_1$. Here, K_1 is the Klein-Jordan-Wigner operator anticommuting with a_1, hence we have

$$[\rho(a_n),\, a_1] \;=\; [\rho(a_n),\, a_1^*] \;=\; 0, \quad n \in \mathbf{N}. \tag{6.43}$$

Hence $\rho(\mathcal{A}_{SR_1})$ is the commutant of the subalgebra generated by a_1 and a_1^*. Likewise, for $\varphi_{[1,4]}$, we obtain

$$\varphi_{[1,4]}(a_n) = (-1)^{n-1}\rho(a_n)^* = (-1)^{n-1}K_1 a_{n+1}^*, \quad n \in \mathbf{N}. \tag{6.44}$$

Therefore, $\varphi_{[2,3,4]}$, $\varphi_{[1,2,4]}$, $\rho = \varphi_{[2,3]}$, and $\varphi_{[1,4]}$ induce the $*$-endomorphisms of the CAR algebra which are expressed in terms of the third order polynomials.

Next, for $\varphi_{[2,4]}$, we have

$$\begin{aligned}
\varphi_{[2,4]}(a_1) &= s_{1,1;\,2,2} + s_{1,2;\,1,2} \\
&= -\, a_1(a_2 + a_2^*), & (6.45)\\
\varphi_{[2,4]}(a_2) &= s_{1,1,1;\,2,2,1} + s_{1,1,2;\,1,2,1} - s_{2,2,1;\,2,1,2} - s_{2,2,2;\,1,1,2} \\
&= -\, (a_1a_1^*a_2 + a_1^*a_1a_2^*)(a_3 + a_3^*), & (6.46)\\
\varphi_{[2,4]}(a_3) &= s_{1,1,1,1;\,2,2,1,1} + s_{1,1,1,2;\,1,2,1,1} - s_{1,2,2,1;\,2,1,2,1} - s_{1,2,2,2;\,1,1,2,1} \\
&\quad - s_{2,2,1,1;\,2,2,2,2} - s_{2,2,1,2;\,1,2,2,2} + s_{2,1,2,1;\,2,1,1,2} + s_{2,1,2,2;\,1,1,1,2} \\
&= -\, a_1a_1^*(a_2a_2^*a_3 + a_2^*a_2a_3^*)(a_4 + a_4^*) + a_1^*a_1(a_2^*a_2a_3 + a_2a_2^*a_3^*)(a_4 + a_4^*). & (6.47)
\end{aligned}$$

In general, we obtain the recurrence formula as follows:

$$\varphi_{[2,4]}(a_n) = a_1a_1^*\, b'_{n-1} - a_1^*a_1\, b''_{n-1} \quad n \geqq 3, \tag{6.48}$$

where b'_{n-1} is obtained from $\varphi_{[2,4]}(a_{n-1})$ by replacing a_k and a_k^* $(k = 1, \ldots, n)$ by a_{k+1} and a_{k+1}^*, respectively, while b''_{n-1} is obtained from b'_{n-1} by exchanging a_2 and a_2^*. It should be noted that $\varphi_{[2,4]}(a_n)$ is expressed in terms of the $(2n)$-th order polynomials. Likewise, for $\varphi_{[1,3]}$, we obtain

$$\varphi_{[1,3]}(a_n) = \begin{cases} (-1)^{\frac{n-1}{2}}\varphi_{[2,4]}(a_n) & \text{for odd } n, \\ (-1)^{\frac{n-2}{2}}\varphi_{[2,4]}(a_n)^* & \text{for even } n. \end{cases} \tag{6.49}$$

Therefore, $\varphi_{[2,4]}$ and $\varphi_{[1,3]}$ induce the $*$-endomorphisms of the CAR algebra which are expressed in terms of even polynomials.

For $\varphi_{[1,2]}$, we have

$$\begin{aligned}
\varphi_{[1,2]}(a_1) &= s_{2,1;\,1,1} + s_{1,2;\,2,2} \\
&= a_1^*a_2a_2^* + a_1a_2^*a_2, & (6.50)\\
\varphi_{[1,2]}(a_2) &= s_{1,2,1;\,1,1,2} + s_{2,1,2;\,2,2,1} - s_{2,2,1;\,1,1,1} - s_{1,1,2;\,2,2,2} \\
&= (a_1^* + a_1)(-a_2^*a_3a_3^* + a_2a_3^*a_3), & (6.51)\\
\varphi_{[1,2]}(a_3) &= s_{2,1,2,1;\,1,1,2,1} + s_{1,2,1,2;\,2,2,1,2} - s_{1,2,2,1;\,1,1,1,2} - s_{2,1,1,2;\,2,2,2,1} \\
&\quad - s_{1,1,2,1;\,1,1,2,2} - s_{2,2,1,2;\,2,2,1,1} + s_{2,2,2,1;\,1,1,1,1} + s_{1,1,1,2;\,2,2,2,2} \\
&= (a_1^* - a_1)(-a_2^* + a_2)(-a_3^*a_4a_4^* + a_3a_4^*a_4). & (6.52)
\end{aligned}$$

In general, we obtain the recurrence formula as follows:

$$\varphi_{[1,2]}(a_n) = (a_1^* + (-1)^n a_1)b_{n-1}, \quad n \geqq 2, \tag{6.53}$$

where b_{n-1} is obtained from $\varphi_{[1,2]}(a_{n-1})$ by replacing a_1, a_1^*, a_k, and a_k^* ($k = 2, \ldots, n$) by a_2, $-a_2^*$, a_{k+1}, and a_{k+1}^*, respectively. It should be noted that $\varphi_{[1,2]}(a_n)$ is expressed in terms of the $(n+2)$-th order polynomials. Likewise, for $\varphi_{[1,3,2,4]}$, we obtain

$$\varphi_{[1,3,2,4]}(a_n) = (-1)^n \varphi_{[1,2]}(a_n)^*, \quad n \in \mathbf{N}. \tag{6.54}$$

Thus, we have completed to clarify restrictions of all the second-order permutation endomorphisms of $\mathcal{O}_2$ to $\mathcal{A}_{SR_1}$.

In summary, the $*$-endomorphisms of the CAR algebra induced by the second order permutation endomorphisms of $\mathcal{O}_2$ are divided broadly into the following:

(1) $*$-automorphisms

$$\begin{aligned} &\text{Identity map:} \quad \varphi_{id}, \\ &\text{Bogoliubov (inner) $*$-automorphism:} \quad \varphi_{[1,4][2,3]}, \\ &\text{Bogoliubov (outer) $*$-automorphisms:} \quad \varphi_{[1,2][3,4]}, \quad \varphi_{[1,3][2,4]}\,; \end{aligned} \tag{6.55}$$

(2) $*$-endomorphisms expressed in terms of the second order binomials

$$\varphi_{[1,2,3]}, \quad \varphi_{[1,3,4]}, \quad \varphi_{[1,4,2]}, \quad \varphi_{[2,4,3]}\,; \tag{6.56}$$

(3) $*$-endomorphisms expressed in terms of the third order polynomials

$$\begin{matrix} \varphi_{[1,4]}, & \varphi_{[2,3]}, & \varphi_{[1,2,4]}, & \varphi_{[1,3,2]}, \\ \varphi_{[1,4,3]}, & \varphi_{[2,3,4]}, & \varphi_{[1,3,4,2]}, & \varphi_{[1,2,4,3]}\,; \end{matrix} \tag{6.57}$$

(4) $*$-endomorphisms expressed in terms of even polynomials

$$\varphi_{[1,3]}, \quad \varphi_{[2,4]}, \quad \varphi_{[1,2,3,4]}, \quad \varphi_{[1,4,3,2]}\,; \tag{6.58}$$

(5) other $*$-endomorphisms expressed in terms of polynomials

$$\varphi_{[1,2]}, \quad \varphi_{[3,4]}, \quad \varphi_{[1,3,2,4]}, \quad \varphi_{[1,4,2,3]}. \tag{6.59}$$

It should be noted that the above division of the induced $*$-endomorphisms is according to their apparent differences only, but not to their intrinsic properties. Indeed, all $*$-endomorphisms of Item (2) and half of Item (3) are expressed as composites of those in Items (4) and (5) as follows:

$$\begin{cases} \varphi_{[1,2,3]} = \varphi_{[1,3]} \circ \varphi_{[1,2]}, & \varphi_{[1,3,4]} = \varphi_{[1,3]} \circ \varphi_{[3,4]}, \\ \varphi_{[1,4,2]} = \varphi_{[2,4]} \circ \varphi_{[1,2]}, & \varphi_{[2,4,3]} = \varphi_{[2,4]} \circ \varphi_{[3,4]}, \end{cases} \tag{6.60}$$

$$\begin{cases} \varphi_{[1,2,4]} = \varphi_{[1,2]} \circ \varphi_{[2,4]}, & \varphi_{[1,3,2]} = \varphi_{[1,2]} \circ \varphi_{[1,3]}, \\ \varphi_{[1,4,3]} = \varphi_{[3,4]} \circ \varphi_{[1,3]}, & \varphi_{[2,3,4]} = \varphi_{[3,4]} \circ \varphi_{[2,4]}. \end{cases} \tag{6.61}$$

On the other hand, the rest of Item 3 are not expressed as above.

In this subsection, we have restricted ourselves to consider the second order permutation endomorphisms only. If we consider more generally the second order homogeneous endomorphisms involving the $*$-automorphism by the $U(2)$ action, we can find a relation between Items (4) and (5) above. Indeed, by using a $*$-automorphism of $\mathcal{O}_2$ given by

$$\begin{aligned} \alpha_\theta(s_1) &= \cos\theta\, s_1 - \sin\theta\, s_2, \\ \alpha_\theta(s_2) &= \sin\theta\, s_1 + \cos\theta\, s_2, \end{aligned} \qquad 0 \leqq \theta < 2\pi, \tag{6.62}$$

we can rewrite $\varphi_{[1,2]}$ as follows:

$$\varphi_{[1,2]} = \alpha_{-\pi/4} \circ \varphi_{[2,4]} \circ \alpha_{\pi/4}, \tag{6.63}$$

where α_θ induces an outer $*$-automorphism of the CAR algebra expressed in terms of a nonlinear transformation as discussed later in Sec. 8.

§§6-2. Even-CAR endomorphisms

As pointed out in Ref. 1), $\varphi_{[2,4]}$ induces the $*$-endomorphism $\tilde{\varphi}_{[2,4]} \equiv {\Phi_{SR_1}}^{-1} \circ \varphi_{[2,4]} \circ \Phi_{SR_1}$ of the CAR algebra onto its even subalgebra[11,12]. Since $\varphi_{[2,4]}$ is nothing but the special case of φ_{σ_p} with $p = 1$ defined by (3.22), we consider it in more general.

Let Γ and $\tilde{\Gamma}$ be the $*$-automorphism of $\mathcal{O}_2$ defined by $\Gamma(s_1) = s_1$, $\Gamma(s_2) = -s_2$ and its induced $*$-automorphism of the CAR algebra defined by $\tilde{\Gamma} \equiv \Phi_{SR_1}^{-1} \circ \Gamma \circ \Phi_{SR_1}$, respectively. Then, from (4.5) with (4.9), we have

$$\tilde{\Gamma}(a_n) = -a_n, \quad n \in \mathbf{N}. \tag{6.64}$$

Hence, from $\Phi_{SR_1}(\mathrm{CAR}) = \mathcal{O}_2^{U(1)}$, we obtain the following $*$-isomorphism:

$$\mathrm{CAR_e} \cong (\mathcal{O}_2^{U(1)})_\mathrm{e}, \tag{6.65}$$

$$\mathrm{CAR_e} \equiv \{X \in \mathrm{CAR} \mid \tilde{\Gamma}(X) = X\}, \tag{6.66}$$

$$(\mathcal{O}_2^{U(1)})_\mathrm{e} \equiv \{X \in \mathcal{O}_2^{U(1)} \mid \Gamma(X) = X\}, \tag{6.67}$$

where $\mathrm{CAR_e}$ is the even subalgebra of the CAR algebra, and $(\mathcal{O}_2^{U(1)})_\mathrm{e}$ is the Γ-fixed point subalgebra of $\mathcal{O}_2^{U(1)}$. Since it is obvious that $\Gamma \circ \varphi_{\sigma_p} = \varphi_{\sigma_p}$ from (3.22), we have

$$\varphi_{\sigma_p}(\mathcal{O}_2^{U(1)}) \subset (\mathcal{O}_2^{U(1)})_\mathrm{e}, \quad p \in \mathbf{N}. \tag{6.68}$$

In the case $p=1$, we also have $\varphi_{\sigma_1}(\mathcal{O}_2^{U(1)}) \supset (\mathcal{O}_2^{U(1)})_\mathrm{e}$, since it is shown inductively that

$$\varphi_{\sigma_1}(\mathcal{D}_k) = \mathcal{E}_k, \quad k \in \mathbf{N}, \tag{6.69}$$

$$\mathcal{D}_k \equiv \{s_{i_1,\dots,i_{k-1},\ell;\,\ell,j_{k-1},\dots,j_1} \mid i_1, \dots, i_{k-1}, j_1, \dots, j_{k-1}, \ell = 1, 2\}, \tag{6.70}$$

$$\mathcal{E}_k \equiv \{X = s_{i_1,\dots,i_k;\,j_k,\dots,j_1} \mid \Gamma(X) = X,\ i_1, \dots, i_k, j_1, \dots, j_k = 1, 2\}, \tag{6.71}$$

where $\{\mathcal{E}_k \mid k \in \mathbf{N}\}$ generates (the dense subset of) $(\mathcal{O}_2^{U(1)})_\mathrm{e}$. Since φ_{σ_1} is injective and $\sharp\,\mathcal{D}_k = 2^{2k-1} = \sharp\,\mathcal{E}_k$, it is sufficient to show $\varphi_{\sigma_1}(\mathcal{D}_k) \subset \mathcal{E}_k$. First, from (3.22), we have

$$\varphi_{\sigma_1}(s_{1;\,1}) = s_{1;\,1}, \quad \varphi_{\sigma_1}(s_{2;\,2}) = s_2 J J^* s_2^* = s_{2;\,2}, \tag{6.72}$$

hence (6.69) is satisfied for $k = 1$. Next, suppose that (6.69) is satisfied for $k = m$, and set $\varphi_{\sigma_1}(s_{i_2,\dots,i_m,\ell;\,\ell,j_m,\dots,j_2}) \equiv s_{i'_2,\dots,i'_{m+1};\,j'_{m+1},\dots,j'_2} \in \mathcal{E}_m$ for a fixed $\ell = 1, 2$. Then, from (3.50), we have

$$\varphi_{\sigma_1}(s_{i_1,i_2,\dots,i_m,\ell;\,\ell,j_m,\dots,j_2,j_1}) = s_{i_1,i''_2,i'_3,\dots,i'_{m+1};\,j'_{m+1},\dots,j'_3,j''_2,j_1}, \tag{6.73}$$

$$i''_2 \equiv \begin{cases} i'_2 & \text{for } i_1 = 1, \\ 3 - i'_2 & \text{for } i_1 = 2, \end{cases} \quad j''_2 \equiv \begin{cases} j'_2 & \text{for } j_1 = 1, \\ 3 - j'_2 & \text{for } j_1 = 2. \end{cases} \tag{6.74}$$

Since the number of 2 in $\{i_1, i_2'', j_1, j_2''\}$ and that in $\{i_2', j_2',\}$ are congruent modulo 2, we have $\varphi_{\sigma_1}(s_{i_1,i_2,\ldots,i_m,\ell;\ell,j_m,\ldots,j_2,j_1}) \in \mathcal{E}_{m+1}$. Thus, (6.69) is obtained. Therefore, we have

$$\varphi_{\sigma_1}(\mathcal{O}_2^{U(1)}) = (\mathcal{O}_2^{U(1)})_{\mathrm{e}}. \tag{6.75}$$

On the other hand, for $p \geqq 2$, $\varphi_{\sigma_p}(\mathcal{O}_2^{U(1)})$ generates a proper subset of $(\mathcal{O}_2^{U(1)})_{\mathrm{e}}$. Indeed, in this case, there exists a proper (i.e., not surjective) $*$-endomorphism φ'_{σ_p} such that

$$\varphi_{\sigma_p} = \varphi'_{\sigma_p} \circ \varphi_{\sigma_1}, \tag{6.76}$$

$$\varphi'_{\sigma_p}(s_1) \equiv s_1, \quad \varphi'_{\sigma_p}(s_2) \equiv s_2 \prod_{k=0}^{p-2} \rho^k(J). \tag{6.77}$$

Since φ'_{σ_p} commutes not only with the $U(1)$ action γ but also with φ_{σ_1}, its restriction to $\varphi_{\sigma_1}(\mathcal{O}_2^{U(1)}) \subset \mathcal{O}_2^{U(1)}$ is also proper. Consequently, we obtain

$$\varphi_{\sigma_p}(\mathcal{O}_2^{U(1)}) = \varphi'_{\sigma_p}\big(\varphi_{\sigma_1}(\mathcal{O}_2^{U(1)})\big) \subsetneqq \varphi_{\sigma_1}(\mathcal{O}_2^{U(1)}) = (\mathcal{O}_2^{U(1)})_{\mathrm{e}}, \quad p \geqq 2. \tag{6.78}$$

Therefore, the restriction of φ_{σ_p} to $\mathcal{A}_{SR_1} = \mathcal{O}_2^{U(1)}$ generally induces a $*$-endomorphism $\tilde{\varphi}_{\sigma_p} \equiv {\Phi_{SR_1}}^{-1} \circ \varphi_{\sigma_p} \circ \Phi_{SR_1}$ of the CAR algebra into its even subalgebra, and only for the case $p = 1$, it gives the $*$-isomorphism between them. In general, the $*$-endomorphism of the CAR algebra whose range is a subset of its even subalgebra is called the *even-CAR endomorphism*. Thus, $\tilde{\varphi}_{\sigma_p}$ ($p \in \mathbf{N}$) are typical examples of the even-CAR endomorphisms.

We write down the explicit expression for $\tilde{\varphi}_{\sigma_p}(a_n)$ in the form of a recurrence formula similar to (6.48) in the following:

$$\tilde{\varphi}_{\sigma_p}(a_n) = \begin{cases} a_n \displaystyle\prod_{\ell=1}^{n+p-1} K_\ell\,(a_{n+p} + a_{n+p}^*) & \text{for } 1 \leqq n \leqq p, \\ b_{m,n} \displaystyle\prod_{\ell=n-p}^{n+p-1} K_\ell\,(a_{n+p} + a_{n+p}^*) & \text{for } mp+1 \leqq n \leqq (m+1)p, \quad m \in \mathbf{N}, \end{cases} \tag{6.79}$$

$$b_{1,n} \equiv a_{n-p} a_{n-p}^* a_n + a_{n-p}^* a_{n-p} a_n^*, \tag{6.80}$$

$$b_{2,n} \equiv a_{n-2p} a_{n-2p}^* b_{1,n} + a_{n-2p}^* a_{n-2p} b'_{1,n}, \tag{6.81}$$

$$b_{m,n} \equiv a_{n-mp} a_{n-mp}^* b_{m-1,n} - a_{n-mp}^* a_{n-mp} b'_{m-1,n}, \qquad m \geqq 3, \tag{6.82}$$

$$K_\ell \equiv a_\ell a_\ell^* - a_\ell^* a_\ell = 1 - 2a_\ell^* a_\ell = \exp(\sqrt{-1}\,\pi\, a_\ell^* a_\ell), \tag{6.83}$$

where $b'_{m,n}$ is obtained from $b_{m,n}$ by exchanging a_{n-mp} and a_{n-mp}^*. It is straightforward to see that (6.45)–(6.48) is reproduced from (6.79)–(6.83) by setting $p = 1$.

§§6-3. Simple examples of higher order permutation endomorphisms

As for other higher order permutation endomorphisms, we give two simple examples in the following.

It is straightforward to generalize the inner $*$-automorphism $\varphi_{[1,4][2,3]}$ so that it should yield the Bogoliubov transformation exchanging a_k and a_k^* for an arbitrary k. For that purpose, we define a linear mapping $\xi : \mathcal{O}_2 \to \mathcal{O}_2$ by

$$\xi(X) \equiv s_2 X s_1^* + s_1 X s_2^*, \quad X \in \mathcal{O}_2. \tag{6.84}$$

Then, it satisfies the following:

$$\xi(X)^* = \xi(X^*), \tag{6.85}$$

$$\xi(X)\xi(Y) = \rho(XY), \quad \xi(X)\rho(Y) = \rho(X)\xi(Y) = \xi(XY), \quad X,\, Y \in \mathcal{O}_2, \tag{6.86}$$

$$\xi(I) = J, \quad J = s_{2;1} + s_{1;2}, \quad J = J^* = J^{-1}. \tag{6.87}$$

As a generalization of the operator J, we introduce J_k ($k \in \mathbf{N}$) as follows:

$$J_k \equiv \xi^k(I), \quad J_k = J_k^* = J_k^{-1}, \quad J_1 = J, \quad k \in \mathbf{N}, \tag{6.88}$$

which satisfies

$$J_k s_i = s_{3-i} J_{k-1}, \quad i = 1,\, 2, \quad k \geqq 2, \tag{6.89}$$

$$J_k J_\ell = J_\ell J_k = \rho^k(J_{\ell-k}), \quad k < \ell, \tag{6.90}$$

$$J_k\, \zeta_1^m(X)\, J_k^* = \begin{cases} (-1)^m \zeta_1^m(J_{k-m} X J_{k-m}^*) & \text{for } m < k, \\ (-1)^k \zeta_1^m(X) & \text{for } m \geqq k, \end{cases} \quad X \in \mathcal{O}_2. \tag{6.91}$$

In terms of J_k, we define an inner $*$-automorphism $\hat{\varphi}_k$ ($k = 1,\, 2,\, \ldots$) of $\mathcal{O}_2$ by

$$\hat{\varphi}_k(X) \equiv \begin{cases} J_1 X J_1^* & \text{for } k = 1, \\ J_{k-1} J_k X J_k^* J_{k-1}^* & \text{for } k \geqq 2, \end{cases} \quad X \in \mathcal{O}_2. \tag{6.92}$$

Then, it is shown that $\hat{\varphi}_k$ is one of the $(k+1)$-th order permutation endomorphisms of $\mathcal{O}_2$. Since $\hat{\varphi}_1 = \varphi_{[1,4][2,3]}$, we consider the case $k \geqq 2$:

$$\begin{aligned} \hat{\varphi}_k(s_i) &= J_{k-1} J_k s_i J_k^* J_{k-1}^* = s_i\, J_{k-2} J_{k-1} J_k J_{k-1} = s_i\, \rho^{k-2}(J_2) \\ &= \sum_{j_1,\ldots,j_{k-2}=1}^{2} s_i s_{j_1,\ldots,j_{k-2}} (s_{2,2;\,1,1} + s_{1,2;\,1,2} + s_{2,1;\,2,1} + s_{1,1;\,2,2}) s_{j_1,\ldots,j_{k-2}}^* \\ &= \sum_{j_1,\ldots,j_k=1}^{2} s_{\sigma(i,j_1,\ldots,j_k)} s_{j_1,\ldots,j_k}^*, \end{aligned}$$

$$\sigma : (i, j_1, \ldots, j_{k-2}, j_{k-1}, j_k) \mapsto (i, j_1, \ldots, j_{k-2}, \hat{j}_{k-1}, \hat{j}_k), \quad \hat{j} \equiv 3 - j. \tag{6.93}$$

From (6.89) and (6.91), it is straightforward to show that $\hat{\varphi}_k$ gives the following Bogoliubov transformation if restricted to $\mathcal{A}_{SR_1}$:

$$\hat{\varphi}_k(a_n) = \begin{cases} a_n & \text{for } n < k, \\ a_k^* & \text{for } n = k, \\ -a_n & \text{for } n > k. \end{cases} \tag{6.94}$$

Another example is the p-th power of the canonical endomorphism of $\mathcal{O}_2$, ρ^p ($p \geqq 1$), which is one of the (p+1)-th order permutation endomorphism. Restricting it to $\mathcal{A}_{SR_1}$, we obtain

$$\rho^p(a_n) = \zeta_1^p(I)\, \zeta_1^p(a_n) = \prod_{m=1}^{p} K_m\, a_{n+p}, \quad K_m \equiv \exp(\sqrt{-1}\, \pi\, a_m^* a_m), \quad n \in \mathbf{N}, \tag{6.95}$$

$$[\rho^p(a_n),\, a_m] = [\rho^p(a_n),\, a_m^*] = 0, \qquad n \in \mathbf{N}, \quad m = 1,\, 2,\, \ldots,\, p. \tag{6.96}$$

Therefore, $\rho^p(\mathcal{A}_{SR_1})$ is the commutant of the $*$-subalgebra generated by $\{a_m \mid 1 \leqq m \leqq p\}$.

§7. Branching of Fock Representation and KMS state

As shown in the previous section, the $*$-endomorphism φ_{σ_p} ($p \in \mathbf{N}$), which is defined by (3.22), of $\mathcal{O}_2$ induces the $*$-endomorphism $\tilde{\varphi}_{\sigma_p} = \Phi_{SR_1}^{-1} \circ \varphi_{\sigma_p} \circ \Phi_{SR_1}$ of the CAR algebra into its even subalgebra. In this section, we consider a branching of the Fock representation of the CAR algebra by $\tilde{\varphi}_{\sigma_p}$, and show that a certain KMS state[13)] of the CAR algebra is obtained.

Let $\pi_p^{\rm even}$ be a representation of the CAR algebra obtained by composing the Fock representation and the even-CAR endomorphism $\tilde{\varphi}_{\sigma_p}$ as follows:

$$\begin{aligned}\pi_p^{\rm even} &\equiv {\rm Fock} \circ \tilde{\varphi}_{\sigma_p} = (\pi_s \circ \Phi_{SR_1}) \circ (\Phi_{SR_1}^{-1} \circ \varphi_{\sigma_p} \circ \Phi_{SR_1}) \\ &= \pi_s \circ \varphi_{\sigma_p} \circ \Phi_{SR_1}. \end{aligned} \tag{7.1}$$

Then, from (3.43), (5.21) and (4.25), it is straightforward to have

$$\begin{aligned}\pi_p^{\rm even} &\cong \bigoplus_{L \in IPR_p} \pi_L \circ \Phi_{SR_1} \\ &\cong \bigoplus_{L \in IPR_p} \bigoplus_{\lambda=0}^{\kappa-1} \pi_{j(\lambda)}^{(\kappa)} \circ \Phi_{SR_\kappa} \\ &\cong \bigoplus_{L \in IPR_p} \bigoplus_{\lambda=0}^{\kappa-1} \pi_{j(\lambda)}^{(\kappa)} \circ \Psi_{\kappa,p} \circ \Phi_{SR_p}, \end{aligned} \tag{7.2}$$

where the label of π_L is set by $L \equiv (j_0, \ldots, j_{\kappa-1})$, $j(\lambda) \equiv \sum_{\ell=1}^{\kappa} (j_{\lambda+\ell-1} - 1) 2^{\ell-1} + 1$ with the subscript of $j_{\lambda+\ell-1}$ taking values in $\mathbf{Z}_\kappa$, and $\Psi_{p,\kappa}$ denoting the homogeneous embedding of $\mathcal{O}_{2^p}$ into $\mathcal{O}_{2^\kappa}$. Substituting (3.18) with $d = 2^\kappa$ and $q = p/\kappa$ into (7.2), we obtain the branching formula as follows:

$$\begin{aligned}\pi_p^{\rm even} &\cong \bigoplus_{L \in IPR_p} \bigoplus_{\lambda=0}^{\kappa-1} \pi_{\tilde{j}(\lambda)}^{(p)} \circ \Phi_{SR_p}, \qquad \tilde{j}(\lambda) \equiv \frac{2^p - 1}{2^\kappa - 1}(j(\lambda) - 1) + 1 \\ &\cong \bigoplus_{i_0=1}^{2^p} \pi_{i_0}^{(p)} \circ \Phi_{SR_p} \\ &\cong \bigoplus_{i_0=1}^{2^p} \Big({\rm Fock} \circ \phi_{i_0} \Big). \end{aligned} \tag{7.3}$$

Here, the vacuum $e_1^{(i_0)}$ of ${\rm Fock} \circ \phi_{i_0}$ is given by (3.33), that is,

$$(\pi_{i_0}^{(p)} \circ \Phi_{SR_p})(a_n^{(i_0)})\, e_1^{(i_0)} = 0, \qquad a_n^{(i_0)} \equiv \phi_{i_0}(a_n), \qquad n \in \mathbf{N}, \tag{7.4}$$

$$\begin{aligned} e_1^{(i_0)} &\equiv \pi_s(s_{i_{0,1},\ldots,i_{0,p}})\, e_1 \\ &= e_{i_0}, \qquad i_0 = 1, 2, \ldots, 2^p, \end{aligned} \tag{7.5}$$

where use has been made of (3.11). We denote the ϕ_{i_0}-Fock space by $\mathcal{H}^{[i_0]}$:

$$\mathcal{H}^{[i_0]} \equiv {\rm Lin}\langle \{ e_1^{(i_0)},\ (\pi_{i_0}^{(p)} \circ \Phi_{SR_p})(a_{n_1}^{(i_0)*} \cdots a_{n_r}^{(i_0)*})\, e_1^{(i_0)},\ n_1 < \cdots < n_r,\ r \geqq 1 \} \rangle. \tag{7.6}$$

Although it is rather complicated to express the basis of $\{e_n^{(i_0)}\}_{n=1}^{\infty}$ of $\mathcal{H}^{[i_0]}$ directly in terms of the basis of $\pi_s^{(p)}$, $\{e_n\}_{n=1}^{\infty}$, of $\mathcal{H}$ except for the vacuum $e_1^{(i_0)}$, we can adopt the similar numbering in (3.11) by an appropriate unitary transformation:

$$e_n^{(i_0)} \equiv (\pi_{i_0}^{(p)} \circ \Phi_{SR_p})(a_{n_1}^{(i_0)\,*} \cdots a_{n_r}^{(i_0)\,*})\, e_1^{(i_0)}, \qquad n \equiv \sum_{k=1}^{r} 2^{n_k} + 1. \tag{7.7}$$

In any way, the total Hilbert space $\mathcal{H}$, which is the representation space of $\pi_s \circ \Phi_{SR_1}$(= $\pi_s^{(p)} \circ \Phi_{SR_p}$), is decomposed into a direct sum of mutually orthogonal subspaces:

$$\mathcal{H} = \bigoplus_{i_0=1}^{2^p} \mathcal{H}^{[i_0]}, \qquad \mathcal{H}^{[i_0]} \perp \mathcal{H}^{[i_0']}, \quad i_0 \neq i_0'. \tag{7.8}$$

Now, we consider a state ω of the CAR algebra defined by the representation π_p^{even} with an appropriate unit vector $\Omega \in \mathcal{H}$ as follows:

$$\omega(X) \equiv \langle\, \Omega \,|\, \pi_p^{\text{even}}(X)\, \Omega \,\rangle, \qquad X \in \text{CAR}. \tag{7.9}$$

If we set

$$\Omega \equiv \sum_{i_0=1}^{2^p} \sqrt{\Lambda_{i_0}}\, e_1^{(i_0)} \tag{7.10}$$

with $\{\Lambda_{i_0}\}_{i_0=1}^{2^p}$ being a set of nonnegative constants satisfying $\sum_{i_0=1}^{2^p} \Lambda_{i_0} = 1$, then, from (7.3) and (7.8), we can rewrite ω as a convex sum of pure states as follows:

$$\omega(X) = \sum_{i_0=1}^{2^p} \Lambda_{i_0}\, \omega_{i_0}(X), \tag{7.11}$$

$$\omega_{i_0}(X) \equiv \langle\, e_1^{(i_0)} \,|\, (\pi_{i_0}^{(p)} \circ \Phi_{SR_p})(X)\, e_1^{(i_0)} \,\rangle, \tag{7.12}$$

where ω_{i_0} is pure since $\pi_{i_0}^{(p)} \circ \Phi_{SR_p} = \text{Fock} \circ \phi_{i_0}$ is irreducible. We parametrize $\{\Lambda_{i_0}\}$ by

$$\Lambda_{i_0} = \prod_{j=1}^{p} \Lambda_{j,\, i_{0,j}}, \tag{7.13}$$

$$\Lambda_{j,\, i_{0,j}} \equiv \begin{cases} 1-\lambda_j & \text{for } i_{0,j} = 1, \\ \lambda_j & \text{for } i_{0,j} = 2 \end{cases} \tag{7.14}$$

with $i_0 = \sum_{j=1}^{p} (i_{0,j} - 1) 2^{j-1} + 1$ $(i_{0,1}, \ldots, i_{0,p} = 1, 2)$ and $0 \leqq \lambda_i \leqq 1$ $(j = 1, \ldots, p)$, so that we have

$$\sum_{i_0 \in A_{j,\,1}} \Lambda_{i_0} = 1 - \lambda_j, \qquad \sum_{i_0 \in A_{j,\,2}} \Lambda_{i_0} = \lambda_j \tag{7.15}$$

with $A_{j,\,1}$ and $A_{j,\,2}$ $(j = 1, \ldots, p)$ being a set of all indices i_0's which satisfy $i_{0,j} = 1$ and that $i_{0,j} = 2$, respectively. Then, we obtain

$$\omega(a_{p(m-1)+j}\, a_{p(n-1)+k}^{*}) = \delta_{m,n}\delta_{j,k} \sum_{i_0 \in A_{j,\,1}} \Lambda_{i_0} = \delta_{m,n}\delta_{j,k}(1 - \lambda_j), \tag{7.16}$$

$$\omega(a_{p(m-1)+j}^{*}\, a_{p(n-1)+k}) = \delta_{m,n}\delta_{j,k} \sum_{i_0 \in A_{j,\,2}} \Lambda_{i_0} = \delta_{m,n}\delta_{j,k}\lambda_j \tag{7.17}$$

with $m, n \in \mathbf{N}$, $j, k = 1, \ldots, p$, and states for products of the same number of a_m's and a_n^*'s expressed in terms of (7.16) and (7.17), while others vanish.

If each of $\{\lambda_j \mid j = 1, \cdots, p\}$ satisfies $0 < \lambda_j < 1/2$, we rewrite it as follows:

$$\lambda_j = \frac{1}{1 + \exp(\beta \varepsilon_j)}, \quad \varepsilon_j > 0, \quad \beta > 0. \tag{7.18}$$

Then, ω is identical with the KMS state of the CAR algebra with the inverse temperature β with respect to the one-parameter group $\{\tau_t^{(0)} \mid t \in \mathbf{R}\}$ of $*$-automorphisms defined by

$$\tau_t^{(0)}(a_{p(m-1)+j}) \equiv \exp(-\sqrt{-1}\, \varepsilon_j\, t)\, a_{p(m-1)+j}, \quad m \in \mathbf{N},\ j = 1, \ldots, p, \tag{7.19}$$

which describes the time evolution of a (quasi-)free fermion system. Here, the superscript (0) stands for the free fermions. Indeed, it is shown that ω satisfies the KMS condition[13] given by

$$\omega(X\, \tau^{(0)}_{\sqrt{-1}\beta}(Y)) = \omega(YX), \quad X, Y \in \mathrm{CAR}. \tag{7.20}$$

It is remarkable that we can induce $\{\tau_t^{(0)} \mid t \in \mathbf{R}\}$ from a one-parameter group $\{\alpha_t^{(0)} \mid t \in \mathbf{R}\}$ of $*$-automorphisms of $\mathcal{O}_{2^p}$ by using the embedding Φ_{SR_p} of the CAR algebra into $\mathcal{O}_{2^p}$ associated with the standard RFS_p as follows:

$$\tau_t^{(0)} = \Phi_{SR_p}^{-1} \circ \alpha_t^{(0)} \circ \Phi_{SR_p}, \tag{7.21}$$

$$\alpha_t^{(0)}(s_i) \equiv \exp\left(\sqrt{-1}\, \varepsilon(i)\, t\right) s_i, \quad \varepsilon(i) \equiv \sum_{j=1}^{p} (i_j - 1)\varepsilon_j + \varepsilon \tag{7.22}$$

with $i = \sum_{j=1}^{p} (i_j - 1) 2^{j-1} + 1$ $(i = 1, \ldots, 2^p;\ i_j = 1, 2)$, ε being an arbitrary real constant. However, one should note that the above ω is *not* induced from the KMS state of $\mathcal{O}_{2^p}$ with respect to the one-parameter group $\{\alpha_t^{(0)}\}$. In contrast with the KMS state for the CAR algebra, as is well-known, the inverse temperature β for the KMS state of the Cuntz algebra is uniquely determined for a one-parameter group of $*$-automorphisms such as $\{\alpha_t^{(0)}\}$.[14–16]

If each of $\{\lambda_j \mid j = 1, \cdots, p\}$ satisfies $1/2 < \lambda_j < 1$, using the Bogoliubov transformation $\phi_{2^p}(a_n) = a_n^*$ $(n \in \mathbf{N})$, we can identify $\omega \circ \phi_{2^p}$ with the KMS state as above. If $\lambda_j = 1/2$ for any $j = 1, \ldots, p$, or equivalently $\Lambda_{i_0} = 1/2^p$ for any $i_0 = 1, \ldots, 2^p$, then, ω is the normalized trace, that is, it satisfies

$$\omega(XY) = \omega(YX), \quad X, Y \in \mathrm{CAR}. \tag{7.23}$$

If $\lambda_j = 0, 1$ for any $j = 1, \ldots, p$, or equivalently $\Lambda_{i_0} = 1$ for one i_0 and all the other Λ_{i_0}'s vanishing, then, ω is nothing but the pure state obtained from the ϕ_{i_0}-Fock representation.

For the case $p = 1$, the above classification with respect to λ_1 is complete, and reproduces the Araki-Woods classification of factors for the CAR algebra.[17] For the case $p \geqq 2$, we decompose the set $\{1, \ldots, p\}$ into three disjoint subsets as follows:

$$\{1, \ldots, p\} = J_1 \cup J_2 \cup J_3, \tag{7.24}$$

$$\begin{aligned} &J_1 \equiv \big\{ j \in \{1, \ldots, p\} \mid \lambda_j = 0,\ 1 \big\}, \\ &J_2 \equiv \big\{ j \in \{1, \ldots, p\} \mid \lambda_j = \tfrac{1}{2} \big\}, \\ &J_3 \equiv J_{3,1} \cup J_{3,2}, \quad \begin{cases} J_{3,1} \equiv \big\{ j \in \{1, \ldots, p\} \mid 0 < \lambda_j < \tfrac{1}{2} \big\}, \\ J_{3,2} \equiv \big\{ j \in \{1, \ldots, p\} \mid \tfrac{1}{2} < \lambda_j < 1 \big\}. \end{cases} \end{aligned} \tag{7.25}$$

Correspondingly, we decompose each monomial in the CAR algebra into a product of elements of the three C^*-subalgebras as follows:

$$\mathcal{A}_k \equiv C^*\langle \{a_{p(m-1)+j} \mid m \in \mathbf{N},\ j \in J_k\} \rangle, \quad k = 1, 2, 3. \tag{7.26}$$

Then, we can factorize ω into the following form:

$$\omega(X_1\, X_2\, X_3) = \prod_{k=1}^{3} \omega_k(X_k), \quad X_k \in \mathcal{A}_k, \tag{7.27}$$

where ω_1, ω_2, and ω_3 are the vector state associated with the Fock-like representation, the normalized trace, and the ϕ_i-transformed KMS state for the one-parameter group $\{\tau_t^{(0)}\}$ of $*$-automorphisms, respectively, where ϕ_i is the Bogoliubov transformation defined by (5.7) with $i \equiv \sum_{j \in J_{3,2}} 2^{j-1} + 1$. Therefore, the state ω given by (7.11) and (7.12) for a generic p may be also understood according to the classification by Araki-Woods.

§8. Induced Automorphism of the CAR Algebra

In this section, we summarize the induced $*$-automorphism[2)] of the CAR algebra from the $U(2^p)$ action on the Cuntz algebra $\mathcal{O}_{2^p}$, and apply it to construct one-parameter groups of $*$-automorphisms of the CAR algebra describing nontrivial time evolutions of fermions.

We consider a $*$-automorphism α_u of $\mathcal{O}_{2^p}$ obtained from the action of $U(2^p)$ as follows:

$$\alpha_u(s_i) \equiv \sum_{k=1}^{2^p} s_k u_{k,i}, \quad u = (u_{k,i}) \in U(2^p), \quad i = 1,\, 2,\, \ldots,\, 2^p, \tag{8.1}$$

Since α_u commutes with the $U(1)$ action γ defined by (2.7), the restriction of α_u to $\mathcal{O}_{2^p}^{U(1)}$ gives a $*$-automorphism of $\mathcal{O}_{2^p}^{U(1)}$. Therefore, from $\mathcal{A}_{SR_p} = \mathcal{O}_{2^p}^{U(1)}$, α_u induces a $*$-automorphism τ_u of the CAR algebra as follows:

$$\begin{aligned} \tau_u : U(2^p) \curvearrowright \mathrm{CAR}, \quad u \in U(2^p), \\ \tau_u \equiv \varPhi_{SR_p}^{-1} \circ \alpha_u \circ \varPhi_{SR_p}. \end{aligned} \tag{8.2}$$

It is straightforward to show that $\tau_u(a_n)$ is expressed in terms of a polynomial in a_k and a_ℓ^* with $k,\, \ell \leqq pm$ for $p(m-1)+1 \leqq n \leqq pm$. Therefore, τ_u gives a nonlinear transformation of the CAR algebra associated with $u \in U(2^p)$. The whole set of τ_u with $u \in U(2^p)$ denoted by

$$\mathrm{Aut}_{U(2^p)}(\mathrm{CAR}) \equiv \{\, \tau_u \mid u \in U(2^p) \,\} \tag{8.3}$$

constitutes a nonlinear realization of $U(2^p)$ on the CAR algebra. Using the homogeneous embedding $\varPsi_{p,q}$ of $\mathcal{O}_{2^q}$ into $\mathcal{O}_{2^p}$ with q being a nontrivial multiple of p, which is defined by (2.16), it is shown that, for any $u \in U(2^p)$, there exists $v \in U(2^q)$ such that

$$\alpha_u \circ \varPsi_{p,q} = \varPsi_{p,q} \circ \alpha_v. \tag{8.4}$$

From (4.25) and (8.2), we obtain

$$\begin{aligned} \tau_u &= \varPhi_{SR_p}^{-1} \circ \alpha_u \circ \varPhi_{SR_p} \\ &= \varPhi_{SR_p}^{-1} \circ (\alpha_u \circ \varPsi_{p,q}) \circ \varPhi_{SR_q} = (\varPhi_{SR_p}^{-1} \circ \varPsi_{p,q}) \circ \alpha_v \circ \varPhi_{SR_q} \\ &= \varPhi_{SR_q}^{-1} \circ \alpha_v \circ \varPhi_{SR_q} \\ &= \tau_v, \end{aligned} \tag{8.5}$$

hence we obtain

$$\mathrm{Aut}_{U(2^p)}(\mathrm{CAR}) \subset \mathrm{Aut}_{U(2^{rp})}(\mathrm{CAR}), \quad r = 2, 3, \ldots. \tag{8.6}$$

Now, it becomes possible to define a product $\tau_{u_1} \circ \tau_{u_2}$ even for $u_i \in U(2^{p_i})$ ($p_1 \neq p_2$) explicitly by $\tau_{v_1} \circ \tau_{v_2} = \tau_{v_1 v_2}$, $\tau_{v_i} = \tau_{u_i}$, $v_i \in U(2^q)$ with q being the least common multiple of p_1 and p_2. Thus, we obtain an infinite-dimensional $*$-automorphism group of the CAR algebra defined by

$$\mathrm{Aut}_U(\mathrm{CAR}) \equiv \{\, \tau_u \mid u \in U(2^p),\ p \in \mathbf{N}\,\}. \tag{8.7}$$

Next, we consider $*$-automorphism subgroups $A_p^{(i)}$ ($i = 1, \ldots, 2^p$) of $\mathcal{O}_{2^p}$ defined by

$$A_p^{(i)} \equiv \{\, \alpha_u \mid u_{i,i} \in U(1),\ u_{j,i} = 0,\ j \neq i\}, \quad i = 1, \ldots, 2^p, \tag{8.8}$$

where α_u is defined by (8.1). Then, it is obvious that $A_p^{(i)} \cong U(1) \times U(2^p - 1)$. Correspondingly, we introduce the subgroup $\mathrm{Aut}^{(i)}_{U(2^p)}(\mathrm{CAR})$ of $\mathrm{Aut}_{U(2^p)}(\mathrm{CAR})$ by

$$\mathrm{Aut}^{(i)}_{U(2^p)}(\mathrm{CAR}) \equiv \{\, \tau_u = \Phi_{SR_p}^{-1} \circ \alpha_u \circ \Phi_{SR_p} \mid \alpha_u \in A_p^{(i)}\,\}. \tag{8.9}$$

Since we have $\pi_L \circ \alpha_u \cong \pi_{L'}$ with $\alpha_u \in A_p^{(i)}$, $L = (i; z)$, $L' = (i; z')$ and $z, z' \in U(1)$, we obtain the unitary equivalence as follows:

$$\mathrm{Rep}^{(p)}[\,i\,] \circ \tau^{(i)} \cong \mathrm{Rep}^{(p)}[\,i\,], \quad \tau^{(i)} \in \mathrm{Aut}^{(i)}_{U(2^p)}(\mathrm{CAR}), \tag{8.10}$$

where $\mathrm{Rep}^{(p)}[\,i\,]$ is defined by (5.9). In general, however, we have $\mathrm{Rep}^{(p)}[j] \circ \tau^{(i)} \not\cong \mathrm{Rep}^{(p)}[j]$ for $j \neq i$, hence $\tau^{(i)} \in \mathrm{Aut}^{(i)}_{U(2^p)}(\mathrm{CAR})$ is generally an outer $*$-automorphism of the CAR algebra.

Now, we consider the subgroup of $\mathrm{Aut}_U(\mathrm{CAR})$ defined by

$$\mathrm{Aut}^{(1)}_U(\mathrm{CAR}) \equiv \bigcup_{p \in \mathbf{N}} \mathrm{Aut}^{(1)}_{U(2^p)}(\mathrm{CAR}). \tag{8.11}$$

Then, as seen from (7.19) with (7.21) and (7.22), any one-parameter group of $*$-automorphisms corresponding to the time evolution of a (quasi-)free fermion system is contained in $\mathrm{Aut}^{(1)}_U(\mathrm{CAR})$. As for a generic one-parameter group of $*$-automorphism in (8.11), because of its nonlinearity, it describes a time evolution in which the particle number changes generally. In the following, we show this property of (8.11) by using a few simple examples.

Example 1. Let $\{\alpha_t \mid t \in \mathbf{R}\}$ be a one-parameter group of $*$-automorphisms of $\mathcal{O}_4$ defined by

$$\begin{cases} \alpha_t(s_i) = s_i, \quad i = 1, 2, \\ \alpha_t(s_3) = \cos\theta_t\, s_3 - \sin\theta_t\, s_4, \quad \alpha_t(s_4) = \sin\theta_t\, s_3 + \cos\theta_t\, s_4, \quad \theta_t \equiv \mu t, \end{cases} \tag{8.12}$$

where μ is a nonvanishing real constant. Then, the corresponding induced one-parameter group $\{\tau_t \mid t \in \mathbf{R}\}$ of $*$-automorphisms of the CAR algebra is obtained as follows:

$$\tau_t(a_{2m-1}) = G_{m-1}\Big[a_{2m-1} - \sin\theta_t\Big(\sin\theta_t\,(a_{2m-1} + a^*_{2m-1}) - \cos\theta_t\,W_{2m-1}\Big)a^*_{2m}a_{2m}\Big], \tag{8.13}$$

$$\tau_t(a_{2m}) = G_{m-1}\Big[\cos\theta_t\,a_{2m} + \sin\theta_t\,W_{2(m-1)}\,(a_{2m-1} - a^*_{2m-1})\,a_{2m}\Big], \tag{8.14}$$

$$G_n \equiv \prod_{k=1}^{n} F_k, \quad G_0 \equiv I, \tag{8.15}$$

$$F_k \equiv I - 2\sin\theta_t\Big(\sin\theta_t\,I - \cos\theta_t\,W_{2(k-1)}\,(a_{2k-1} - a^*_{2k-1})\Big)a^*_{2k}a_{2k}, \tag{8.16}$$

$$W_n \equiv \prod_{\ell=1}^{n} K_\ell, \quad W_0 \equiv I, \quad K_\ell \equiv a_\ell a^*_\ell - a^*_\ell a_\ell = \exp(\sqrt{-1}\,\pi\,a^*_\ell a_\ell), \tag{8.17}$$

where $m \in \mathbf{N}$, and F_k's satisfy $[F_k,\, F_\ell] = 0$. Since the vacuum e_1 in the Fock representation at $t = 0$, which is defined by (5.1) with (5.2), satisfies

$$\tau_t(a_n)\,e_1 = 0, \quad t \in \mathbf{R}, \quad n \in \mathbf{N}, \tag{8.18}$$

as it should be, we can adopt e_1 as the vacuum at any t. Then, τ_t does *not* conserve the particle number due to the nonvanishing term proportional to $a_{2m-1}a_{2m}$ in $\tau_t(a_{2m})$ ($m \in \mathbf{N}$). To show this in detail, we define the (formal) particle number operator at t, N_t, as follows:

$$N_t \equiv \sum_{n=1}^{\infty} \tau_t(a^*_n a_n), \tag{8.19}$$

$$\tau_t(a^*_{2m-1}a_{2m-1}) = H_{m-1}\Big[a^*_{2m-1}a_{2m-1} + \sin\theta_t\Big(\sin\theta_t\,K_{2m-1} + \cos\theta_t(a_{2m-1} + a^*_{2m-1})\Big)a^*_{2m}a_{2m}\Big], \tag{8.20}$$

$$\tau_t(a^*_{2m}a_{2m}) = H_{m-1}\,a^*_{2m}a_{2m}, \tag{8.21}$$

$$H_n \equiv \prod_{k=1}^{n}\Big(I + \sin^2(2\theta_t)\,a^*_{2k}a_{2k}\Big), \quad H_0 \equiv I. \tag{8.22}$$

Obviously, N_t is explicitly dependent on t. An eigenvector of N_t with an eigenvalue k is called a *k-particle state vector* at t. A generic k-particle state vector at t is, of course, a linear combination of vectors in the form of $\tau_t(a^*_{n_1}\cdots a^*_{n_k})\,e_1$ ($n_1 < \cdots < n_k$), and is orthogonal to any k'-particle state vector at t with $k' \neq k$. Then, since, from (8.14), we have

$$\tau_t(a^*_{2m})\,e_1 = (\cos\theta_t\,I - \sin\theta_t\,a^*_{2m-1})a^*_{2m}\,e_1, \qquad m \in \mathbf{N}, \tag{8.23}$$

a one-particle state vector at $t \neq \pi c/\mu$ ($c \in \mathbf{Z}$) given by $\tau_t(a^*_{2m})\,e_1$ is *not* orthogonal to a two-particle state vector at $t = 0$ given by $\tau_0(a^*_{2m-1}a^*_{2m})\,e_1 = a^*_{2m-1}a^*_{2m}\,e_1$ ($m \in \mathbf{N}$):

$$\langle\,\tau_t(a^*_{2m})\,e_1 \mid \tau_0(a^*_{2m-1}a^*_{2m})\,e_1\,\rangle = -\sin\theta_t. \tag{8.24}$$

Therefore, the one-parameter group $\{\tau_t\}$ of $*$-automorphisms given by (8.13)–(8.16) does not conserve the particle number.

Next, we consider the expectation value of N_t by a k-particle state at $t=0$, v_k, as follows:

$$\omega(N_t;\, v_k) \equiv \langle\, v_k \mid N_t\, v_k \,\rangle, \quad N_0\, v_k = k\, v_k, \quad k \in \mathbf{N}, \tag{8.25}$$

which may give the particle number of v_k at t. From (8.19)–(8.22), it is straightforward to calculate $\omega(N_t;\, v_k)$ for each v_k. More precisely, we denote a k-particle state for a specific set of the creation operators by

$$v_{n_1,n_2,\ldots,n_k} \equiv a^*_{n_1} a^*_{n_2} \cdots a^*_{n_k}\, e_1, \quad n_1 < n_2 < \cdots < n_k. \tag{8.26}$$

Then, for one-particle state vectors and two-particle state vectors, we obtain

$$\omega(N_t;\, v_{n_1}) = \begin{cases} 1 & \text{for } n_1 = 2m_1 - 1, \\ 1 + \sin^2\theta_t & \text{for } n_1 = 2m_1, \end{cases} \tag{8.27}$$

$$\omega(N_t;\, v_{n_1,n_2}) = \begin{cases} 2 & \text{for } n_1 = 2m_1 - 1,\ n_2 = 2m_2 - 1, \\ 2 - \sin^2\theta_t & \text{for } n_1 = 2m_1 - 1,\ n_2 = 2m_1, \\ 2 + \sin^2\theta_t & \text{for } n_1 = 2m_1 - 1,\ n_2 = 2m_2, \\ 2 + \sin^2\theta_t + \sin^2(2\theta_t) & \text{for } n_1 = 2m_1, \quad n_2 = 2m_2 - 1, \\ (2 + \sin^2(2\theta_t))(1 + \sin^2\theta_t) & \text{for } n_1 = 2m_1, \quad n_2 = 2m_2, \end{cases} \tag{8.28}$$

where $m_1, m_2 \in \mathbf{N}$ $(m_1 < m_2)$. As for k-particle state vectors, in particular case in which either all of $n_1, \ldots, n_k$ $(n_1 < \cdots < n_k)$ are odd or they are even, it is easy to obtain the following:

$$\omega(N_t;\, v_{2m_1-1,\ldots,2m_k-1}) = k, \tag{8.29}$$

$$\omega(N_t;\, v_{2m_1,\ldots,2m_k}) = \frac{(1 + \sin^2(2\theta_t))^k - 1}{\sin^2(2\theta_t)}\,(1 + \sin^2\theta_t). \tag{8.30}$$

In this way, $\omega(N_t;\, v_k)$ is, in general, dependent on t nontrivially.

Example 2. Let $\{\alpha_t \mid t \in \mathbf{R}\}$ be a one-parameter group of $*$-automorphisms of $\mathcal{O}_8$ defined by

$$\begin{cases} \alpha_t(s_i) = s_i, \quad i = 1,2,3,4,6,7, \\ \alpha_t(s_5) = \cos\theta_t\, s_5 - \sin\theta_t\, s_8, \quad \alpha_t(s_8) = \sin\theta_t\, s_5 + \cos\theta_t\, s_8, \quad \theta_t \equiv \mu t, \end{cases} \tag{8.31}$$

where μ is a nonvanishing real constant. Then, the corresponding induced one-parameter group $\{\tau_t \mid t \in \mathbf{R}\}$ of $*$-automorphisms of the CAR algebra is much simpler than (8.13)–(8.17), since we have $\alpha_t(\zeta_3(X)) = \zeta_3(\alpha_t(X))$ $(X \in \mathcal{O}_8)$ for the recursive map ζ_3 defined by (4.11), and $\zeta_p(X_1 \cdots X_n) = \zeta_p(X_1) \cdots \zeta_p(X_n)$ for an odd integer n and $p \in \mathbf{N}$. We obtain the following:

$$\tau_t(a_{3m-2}) = a_{3m-2} + \big((\cos\theta_t - 1)a_{3m-2} + \sin\theta_t\, a^*_{3m-1}\big)a^*_{3m}a_{3m}, \tag{8.32}$$

$$\tau_t(a_{3m-1}) = a_{3m-1} + \big(-\sin\theta_t\, a^*_{3m-2} + (\cos\theta_t - 1)a_{3m-1}\big)a^*_{3m}a_{3m}, \tag{8.33}$$

$$\tau_t(a_{3m}) = \Big[\cos\theta_t\, I + (1 - \cos\theta_t)(a^*_{3m-2}a_{3m-2} - a^*_{3m-1}a_{3m-1})^2 + \sin\theta_t(a_{3m-2}a_{3m-1} + a^*_{3m-2}a^*_{3m-1})\Big]a_{3m}, \tag{8.34}$$

where $m \in \mathbf{N}$. As seen from the term $a_{3m-2}a_{3m-2}a_{3m}$ in (8.34), a one-particle state vector at $t \neq \pi c/\mu$ ($c \in \mathbf{Z}$) given by $\tau_t(a^*_{3m})\, e_1$ is *not* orthogonal to a three-particle state vector at $t = 0$ given by $\tau_0(a^*_{3m-2}a^*_{3m-1}a^*_{3m})\, e_1 = a^*_{3m-2}a^*_{3m-1}a^*_{3m}\, e_1$ as follows:

$$\langle\, \tau_t(a^*_{3m})\, e_1 \mid \tau_0(a^*_{3m-2}a^*_{3m-1}a^*_{3m})\, e_1 \,\rangle = -\sin\theta_t. \tag{8.35}$$

The particle number operator N_t in the present case is given by

$$N_t = \sum_{n=1}^{\infty} \tau_t(a^*_n a_n), \tag{8.36}$$

$$\begin{aligned}\tau_t(a^*_{3m-2}a_{3m-2}) &= a^*_{3m-2}a_{3m-2} - \sin^2\theta_t\,(a^*_{3m-2}a_{3m-2} + a^*_{3m-1}a_{3m-1} - I)a^*_{3m}a_{3m} \\ &\quad + \sin\theta_t\, \cos\theta_t\,(a^*_{3m-2}a^*_{3m-1} + a_{3m-1}a_{3m-2})a^*_{3m}a_{3m},\end{aligned} \tag{8.37}$$

$$\begin{aligned}\tau_t(a^*_{3m-1}a_{3m-1}) &= a^*_{3m-1}a_{3m-1} - \sin^2\theta_t\,(a^*_{3m-2}a_{3m-2} + a^*_{3m-1}a_{3m-1} - I)a^*_{3m}a_{3m} \\ &\quad + \sin\theta_t\, \cos\theta_t\,(a^*_{3m-2}a^*_{3m-1} + a_{3m-1}a_{3m-2})a^*_{3m}a_{3m},\end{aligned} \tag{8.38}$$

$$\tau_t(a^*_{3m}a_{3m}) = a^*_{3m}a_{3m}, \tag{8.39}$$

where $m \in \mathbf{N}$. It is easy to obtain the expectation values of N_t by k-particle state vectors at $t = 0$ as follows:

$$\omega(N_t;\, v_{n_1}) = \begin{cases} 1 & \text{for } n_1 = 3m-2,\ 3m-1, \\ 1 + 2\sin^2\theta_t & \text{for } n_1 = 3m, \end{cases} \tag{8.40}$$

$$\omega(N_t;\, v_{n_1,n_2}) = \omega(N_t;\, v_{n_1}) + \omega(N_t;\, v_{n_2}), \tag{8.41}$$

$$\omega(N_t;\, v_{n_1,n_2,n_3}) = \begin{cases} 3 - 2\sin^2\theta_t & \text{for } n_1 {=} 3m{-}2,\ n_2 {=} 3m{-}1,\ n_3 {=} 3m, \\ \displaystyle\sum_{i=1}^{3} \omega(N_t;\, v_{n_i}) & \text{otherwise}, \end{cases} \tag{8.42}$$

where $m \in \mathbf{N}$, and likewise in the case of $k \geqq 4$.

Example 3. Let $\{\alpha_t \,|\, t {\in} \mathbf{R}\}$ be a one-parameter group of $*$-automorphisms of $\mathcal{O}_{16}$ defined by

$$\begin{cases} \alpha_t(s_i) = s_i, \quad i = 1, 4, 6, 7, 10, 11, 13, 16, \\ \alpha_t(s_2) = \cos\theta_t\, s_2 - \sin\theta_t\, s_{15}, \quad \alpha_t(s_{15}) = \sin\theta_t\, s_2 + \cos\theta_t\, s_{15}, \\ \alpha_t(s_3) = \cos\theta_t\, s_3 - \sin\theta_t\, s_{14}, \quad \alpha_t(s_{14}) = \sin\theta_t\, s_3 + \cos\theta_t\, s_{14}, \\ \alpha_t(s_5) = \cos\theta_t\, s_5 - \sin\theta_t\, s_{12}, \quad \alpha_t(s_{12}) = \sin\theta_t\, s_5 + \cos\theta_t\, s_{12}, \\ \alpha_t(s_9) = \cos\theta_t\, s_9 - \sin\theta_t\, s_8, \quad \alpha_t(s_8) = \sin\theta_t\, s_9 + \cos\theta_t\, s_8, \\ \theta_t \equiv \mu t, \end{cases} \tag{8.43}$$

where μ is a nonvanishing real constant. In this case, it is also possible to introduce one to four mutually different constants μ's for the $SO(2)$ action on four pairs (s_2, s_{15}), (s_3, s_{14}), (s_5, s_{12}), and (s_9, s_8). If we have introduced more than one constants μ's whose ratios are irrational for at least one pair of them, $\{\alpha_t\}$ would become nonperiodic in contrast with the previous examples. Anyway, for simplicity, we have introduced only one constant μ.

Then, the corresponding one-parameter group $\{\tau_t \mid t \in \mathbf{R}\}$ of $*$-automorphisms of the CAR algebra is obtained as follows:

$$\tau_t(a_{m,j_1}) = \cos\theta_t\, a_{m,j_1} + \sin\theta_t\, b_{m,j_1}, \quad a_{m,j_1} \equiv a_{4(m-1)+j_1}, \tag{8.44}$$

$$b_{m,j_1} \equiv a_{m,j_2}\, a_{m,j_3}\, a_{m,j_4} + a^*_{m,j_2}\, a^*_{m,j_3}\, a_{m,j_4} + a^*_{m,j_3}\, a^*_{m,j_4}\, a_{m,j_2} - a^*_{m,j_4}\, a^*_{m,j_2}\, a_{m,j_3}, \tag{8.45}$$

where $m \in \mathbf{N}$, (j_1, j_2, j_3, j_4) is a cyclic permutation of $(1,2,3,4)$. From the existence of $a_{m,j_2}\, a_{m,j_3}\, a_{m,j_4}$ in (8.44) with (8.45), a one-particle state vector at $t \neq \pi c/\mu$ $(c \in \mathbf{Z})$ is not orthogonal with a three-particle state vector at $t = 0$ as in Example 2. The particle number operator N_t is given by

$$\begin{aligned} N_t =& \sum_{n=1}^{\infty} \tau_t(a_n^* a_n) \\ =& \sum_{m=1}^{\infty} \sum_{j_1=1}^{4} \Big[(1 + 2\sin^2\theta_t) a^*_{m,j_1}\, a_{m,j_1} + 2\sin^2\theta_t \big(2a^*_{m,j_2} a_{m,j_2}\, a^*_{m,j_3} a_{m,j_3}\, a^*_{m,j_4} a_{m,j_4} \\ & - a^*_{m,j_2} a_{m,j_2}\, a^*_{m,j_3} a_{m,j_3} - a^*_{m,j_3} a_{m,j_3}\, a^*_{m,j_4} a_{m,j_4} - a^*_{m,j_4} a_{m,j_4}\, a^*_{m,+j_2} a_{m,j_2} \big) \\ & - 2\sin\theta_t\, \cos\theta_t \big(a_{m,j_1}\, a^*_{m,j_2}\, a^*_{m,j_3}\, a^*_{m,j_4} + a^*_{m,j_1}\, a_{m,j_2}\, a_{m,j_3}\, a_{m,j_4} \big) \Big], \end{aligned} \tag{8.46}$$

where (j_1, j_2, j_3, j_4) is a cyclic permutation of $(1,2,3,4)$. The expectation values of N_t by k-particle state vectors at $t = 0$ are obtained as follows:

$$\omega(N_t;\, v_{n_1}) = 1 + 2\sin^2\theta_t, \tag{8.47}$$

$$\omega(N_t;\, v_{n_1,n_2}) = 2 + 4(1 - \delta_{m_1,m_2}) \sin^2\theta_t, \tag{8.48}$$

$$\omega(N_t;\, v_{n_1,n_2,n_3}) = 3 + \big(6 + 4(\delta_{m_1,m_2}\delta_{m_2,m_3} - \delta_{m_1,m_2} - \delta_{m_2,m_3} - \delta_{m_3,m_1})\big) \sin^2\theta_t, \tag{8.49}$$

$$\begin{aligned} \omega(N_t;\, v_{n_1,n_2,n_3,n_4}) =& 4 + 4\big(2 + \delta_{m_1,m_2}\delta_{m_2,m_3} + \delta_{m_2,m_3}\delta_{m_3,m_4} + \delta_{m_3,m_4}\delta_{m_4,m_1} + \delta_{m_4,m_1}\delta_{m_1,m_2} \\ & - \delta_{m_1,m_2} - \delta_{m_1,m_3} - \delta_{m_1,m_4} - \delta_{m_2,m_3} - \delta_{m_2,m_4} - \delta_{m_3,m_4} \big) \sin^2\theta_t, \end{aligned} \tag{8.50}$$

where $n_i \equiv 4(m_i - 1) + j_i$, $m_i \in \mathbf{N}$, $j_i = 1,2,3,4$ with $n_{i_1} < n_{i_2}$ for $i_1 < i_2$, and likewise in the case of $k \geqq 5$.

It is straightforward to generalize the one-parameter group $\{\alpha_t\}$ of $*$-automorphisms in the above examples to the case of $\mathcal{O}_{2^p}$. The corresponding induced one-parameter group $\{\tau_t\}$ of $*$-automorphisms of the CAR algebra may, in general, contain mixing of one-particle states and r-particle states with $r \leqq p$. By composing such $*$-automorphisms which mutually commute, it is possible to make various one-parameter group of $*$-automorphisms of the CAR algebra, which change the particle number with keeping the Fock vacuum invariant.

§9. Discussion

In the present paper, we have shown that it is possible to reveal some nontrivial structures of the CAR algebra concretely in terms of generators without difficulties by transcribing various properties of the Cuntz algebra through our recursive fermion system.

As far as the permutation representations of the Cuntz algebra are concerned, the standard recursive fermion system yields irreducible representations only from those with central cycles of length 1. It is remarkable that, from any irreducible permutation representation with a central cycle of length greater than 1, a suitable nonstandard recursive fermion system, which is obtained from the standard one by using an inhomogeneous endomorphism, yields an irreducible one. According to the recent study in C^*-algebra,[18] for any two pure states (or irreducible representations) in a rather wide class of C^*-algebras including the Cuntz algebra, there exists a $*$-automorphism connecting them. However, it seems still unknown how to construct such a $*$-automorphism explicitly in terms of generators. If it becomes to possible to construct it, we may apply it in the recursive fermion system by replacing the above inhomogeneous endomorphisms.

As shown in Sec. 3-2, a set of permutation endomorphisms φ_{σ_p} ($p \in \mathbf{N}$), which induce $*$-homomorphisms of the CAR algebra to its even subalgebra, has interesting properties such as (3.23), (3.24) and (3.43), although it is not closed with respect to the composition. It may be useful for a systematic study of such even-CAR endomorphisms to extend the set of φ_{σ_p} ($p \in \mathbf{N}$) so as to constitute an abelian semigroup. Let r be an odd positive integer and $P \equiv (p_1, p_2, \dots, p_r)$ with $p_1, \dots, p_r \in \mathbf{N}$ and $p_1 < p_2 < \cdots < p_r$. We define a family of the (p_r+1)-th order permutation endomorphisms φ_{σ_P} of $\mathcal{O}_2$ by (2.41) with $\sigma_P \in \mathfrak{S}_{2^{p_r+1}}$ being given by

$$\begin{cases} \sigma_P(1, j_1, \dots, j_{p_r}) \equiv (1, j_1, \dots, j_{p_r}), \\ \sigma_P(2, j_1, \dots, j_{p_1-1}, j_{p_1}, \dots, j_{p_2-1}, j_{p_2}, \dots, j_{p_r-1}, j_{p_r}) \\ \qquad \equiv (2, j_1, \dots, j_{p_1-1}, \hat{j}_{p_1}, \dots, j_{p_2-1}, \hat{j}_{p_2}, \dots, j_{p_r-1}, \hat{j}_{p_r}) \end{cases} \tag{9.1}$$

with $\hat{j} \equiv 3 - j$. Then, φ_{σ_P} is written as follows:

$$\begin{cases} \varphi_{\sigma_P}(s_1) = s_1, \\ \varphi_{\sigma_P}(s_2) = s_2 \displaystyle\prod_{k=1}^{r} \rho^{p_k - 1}(J), \quad J \equiv s_{2;1} + s_{1;2}, \end{cases} \tag{9.2}$$

where ρ is the canonical endomorphism of $\mathcal{O}_2$. Let $\mathcal{P}$ be the whole set of P's, each of which consists of an odd number of mutually different positive integers. Then, it is straightforward to show that $\{\varphi_{\sigma_P} \mid P \in \mathcal{P}\}$ constitutes an abelian semigroup with respect to the composition, that is, for any $P, Q \in \mathcal{P}$, there exists an element $R \in \mathcal{P}$ such that

$$\varphi_{\sigma_R} = \varphi_{\sigma_P} \circ \varphi_{\sigma_Q} = \varphi_{\sigma_Q} \circ \varphi_{\sigma_P}. \tag{9.3}$$

As for the KMS state of the CAR algebra given in Sec. 7, we have constructed it from Rep(1) of $\mathcal{O}_2$ by using the above even-CAR endomorphism φ_{σ_p} but not from a KMS state of the Cuntz algebra. Since it is known that the inverse temperature for the KMS state of the Cuntz algebra with respect to a certain type of one-parameter group of $*$-automorphisms is unique,[14–16] it does not seem desirable to construct a KMS state of the CAR algebra by restricting that of the Cuntz algebra with a unique inverse temperature. For more study of KMS states of the CAR algebra in view of the Cuntz algebra, it seems important to clarify in what class of one-parameter groups of $*$-automorphisms the uniqueness of the KMS state of the Cuntz algebra is valid.

In Sec. 8, we have constructed nontrivial one-parameter groups $\{\tau_t\}$ of $*$-automorphisms of the CAR algebra which preserve the Fock vacuum invariant. For rather simple cases such as Examples 2 and 3, we can also construct the generator of τ_t, that is, the Hamiltonian. From the unitary $u_t \in U(1, \mathcal{O}_{2^p})$ associated with the $*$-automorphism

α_t of $\mathcal{O}_{2^p}$, which is defined by

$$u_t \equiv \sum_{i=1}^{2^p} \alpha_t(s_i)\, s_i^*, \tag{9.4}$$

we obtain

$$\tau_t(a_j) = \Phi_{SR_p}^{-1}(u_t\, \boldsymbol{a}_j\, u_t^*), \quad j = 1, \ldots, p, \tag{9.5}$$

where $\boldsymbol{a}_j$'s are the seeds of the standard RFS$_p$. In the case of Examples 2 and 3, since $\{a_{p(m-1)+j} \mid j = 1, \ldots, p\}$ for a fixed $m \in \mathbf{N}$ transform in the same way as the seeds, by extrapolating the expression for u_t in terms of the seeds to that of $\{a_{p(m-1)+j} \mid m \in \mathbf{N},\ j = 1, \ldots, p\}$, it is straightforward construct the corresponding Hamitonians H generating τ_t as follows:

$$\tau_t(a_n) = e^{\sqrt{-1}\,H\,t}\, a_n\, e^{-\sqrt{-1}\,H\,t}, \quad H^* = H, \tag{9.6}$$

$$\text{Example 2: } H = \sqrt{-1}\,\mu \sum_{m=1}^{\infty} \big(a_{3m-2}^*\, a_{3m-1}^* - a_{3m-1}\, a_{3m-2}\big) a_{3m}^* a_{3m}, \tag{9.7}$$

$$\begin{aligned} \text{Example 3: } H = \sqrt{-1}\,\mu \sum_{m=1}^{\infty} \sum_{j_1=1}^{4} \big(& a_{4(m-1)+j_1}^*\, a_{4(m-1)+j_2}^*\, a_{4(m-1)+j_3}^*\, a_{4(m-1)+j_4} \\ & - a_{4(m-1)+j_4}^*\, a_{4(m-1)+j_3}\, a_{4(m-1)+j_2}\, a_{4(m-1)+j_1}\big), \end{aligned} \tag{9.8}$$

where (j_1, j_2, j_3, j_4) is a cyclic permutation of $(1, 2, 3, 4)$. Here, it should be noted that the above H's are well-defined only in the Fock representation, or more precisely, in the representations induced from those of the Cuntz algebra which are kept invariant under the corresponding α_t.

It is interesting to consider expectation values of products of $\tau_t(a_m)$'s and $\tau_t(a_m^*)$'s with mutually different time variables by the vacuum e_1, which is written as

$$\omega\big(a_{m_1}^\sharp(t_1)\, a_{m_2}^\sharp(t_2) \cdots a_{m_n}^\sharp(t_n)\big) \equiv \langle\, e_1 \mid a_{m_1}^\sharp(t_1)\, a_{m_2}^\sharp(t_2) \cdots a_{m_n}^\sharp(t_n)\, e_1 \,\rangle, \tag{9.9}$$

where $a_m^\sharp(t)$ denotes $\tau_t(a_m)$ or $\tau_t(a_m^*)$. In contrast with those obeying linear transformations such as (7.19), we found there are nontrivial truncated n-point functions, where truncation means subtraction of contributions from lower-point functions. As illustration, we consider those consisting only of a_1, a_1^*, a_2 and a_2^* in the case of Example 1. Since all one-point functions vanish, there is no difference between truncated functions and untruncated ones in the case of $n = 2, 3$. From (8.13) and (8.14), we obtain the following two-point functions and three-point ones:

$$\omega\big(a_1(t_1)a_1^*(t_2)\big) = 1, \tag{9.10}$$

$$\omega\big(a_2(t_1)a_2^*(t_2)\big) = \cos(\theta_1 - \theta_2), \tag{9.11}$$

$$\omega\big(a_2(t_1)a_2^*(t_2)a_1^*(t_3)\big) = \omega\big(a_1(t_2)a_2(t_4)a_2^*(t_3)\big)^* = \sin(\theta_1 - \theta_2), \tag{9.12}$$

$$\omega\big(a_2(t_1)a_1^*(t_2)a_2^*(t_3)\big) = \omega\big(a_2(t_3)a_1(t_2)a_2^*(t_1)\big)^* = \sin(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3), \tag{9.13}$$

and others vanish, where $\theta_i \equiv \theta_{t_i}$. Here, (9.13) reproduces (8.24) with $n = 1$ by setting

$t_1 = t$, $t_2 = t_3 = 0$. As for the truncated four-point functions, they are obtained as follows:

$$\begin{aligned}
&\omega\big(a_1(t_1)a_2(t_2)a_1^*(t_3)a_2^*(t_4)\big)_{\mathrm{T}} = \omega\big(a_2(t_4)a_1(t_3)a_2^*(t_2)a_1^*(t_1)\big)_{\mathrm{T}}^* \\
&\qquad = \omega\big(a_1(t_1)a_2(t_2)a_1^*(t_3)a_2^*(t_4)\big) + \omega\big(a_1(t_1)a_1^*(t_3)\big)\,\omega\big(a_2(t_2)a_2^*(t_4)\big) \\
&\qquad = -\cos(\theta_3-\theta_2)\cos(\theta_3-\theta_4) + \cos(\theta_2-\theta_4) \\
&\qquad = \sin(\theta_2-\theta_3)\sin(\theta_3-\theta_4), \qquad (9.14)
\end{aligned}$$

$$\begin{aligned}
&\omega\big(a_2(t_1)a_1(t_2)a_1^*(t_3)a_2^*(t_4)\big)_{\mathrm{T}} \\
&\qquad = \omega\big(a_2(t_1)a_1(t_2)a_1^*(t_3)a_2^*(t_4)\big) - \omega\big(a_1(t_2)a_1^*(t_3)\big)\,\omega\big(a_2(t_1)a_2^*(t_4)\big) \\
&\qquad = \cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3)\cos(\theta_3-\theta_4) - \cos(\theta_1-\theta_4) \\
&\qquad = \sin(\theta_1-\theta_2)\sin(\theta_2-\theta_3)\cos(\theta_3-\theta_4) + \sin(\theta_1-\theta_3)\sin(\theta_3-\theta_4), \qquad (9.15)
\end{aligned}$$

$$\omega\big(a_2(t_1)a_1^*(t_2)a_1(t_3)a_2^*(t_4)\big)_{\mathrm{T}} = \sin(\theta_1-\theta_2)\cos(\theta_2-\theta_3)\sin(\theta_3-\theta_4), \qquad (9.16)$$

and others vanish, where a subscript T denotes truncation. Likewise, we can show there are nonvanishing truncated five-point functions and six-point ones. To clarify the physical meaning of these results, it is necessary to study in more detail the one-parameter group $\{\tau_t\}$ of $*$-automorphisms as time evolutions of quantum field theoretical dynamical systems.

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